

Advanced Finite Element Methods

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Overview

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This book covers advanced techniques for the analysis of linear elastic structures by the Finite Element Method (FEM). It has been constructed from Notes prepared for the course **Advanced to Finite Element Methods** or AFEM. This course has been taught at the Department of Aerospace Engineering Sciences, University of Colorado at Boulder since 1990. It is offered every 2 or 3 years. AFEM is a continuation of **Introduction to Finite Element Methods**, or IFEM.

§1.1. CONTENTS

The course embodies five Parts:

- I **Review of Advanced Variational Methods.** The formulation of problems of engineering and physics in Strong, Weak and Variational Form.
- II. **Three-Dimensional Finite Elements:** Axisymmetric iso-P elements. Solid elements: bricks, wedges, tetrahedra, pyramids. Infinite elements.
- III. **High Performance Element Formulations.** The free formulation. The assumed natural strain (ANS) formulation and its variants. The patch test. Variational crimes. Drilling freedoms.
- IV. **Beams, Plates and Shells.** C^1 and C^0 beams. Kirchhoff plate bending elements. Reissner-Mindlin (C^0) plate bending elements. Facet and quadrilateral shell elements. Treatment of junctures. Transition elements.
- V. **Miscellaneous and Special Project Topics.** Discussion of term projects by students.

Understanding Advanced Finite Element Methods require deeper knowledge of Variational Calculus than the “recipe” level of IFEM. Accordingly, Part I of this course deals with that topic. Some of the material has been extracted from the course **Variational Methods in Mechanics** (ASEN 5637), which is no longer offered.

§1.2. WHERE THE MATERIAL FITS

The field of Mechanics can be subdivided into four major areas:

$$\text{Mechanics} \left\{ \begin{array}{l} \textit{Theoretical} \\ \textit{Applied} \\ \textit{Computational} \\ \textit{Experimental} \end{array} \right.$$

Theoretical Mechanics deals with fundamental laws and principles of mechanics studied for their intrinsic value. *Applied Mechanics* transfers this theoretical knowledge to scientific and engineering applications, especially as regards the construction of mathematical models of physical phenomena. *Computational Mechanics* solves specific problems by combining mathematical models with numerical methods implemented on digital computers, a process called *simulation*. *Experimental Mechanics* puts physical laws, mathematical models and numerical simulations to the ultimate test of observation.

Computational Mechanics is strongly interdisciplinary. The major contributing disciplines are pictured in Figure 1.1. This course will focus on Finite Element Methods. Aspects of the other

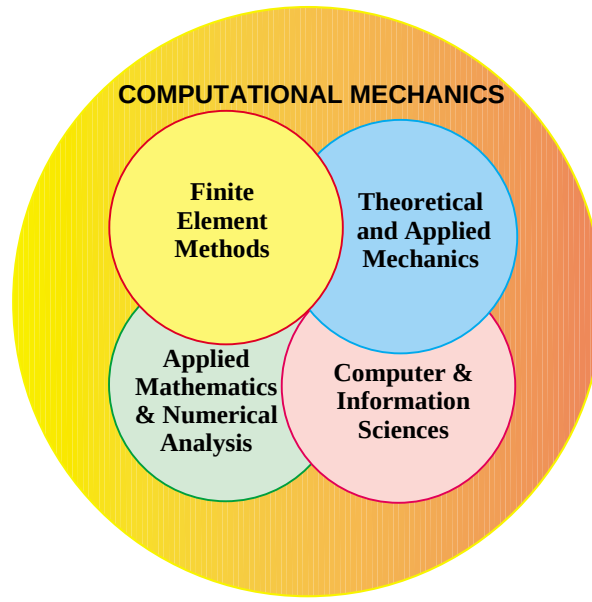


Figure 1.1. The “pizza slide:” Computational Mechanics integrates aspects of four disciplines.

contributing disciplines: Applied Mathematics and Numerical Analysis, Computer Sciences, and Applied Mechanics, will be covered as necessary, but they will not represent the major focus.

The initial Chapters deal in fact with a branch of Applied Mathematics called Variational Methods. More precisely, the application of those methods to the construction of mathematical models of mechanical systems. Various formulations that differ on the selection of primary variables are presented. Subsequent Chapters will use such formulations for building numerical approximation schemes in the form of discrete models.

The theoretical basis of Variational Methods is *Variational Calculus* or VC. Two VC “flavors”, called *standard variational calculus* or SVC and *nonstandard variational calculus* or NSVC, are mentioned below. Explanation of the technical differences between the two, however, is left for specialized courses.

§1.3. THE ANALYSIS PROCESS

Recall from IFEM that the analysis process by computer methods can be characterized by the stages diagrammed in Figure 1.2. This is an expansion of a similar figure in IFEM. The stages are *idealization*, *discretization* and *solution*.

Idealization, also called *mathematical modeling*, leads to a *mathematical model* of the physical system. In Figure 1.1 this model has been subdivided into three broad classes: Strong Form (SF), Weak Form (WF) and Variational Form (VF). These are discussed further in the rest of this Chapter.

§1.4. THE BIG PICTURE

Figure 1.3 depicts three alternative forms of a mathematical model. The yellow circles zoom into the three smaller circles of Figure 1.2.

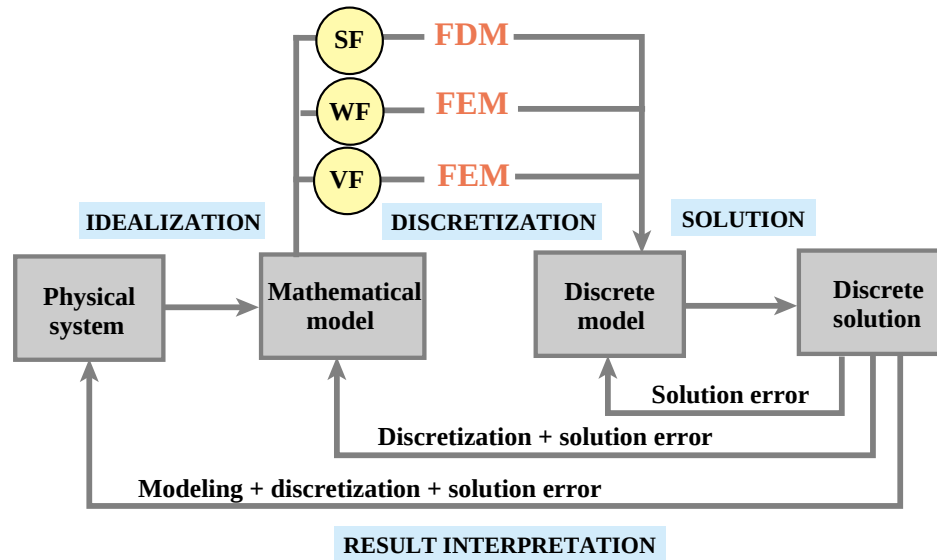


Figure 1.2. The main stages of computer-based simulation: idealization, discretization and solution. This is a slightly expanded version of a similar picture shown in Chapter 1 of IFEM.

- SF Strong Form.** Presented as a system of *ordinary or partial differential equations* in space and/or time, complemented by appropriate boundary conditions. Occasionally this form may be presented in integrodifferential form, or reduce to algebraic equations
- WF Weak Form.** Presented as a *weighted integral equation* that “relaxes” the strong form into a domain-averaging statement.
- VF Variational Form.** Presented as a *functional* whose stationary conditions generate the weak and strong forms.

Variational Calculus or VC is a set of rules and techniques by which one can pass from one of these forms to another.

§1.5. FORM TRANSFORMATIONS

Much of variational theory and practice is concerned with the *transformation* of one form into another. As diagrammed in Figure 1.4, the following transformation paths are always possible:

From SF to WF and vice-versa.

From VF to WF, or from VF to SF.

The last two transformation constitute an important part of standard variational calculus (SVC). The rules to pass from VF to SF essentially represent a generalization of the differentiation rules of ordinary calculus.

The following transformation paths are generally impossible under the framework of *standard variational calculus* (SVC):

From SF to VF.

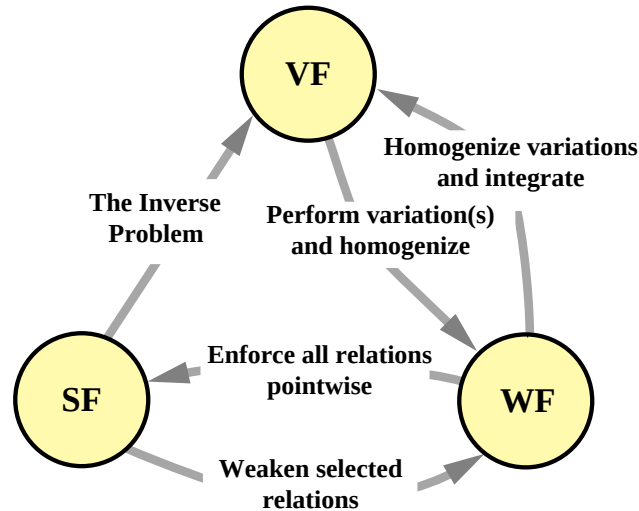


Figure 1.3. Diagram sketching Strong, Weak and Variational Forms, and relationships between form pairs. Weak Forms are also called weighted-residual equations, Galerkin equations, variational equations, variational statements, and integral statements in the literature.

From WF to VF.

Passing from a given SF to a VF is called the Inverse Problem of Variational Calculus, and may be viewed as a generalization of the problem of integrating arbitrary functions. It is therefore understandable that no general solution to this problem exists. Under *extended variational calculus* (EVC), however, such paths become possible.

§1.6. WHY VARIATIONAL METHODS?

The Strong Form (SF) states problems in ordinary or partial differential equation format. This is an old and well studied branch of calculus and mathematical physics. For example the famous Newton's Second Law: $F = ma$, is a Strong Form.

Why then the interest in Weak and Variational Forms? The following reasons may be offered.

1. **Unification:** the functional of the VF embodies all properties of the modeled system, including field equations, natural boundary conditions and conservation laws. Since functionals are scalars, and scalars are invariant with respect to coordinate transformations, the VF provides automatically for transformations between different coordinate frames.
2. VFs and WFs are the basis for technically important computer-based discrete methods of approximation, notably the Finite Element Method (FEM).
3. VFs, and to less extent WFs, directly characterizes "overall" quantities of interest to scientists and engineers. Examples: mass, momentum, energy. Mathematically these forms are said to lead naturally into *conservation laws*.
4. VFs clarify and systematize the treatment of boundary and interface conditions, particularly in connection with discretization schemes. [WFs are also useful in the handling of BCs, but no so powerful.]

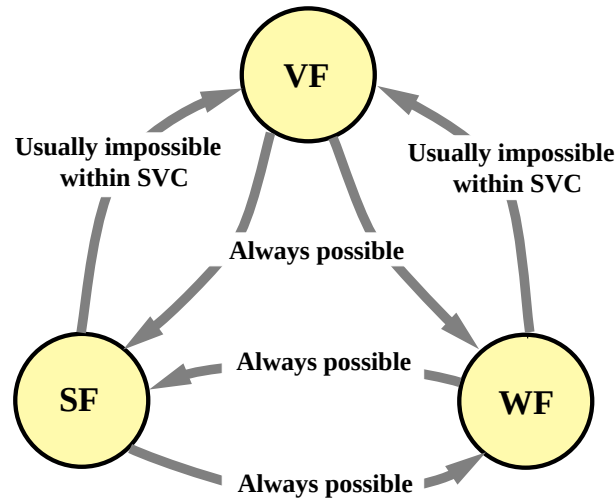


Figure 1.4. Feasibility of transformations between SF, WF and VF.

5. VFs permit a deeper and more powerful mathematical treatment of questions of existence, stability, error bounds, convergence of numerical solutions, etc. More importantly, they provide general guidelines on how to achieve desirable behavior of the related discrete schemes. [WFs are better than SFs in this regard but not as satisfactory as VFs.]

§1.7. METHODS OF APPROXIMATION: DISCRETIZATION

Transforming a SF to WF or VF does not make a problem easy to solve. Complicated problems still have to be treated by *methods of approximation*. These may be hand-based or (since the advent of the digital computer) computer-based.

The essence of all approximation methods is *discretization*. Continuum mathematical models stated in SF, WF or VF have an infinite number of degrees of freedom. Through a discretization method this is reduced to a finite number, yielding *algebraic* equations than can be solved in a reasonable time.

Each form: SF, WF and VF has a *natural* class of discretization methods than can be constructed from it. This attribute is illustrated in Figure 1.5.

§1.7.1. Finite Difference Method

The natural discretization class for SFs is the *finite difference method* (FDM). These are constructed by replacing derivatives by differences. This class is easy to generate and program for regular domains and boundary conditions, but runs into difficulties when geometry or boundary conditions become arbitrary. The other problem with conventional FDM is that the approximate solution is only obtained at the grid points, and extension to other points is not always obvious or even possible. Nevertheless the FDM class is theoretically *general* in that any problem stated in WF or VF can be put into SF.

§1.7.2. Weighted Residual Methods

The natural discretization class for WFs is the *weighted residual method* (WRM). There are well known WRM subclasses: Galerkin, Petrov-Galerkin, collocation, subdomain, finite-volume, least-squares. Sometimes these subclasses, excluding collocation, are collectively called *trial function methods*, an alternative name that accurately reflects the discretization technique. Unlike the FDM, trial-function methods yield approximate solutions defined *everywhere*. Before computers such analytical solutions were obtained by hand, a restriction that limited considerably the scope and accuracy of the approximations. That barrier was overcome with the development of the Finite Element Method (FEM) on high speed computers.

One particularly important subclass of WRM is the Finite Volume Method or FVM, which is used extensively in computational gas dynamics.

§1.7.3. Rayleigh-Ritz Methods

The natural discretization class for VFs is the *Rayleigh-Ritz method* (RRM). Although historically this was the first trial-function method, it is in fact a special subclass of the Galerkin weighted-residual method. The Finite Element Method was originally developed along these lines, and remains the most powerful computer based RRM.

Note that FEM, like FDM, can be viewed as an *universal* approximation method, because any problem can be placed in WF. This statement is no longer true, however, if one restricts FEM to the subclass of Rayleigh-Ritz method, which relies on the VF.

REMARK 1.1

In complex problems treatable within today's computer technology, combinations of these numerical methods, sometimes with a "sprinkling" of analytical techniques, are common. Some examples serve to illustrate the richness of possibilities:

1. Fluid-structure interaction: FEM for the structure, FDM or FVM for the fluid.
2. Structural dynamics: FEM in space, FDM in time.
3. Semi-analytical methods: some space directions are treated by FEM (or FDM), while others are treated analytically. The so-called *methods of lines* is a prime example.
4. Finite difference schemes may be constructed from VF and WF in combination with some FEM ideas. The resulting schemes are collectively known as Finite Difference Energy Methods (FDEM). More recently the so-called *mesh free method* has emerged through a blend of FDM and FEM techniques.

§1.8. *BOUNDARY ELEMENT METHODS: WHERE ARE YOU?

In addition to FEM and FDM, Boundary Element Methods (BEM) represents a third important class of computer-based discretization methods. The BEM is essentially a dimensionality-reducing technique that combines analytical reduction of one space dimension with the FEM discretization of the remaining space dimension(s). It does not have the generality of FEM or FDM, as it is primarily restricted (in its "pure" form) to linear problems with known fundamental solutions.

Originally BEMs were based on a fourth form not shown in Figures 1.3–1.5: the *integro-differential form* or IDF. Over the past decade substantial attention has been given to "merging" BEMs within the framework of the Finite Element Method. The effort has been motivated by the idea of integrating FEM and BEM in

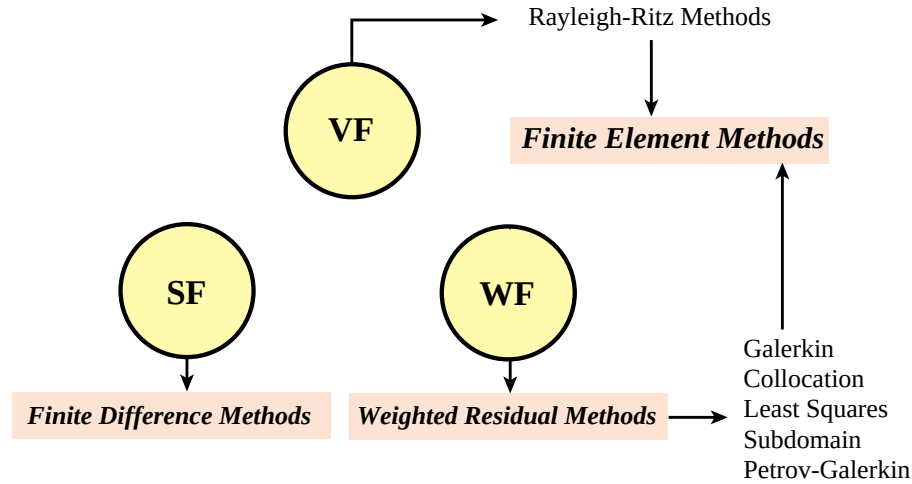


Figure 1.5. Strong, Weak and Variational Forms as source of numerical approximation methods.

the same programming framework. Thus a subclass of BEM called Variational Boundary Element Methods (VBEM) has emerged. These methods can be constructed from VFs and WFs with nonstandard application of trial functions. As of this writing, the future and importance of such methods is not clear.

§1.9. AN EXAMPLE

The following simple example serve to illustrate the three forms introduced in §1.2 and connect them with additional terminology common in applied mathematics.

Consider a function $y = y(x)$, sketched in Figure 1.6, that satisfies the ordinary differential equation

$$y'' = y + 2 \quad \text{in } 0 \leq x \leq 2. \quad (1.1)$$

Here primes denote derivative with respect to x . This is a Strong Form because (1.1) is to be satisfied *at each point* of the interval $0 \leq x \leq 2$. This interval is called the *problem domain*. By itself (1.1) is not sufficient to determine $y(x)$ and must be complemented with two boundary conditions. Two examples:

$$y(0) = 1, \quad y(2) = 4, \quad (1.2)$$

$$y(0) = 1, \quad y'(0) = 0. \quad (1.3)$$

(The first one is that pictured in Figure 1.6.) Equation (1.1) together with (1.2) defines a *boundary value problem* or BVP. Equation (1.1) together with (1.3) defines an *initial value problem* or IVP. BVPs usually model problems in spatial domains whereas IVPs model problems in the time domain.

A *residual function* associated to (1.1) is $r(x) = y'' - y - 2$. The SF (1.1) is equivalent to saying that $r(x) = 0$ at each point in the problem domain $x \in [0, 2]$. The boundary condition residual for (1.2) is $r_0 = y(0) - 1$, $r_2 = y(2) - 4$. Multiply the ODE residual $r(x)$ by a *weight function* $w(x)$ and integrate over $[0, 2]$. Multiply r_0 and r_2 by weights w_0 and w_2 and add the three terms to get

$$\int_0^2 r(x)w(x) dx + r_0w_0 + r_2w_2 = 0. \quad (1.4)$$

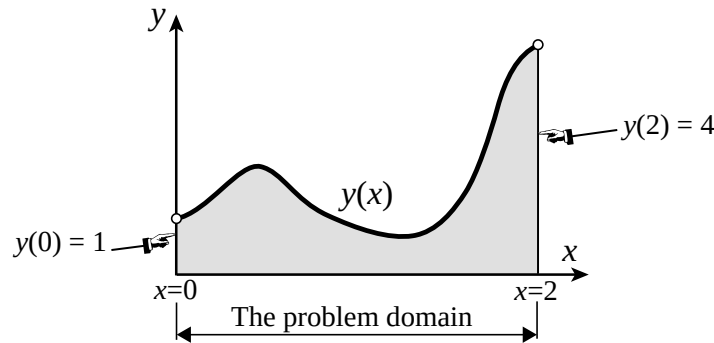


Figure 1.6. Function $y(x)$ for the example in §1.9. This function has to satisfy the boundary conditions $y(0) = 1$ and $y(2) = 4$, which together with the ODE (1.1) represents a *boundary value problem*, or BVP.

This is a *weighted integral form*. It is a Weak Form statement. Obviously a solution of the BVP (1.1)–(1.2) satisfies (1.4). However the possibility is open that other functions not satisfying that BVP may verify (1.4). Thus the qualifier “weak.”

If w , w_0 and w_2 are formally written as the variations of functions v , v_0 and v_2 , respectively, (we have not defined what a variation is, so what follows has to be accepted on faith) then (1.4) becomes

$$\int_0^2 r(x) \delta v(x) dx + r_0 \delta v_0 + r_2 \delta v_2 = 0. \quad (1.5)$$

Here δ is the variation symbol. The v ’s are technically called *test functions*. Equation (1.5) is called a *variational statement*, which leads directly to the important Galerkin and Petrov-Galerkin forms.

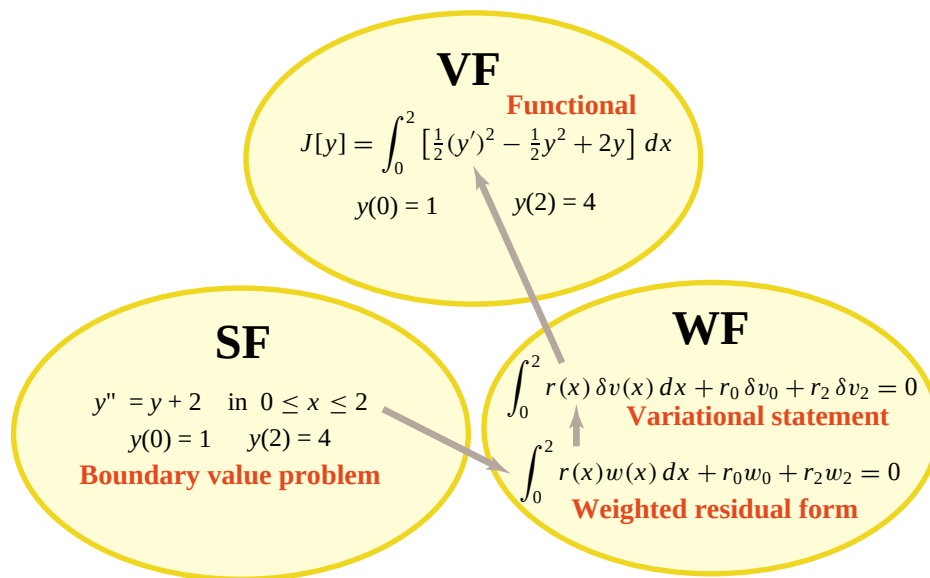


Figure 1.7. Diagrammatic representation of the example forms (1.0)–(1.5).

Finally for the BVP (1.1)-(1.2) the Inverse Problem of VC has a solution. The functional

$$J[y] = \int_0^2 \left[\frac{1}{2}(y'')^2 - \frac{1}{2}y^2 + 2y \right] dx \quad (1.6)$$

when restricted to the class of functions satisfying $y(0) = 1$ and $y(2) = 4$ becomes stationary in the VC sense when $y(x)$ satisfies (1.1), which is called the *Euler-Lagrange equation* of (1.6). This is an example of a Variational Form.

2

Decomposition of Poisson Problems

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§2.1. Introduction

In the first Chapter it was emphasized that the classical formulation of mathematical models in mechanics and physics leads to the Strong Form (SF). These *field equations* are ordinary or partial differential equations in space or spacetime in terms of a *primary variable*. They are complemented by *boundary* and/or *initial conditions*. Field equations and boundary plus initial conditions are collectively called the *governing equations*. The distinguishing property of the SF is that governing equations and conditions hold at each point of the problem domain.

Passing from the Strong Form to Weak and Variational Forms is simplified if the governing equations are presented through a scheme called *Tonti decompositions*.¹ Such schemes introduce two *auxiliary variables* that often have physical significance. One is called the *intermediate variable* and the other the *flux variable*. Examples of such variables are stresses, strains, pressures and heat fluxes. The equations that connect the primary and auxiliary variables in the decomposed SF are called *Strong Links* or *Strong Connectors*.

Tonti decompositions offer two important advantages for further development:

- (I) The construction of various types of Weak and Variational Forms can be graphically explained as weakening selected links.
- (II) The interpretation of the so-called *natural boundary conditions* is facilitated.

This Chapter illustrates the construction of the Tonti decomposition for boundary value problems modeled by the scalar Poisson's equation. We start with these problems because the governing equations are considerably simpler than for the elasticity problem of structural and solid mechanics. The simplicity is due to the fact that the primary variable is a *scalar* function whereas the intermediate variables are vectors. On the other hand, in elasticity the primary variable: displacements, is a vector, whereas the intermediate variables: strains and stresses, are tensors.

Despite this simplicity the Poisson equation governs several interesting problems in engineering and physics. The next Chapter illustrates the construction of Weak and Variational Forms for that equation.

§2.2. The Poisson Equation

Many steady-state application problems in solid, fluid and thermo mechanics, as well as electromagnetics, can be modeled by the generalized Poisson's partial differential equation. This includes the famous Laplace equation as a special case.

Suppose that $u = u(x_1, x_2, x_3)$ is a *primary* scalar function that solves a linear, steady-state (time-independent) application problem involving an *isotropic* medium. The problem is posed in a three-dimensional space spanned by the Cartesian coordinates x_1, x_2, x_3 . (The physical meaning of u changes with the application; for example in thermal conduction problems it is the temperature).

The generalized Poisson's equation is

$$\nabla \cdot (k \nabla u) = s, \quad (2.1)$$

¹ A name suggested by a graphical representation introduced by the mathematician Enzo Tonti.

where the first ∇ is the divergence operator, the second ∇ is the gradient operator, s is a given source function, and ρ is a constitutive coefficient. (This coefficient becomes a tensor ρ_{ij} in anisotropic media).

Both ρ and s may depend on the spatial coordinates, that is, $\rho = \rho(x_1, x_2, x_3)$ and $s = s(x_1, x_2, x_3)$. Equation (2.1) must be complemented by appropriate boundary conditions. These are examined in further detail in connection with the specific examples in §2.3.

If ρ is constant in space, (2.1) reduces to the standard Poisson's equation

$$\rho \nabla^2 u = s. \quad (2.2)$$

where ∇^2 is the Laplace operator. Furthermore if the source term s vanishes this reduces to the familiar Laplace's equation

$$\nabla^2 u = 0. \quad (2.3)$$

Solutions of (2.3) are called *harmonic functions*, which have been extensively studied over the past two centuries.

The extension of the foregoing equations to an unknown *vector* function $\mathbf{u}$ is straightforward. In such a case the first ∇ in (2.1) is the gradient operator and the second ∇ the divergence operator.

Remark 2.1. In unabridged component notation the Poisson's equation (2.1) in one, two, and three dimensions takes the following form:

$$\begin{aligned} \frac{\partial}{\partial x_1} \left(k \frac{\partial u}{\partial x_1} \right) &= s, \\ \frac{\partial}{\partial x_1} \left(k \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(k \frac{\partial u}{\partial x_2} \right) &= s, \\ \frac{\partial}{\partial x_1} \left(k \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(k \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left(k \frac{\partial u}{\partial x_3} \right) &= s. \end{aligned} \quad (2.4)$$

If k is not space dependent:

$$k \frac{\partial^2 u}{\partial x_1^2} = s, \quad k \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) = s, \quad k \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) = s. \quad (2.5)$$

By specializing the primary variable u to various physical quantities, we obtain models for various problems in mechanics, thermomechanics and electromagnetics. Three specific problems: thermal conduction, potential flow, electro and magnetostatics are examined below. Other applications are given as Exercises.

§2.3. Steady-State Linear Heat Conduction

Consider a thermally conducting isotropic body of volume V that obeys Fourier's law of heat conduction, as illustrated in Figure 2.1. The body is bounded by a surface S with external unit normal $\mathbf{n}$. The body is in *thermal equilibrium*, meaning that the temperature distribution $T = T(x_1, x_2, x_3)$ is independent of time. The temperature is the *primal* variable of this formulation so we will replace the u of the foregoing section by T . If the body is thermally isotropic, the k of the previous section

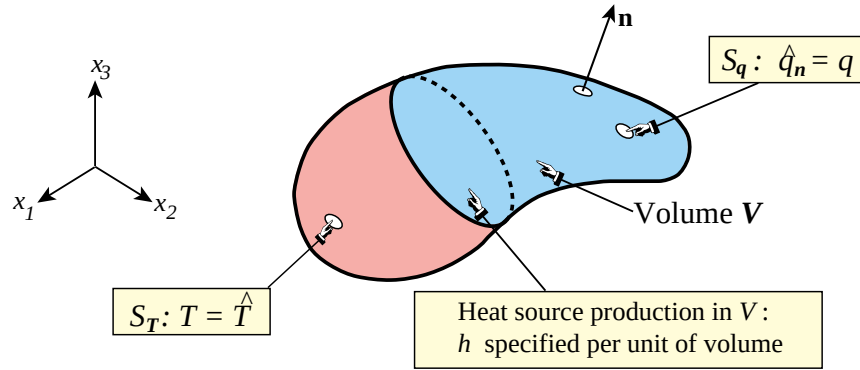


FIGURE 2.1. A heat conducting body obeying Fourier's law, in thermal equilibrium.

becomes the thermal conductivity coefficient k , with a $-$ sign to account for the positive flux sense definition. This coefficient may be a function of position.

The *source* field called s in the previous section is the distributed heat production $h = h(x_1, x_2, x_3)$ in V measured per unit of volume. This heat may be generated, for instance, by combustion or by a chemical reaction.² A negative h would indicate a volumetric heat dissipation or “sink.”

§2.3.1. The Field Equations

The *temperature gradient* vector is called $\mathbf{g} = \nabla T$, which written in full is

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \partial T / \partial x_1 \\ \partial T / \partial x_2 \\ \partial T / \partial x_3 \end{bmatrix}. \quad (2.6)$$

The *heat flux vector* $\mathbf{q}$ is defined by the constitutive equation $\mathbf{q} = -k\mathbf{g} = -k\nabla T$, which is Fourier's law of heat conduction. In full this is

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = -k \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}. \quad (2.7)$$

The heat flux along a direction d defined by the unit vector $\mathbf{d}$ is denoted by $q_d = \mathbf{q} \cdot \mathbf{d} = \mathbf{q}^T \mathbf{d}$. This is a scalar that characterizes the transport of thermal energy along that direction. It is measured in heat units per unit area. As a special case, the boundary-normal heat flux is $q_n = \mathbf{q} \cdot \mathbf{n} = \mathbf{q}^T \mathbf{n}$ evaluated on S .

The *balance equation*, which characterizes steady-state thermal equilibrium, is $\mathbf{div} \mathbf{q} + h = 0$. Written in full component notation:

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} + h = 0. \quad (2.8)$$

Equations (2.6), (2.7) and (2.8) complete the *field equations* of the heat conduction problem.

² In the human body, the heat source are calories produced from food intake.

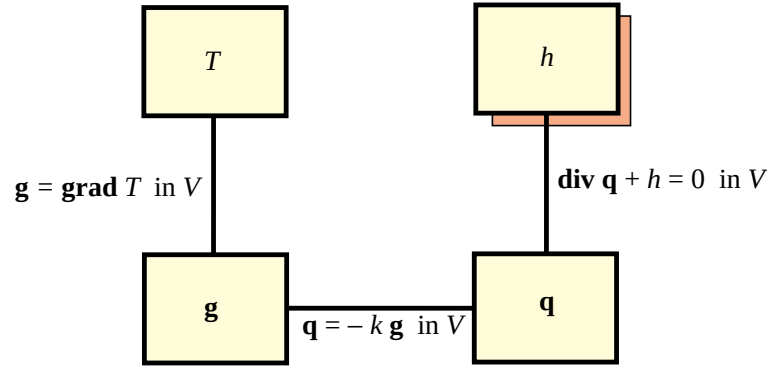


FIGURE 2.2. Tonti diagram for steady-state heat conduction problem, showing only the field equations.

§2.3.2. The Boundary Conditions

The classical *boundary conditions* for this problem are of two types:

1. The temperature T is prescribed to be equal to $\hat{T}$ over a portion S_T of the boundary S (S_T is colored red in Figure 2.1).
2. The boundary-normal heat flux $q_n = \mathbf{q} \cdot \mathbf{n} = -k(\partial T / \partial n)$ is prescribed to be equal to $\hat{q}_n$ over the complementary portion S_q of the boundary $S : S_T \cup S_q$ (S_q is colored blue in Figure 2.1).

Other boundary conditions that occur in practice are those due to *radiation* and to *convection*. Those are more complex (in fact, they are nonlinear) and are not considered here.

§2.3.3. Summary of Governing Equations

The field equations, expressed in direct notation, are now summarized and labeled:

KE:	$\nabla T = \mathbf{g}$	in V ,
CE:	$-k\mathbf{g} = \mathbf{q}$	in V ,
BE:	$\nabla \cdot \mathbf{q} + h = 0$	in V .

(2.9)

The *kinematic equation* (KE) is simply the definition of the temperature gradient vector $\mathbf{g}$. The *constitutive equation* (CE) is Fourier's law of thermal conduction. The *balance equation* (BE) is the law of thermal equilibrium: the heat flux gradient must equal to the heat created (or dissipated) per unit volume. These three labels: KE, CE and BE, will be used throughout this course for wide classes of problems governed by differential equations in space variables.

Fields $\mathbf{g}$ and $\mathbf{q}$ are called the *intermediate variable* and the *flux variable*, respectively.

Elimination of the intermediate variables $\mathbf{g}$ and $\mathbf{q}$ in (2.9) yields the scalar Poisson's equation

$$\nabla \cdot (k \nabla T) = h. \quad (2.10)$$

This shows that steady-state Fourier heat conduction pertains to the "Poisson-problem" class typified by (2.1), in which k remains k , $u \rightarrow T$ and $s \rightarrow h$.

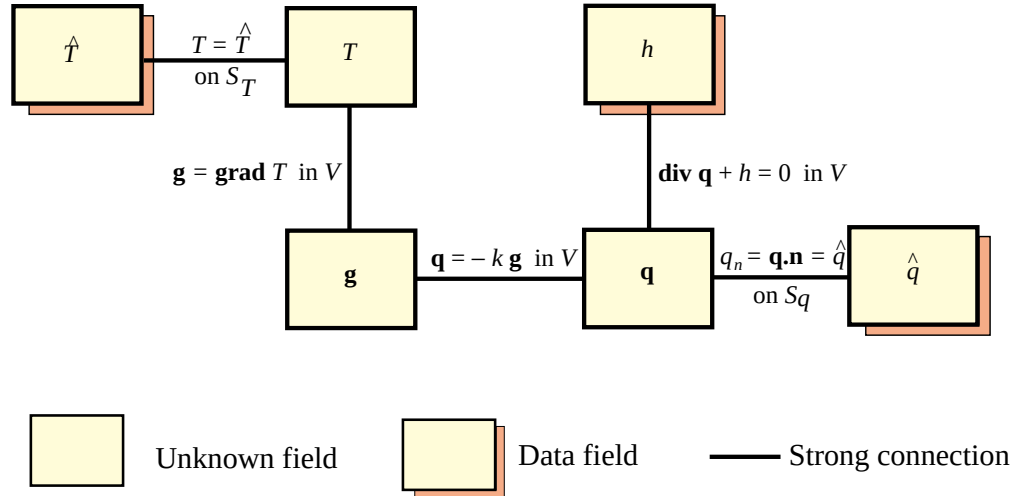


FIGURE 2.3. Tonti diagram for steady-state heat conduction problem, showing both field equations and boundary conditions.

The two classical boundary conditions are labeled as

$$\begin{array}{lll}
 \text{PBC:} & T = \hat{T} & \text{on } S_T, \\
 \text{FBC:} & \mathbf{q}^T \mathbf{n} = q_n = \hat{q}_n & \text{on } S_q.
 \end{array} \tag{2.11}$$

Here labels PBC and FBC denote *primary boundary conditions* and *flux boundary conditions*, respectively. These labels will be also used throughout the course.

The set of field equations: KE, CE, BE, and boundary conditions: PBC and FBC, are collectively called the *governing equations*. These equations constitute the statement of the mathematical model for this particular problem. This formulation is called a *boundary value problem*, or BVP.

§2.3.4. Tonti Diagrams

A convenient graphical representation of the three field equations is the so-called Tonti-diagram, which is drawn in Figure 2.2. This diagram can be expanded as illustrated in Figure 2.3 to include the boundary conditions. Graphical conventions for this expanded diagram are explained in this figure. The term “strong connection” for a relation means that it applies point by point. A “data field” is one that is given as part of the problem specification.

The expanded Tonti diagram has been found to be more convenient from the instructional standpoint than the reduced diagram, and will be adopted from now on.

Figure 2.4 shows the generic names of the components of the expanded Tonti diagram.

§2.3.5. Alternative Notations

To facilitate comparison with reference works, the governing equations are restated below in three alternative forms: in ‘grad/div’ notation, in compact indicial notation and in full (unabridged) component notation. For the second form the summation convention is implied. Indices run from

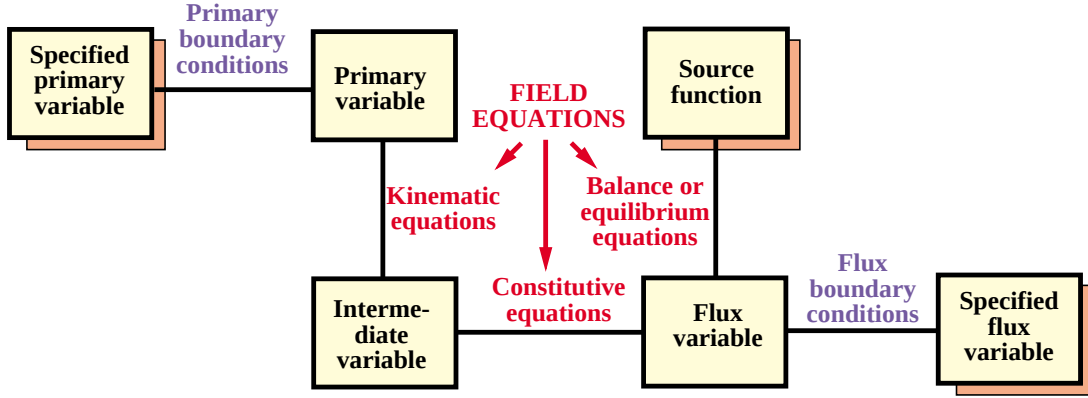


FIGURE 2.4. Generic names for the components (boxes and links) of a Tonti diagram.

1 through 3 for the three-dimensional case. The range is reduced to 2 or 1 if the number of space dimensions is reduced to two and one, respectively.

Matrix/vector form:

KE:	$\mathbf{grad} T = \mathbf{g}$	in V ,	(2.12)
CE:	$-k\mathbf{g} = \mathbf{q}$	in V ,	
BE:	$\mathbf{div} \mathbf{q} + h = 0$	in V ,	
PBC:	$T = \hat{T}$	on S_T ,	
FBC:	$\mathbf{q} \cdot \mathbf{n} = q_n = \hat{q}_n$,	on S_q .	

Indicial form:

KE:	$T_{,i} = g_i$	in V ,	(2.13)
CE:	$-kg_i = q_i$	in V ,	
BE:	$q_{i,i} + h = 0$	in V ,	
PBC:	$T = \hat{T}$	on S_T ,	
FBC:	$q_i n_i = q_n = \hat{q}_n$,	on S_q .	

Unabridged:

KE:	$\frac{\partial T}{\partial x_1} = g_1, \quad \frac{\partial T}{\partial x_2} = g_2, \quad \frac{\partial T}{\partial x_3} = g_3,$	in V ,	(2.14)
CE:	$-kg_1 = q_1, \quad -kg_2 = q_2, \quad -kg_3 = q_3,$	in V ,	
BE:	$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} + h = 0$	in V ,	
PBC:	$T = \hat{T}$	on S_T ,	
FBC:	$q_1 n_1 + q_2 n_2 + q_3 n_3 = \hat{q}_n$	on S_q .	

§2.4. Steady Potential Flow

As next example consider the potential flow of a fluid of mass density ρ that occupies a volume V .³ The fluid volume is bounded by a surface S with external unit normal $\mathbf{n}$. The flow is characterized by the *velocity field* 3-vector $\mathbf{v}(x_1, x_2, x_3)$, which is independent of time. For irrotational flow this field can be expressed as the gradient $\mathbf{v} = -\nabla\phi$ of a scalar function $\phi(x_1, x_2, x_3)$ called the *velocity potential*.

This potential is chosen as primal variable. Note the physical contrast with the thermal conduction problem discussed in §2.3. In heat conduction the primal field — the temperature — has immediate physical meaning whereas the temperature gradient $\mathbf{g}$ is a convenient intermediate variable. On the other hand, in potential flow the field of primary significance — fluid velocity — is an intermediate variable whereas the primal field — the velocity potential — has no physical significance. Despite this contrast the two problems share the same mathematical formulation as explained below.

The forcing and boundary conditions are as follows:

1. The source field is σ , the fluid mass production per unit of volume. Such production is rare in applications. Thus for most potential flow problems $\sigma = 0$.
2. The potential ϕ is prescribed to be equal to $\hat{\phi}$ over a portion S_ϕ of the boundary S .
3. The fluid momentum density $m_n = \rho\mathbf{v}\cdot\mathbf{n}$ is prescribed to be equal to $\hat{m}_n$ over the complementary portion S_m of the boundary $S : S_\phi) \cap S_m$.

In practice the most common boundary condition is that of prescribed normal velocity $\mathbf{v}\cdot\mathbf{n} = v_n = \hat{v}_n$. This can be easily transformed to the prescribed momentum density B.C. on multiplying by the density. Mathematically the momentum density B.C. is the correct one.

The field equations, expressed in direct notation, are:

KE:	$-\nabla\phi = \mathbf{v}$	in V ,	(2.15)
CE:	$\rho\mathbf{v} = \mathbf{m}$	in V ,	
BE:	$\nabla \cdot \mathbf{m} = \sigma$	in V .	

The *kinematic equation* (KE) is simply the definition of the velocity potential. The constitutive equation (CE) is the definition of momentum density. The *balance equation* (BE) expresses conservation of mass.

Elimination of the intermediate variables $\mathbf{v}$ and $\mathbf{m}$ in (2.15) yields the scalar Poisson's equation

$$\nabla \cdot (\rho \nabla \phi) = \sigma. \quad (2.16)$$

This shows that steady potential flow pertains to the “Poisson-problem” class (2.1), in which $k \rightarrow \rho$, $u \rightarrow \phi$ and $s \rightarrow \sigma$. As noted above, usually $\sigma = 0$ whereas ρ is constant, whereupon (2.16) reduces to the Laplace's equation $\nabla^2\phi = 0$.

The boundary conditions are:

³ In fluid mechanics, *potential flow* is short for steady barotropic irrotational flow of a perfect fluid.

PBC:	$\phi = \hat{\phi}$	on S_ϕ ,
FBC:	$\mathbf{m} \cdot \mathbf{n} = m_n = \hat{m}_n$,	on S_m .

(2.17)

It should be obvious now that steady potential flow and steady heat conduction are *mathematically equivalent problems*, despite the great disparity in the physical interpretation of primal and intermediate quantities.

§2.5. Electrostatics

Electrostatics is concerned with the calculation of the steady-state *electrical field* 3-vector $\mathbf{E}(x_1, x_2, x_3)$ in a volume V filled by a dielectric material or medium of permittivity ϵ (this property measures the inductive capacity of the medium; it is also called the dielectric constant).

As in the case of potential flow, $\mathbf{E}$ is not the primal field but is derived from the *electric potential* $\Phi(x_1, x_2, x_3)$ as $\mathbf{E} = -\nabla\Phi$. Thus $\mathbf{E}$ plays the role of intermediate variable. The flux-like variable is the 3-vector $\mathbf{D} = \epsilon\mathbf{E}$, which receives the names of *electric field intensity* or the *electric flux density*.

The forcing and boundary conditions are as follows:

1. The source field is ρ , the electric charge per unit of volume. (This symbol should not be confused with mechanical density). For many electrostatic problems all charges migrate to the surface S , thus $\rho = 0$ in the volume.
2. The potential Φ is prescribed to be equal to $\hat{\Phi}$ over a portion S_Φ of the boundary S .
3. The normal electric flux $D_n = \mathbf{D} \cdot \mathbf{n}$ is prescribed to be equal to $\hat{D}_n$ over the complementary portion S_D of the boundary $S : S_\Phi \cap S_D$.

The electric potential has more physical significance than the (mathematically equivalent) velocity potential in potential flow. In electric circuits this potential can be directly measured as *voltage*. Similarly the flux condition has direct physical interpretation as electric flow, or current. Thus both boundary conditions are physically important.

The field equations, expressed in direct notation, are:

KE:	$-\nabla\Phi = \mathbf{E}$	in V ,
CE:	$\epsilon\mathbf{E} = \mathbf{D}$	in V ,
BE:	$\nabla \cdot \mathbf{D} = \rho$	in V .

(2.18)

The *kinematic equation* (KE) is the definition of the electric potential. The constitutive equation (CE) relates electric intensity and flux through the dielectric constant. The *balance equation* (BE) expresses conservation of charge (this last relation is also called Gauss' law and is one of the famous Maxwell equations).

Elimination of the intermediate variables $\mathbf{E}$ and $\mathbf{D}$ in (2.18) yields the scalar Poisson's equation

$$\nabla \cdot (\epsilon \nabla \Phi) = -\rho \quad (2.19)$$

This shows that electrostatics pertains to the “Poisson-problem” class. Often $\rho = 0$ and ϵ is constant, whereupon (2.19) reduces to the Laplace’s equation $\nabla^2 \Phi = 0$.

The boundary conditions are:

PBC:	$\Phi = \hat{\Phi}$	on S_Φ ,
FBC:	$\mathbf{D} \cdot \mathbf{n} = D_n = \hat{D}_n$,	on S_D .

(2.20)

§2.6. *Magnetostatics

Magnetostatics is concerned with the calculation of the steady-state *magnetic flux density* 3-vector $\mathbf{B}(x_1, x_2, x_3)$ in a volume V filled by a material or medium of permeability μ .

The magnetic field $\mathbf{B}$ is a *solenoidal* vector (meaning that its divergence is zero). Thus it can be derived from the 3-vector *magnetic potential* $\mathbf{A}(x_1, x_2, x_3)$ as $\mathbf{B} = \nabla \times \mathbf{A}$. Hence $\mathbf{B}$ plays the role of intermediate variable, but unlike the three previous examples, the primal variable $\mathbf{A}$ is a vector and not a scalar. The flux-like variable is the 3-vector $\mathbf{H} = (1/\mu)\mathbf{B}$, which receives the name of *magnetic field intensity*.

The forcing and boundary conditions are as follows:

1. The source field is $\mathbf{J}$, the electric current density, which is a 3-vector.
2. The quantity $\mathbf{A} \times \mathbf{n} = A_{\times n}$ is prescribed to be equal to $\hat{A}_{\times n}$ over a portion S_A of the boundary S .
3. The quantity $H_{\times n} = \mathbf{H} \times \mathbf{n}$ is prescribed to be equal to $\hat{H}_{\times n}$ over the complementary portion S_H of the boundary $S : S_A \cup S_H$.

The field equations, expressed in direct notation, are:

KE:	$\nabla \times \mathbf{A} = \mathbf{B}$	in V ,
CE:	$\mu^{-1}\mathbf{B} = \mathbf{H}$	in V ,
BE:	$\nabla \times \mathbf{H} = \mathbf{J}$	in V .

(2.21)

The *kinematic equation* (KE) is the definition of the magnetic potential. The *constitutive equation* (CE) relates magnetic intensity and flux through the permeability constant. The *balance equation* (BE) expresses conservation of current (this last relation is one of the famous Maxwell equations).

Elimination of the intermediate variables $\mathbf{B}$ and $\mathbf{H}$ in (2.21) yields

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) = \mathbf{J}, \quad (2.22)$$

This can be transformed into a vector Poisson’s equation given in any book on field electromagnetics. Finally, the boundary conditions are:

PBC:	$\mathbf{A} \times \mathbf{n} = A_{\times n} = \hat{A}_{\times n}$	on S_A ,
FBC:	$\mathbf{H} \times \mathbf{n} = H_{\times n} = \hat{H}_{\times n}$,	on S_H .

(2.23)

Homework Exercises for Chapter 2

Decomposition of Poisson Problems

EXERCISE 2.1 [A:5] Show that **div** is the transpose of **grad** when these operators are treated as vectors.

EXERCISE 2.2 [A:20=15+5] A bar of length L , elastic modulus E and variable cross-sectional area $A(x)$ is aligned along the x axis, extending from $x = 0$ through $x = L$. The bar axial displacement is $u(x)$. It is loaded by a force $q(x)$ along its length. At $x = 0$ the displacement $u(0)$ is prescribed to be $\hat{u}_0$. At $x = L$ the bar is loaded by axial force $\hat{N}_L$, positive towards $x > 0$. The field equations are $e = du/dx$, $N = EAe$, $dN/dx + q = 0$, and the boundary conditions are $u(0) = \hat{u}_0$ and $N(L) = \hat{N}_L$.

- Is this problem governed by the Poisson equation, and if so, what is the correspondence with, say, the first of (2.4)?
- Draw the expanded Tonti diagram for this problem.

EXERCISE 2.3 [A:25=10+10+5] A steady-state heat conduction problem is posed over the “cylindrical” two-dimensional domain $ABCD$ depicted in Figure E2.1. with dimensions and boundary conditions as shown. (Axis x_3 comes out of the plane of the paper. Domain $ABCD$ extends indefinitely along x_3 , and all conditions are independent of that dimension.) The conductivity k is uniform over the domain $ABCD$.

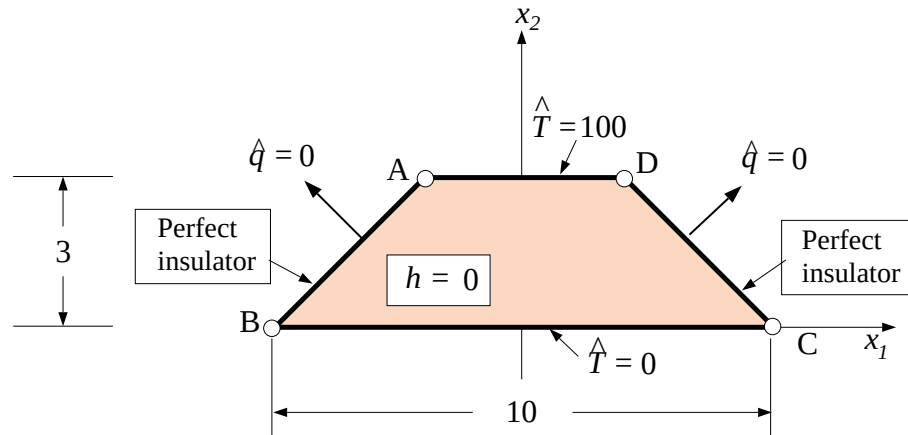


Figure E2.1. A steady-state heat conduction problem.

- Indicate which portions of the boundary form S_T and S_q . Can something clever be said about the symmetry plane $x_1 = 0$ that may allow the problem to be posed on only half of the domain?
- Does the temperature distribution satisfy the Laplace equation $\nabla^2 T = 0$?
- Does the “guess solution” $T = 100x_2/3$ satisfy the field equations and boundary conditions?

EXERCISE 2.4 [A:25] The Saint Venant theory of torsion of a cylindrical bar of arbitrary cross section (cf. Figure E2.2) may be posed as follows.⁴ The problem domain is the bar cross section A , delimited by boundary B . This domain is assumed simple connected, i.e. the section is not hollow. The bar material is isotropic with shear modulus G . The primary variable is the two-dimensional *stress function* $\phi(x_1, x_2)$. This function satisfies the standard Poisson equation

$$\nabla^2 \phi = -2G\theta, \quad (\text{E2.1})$$

⁴ See, for example, Chapter 11 of Timoshenko and Goodier *Theory of Elasticity*, McGraw-Hill, 1951.

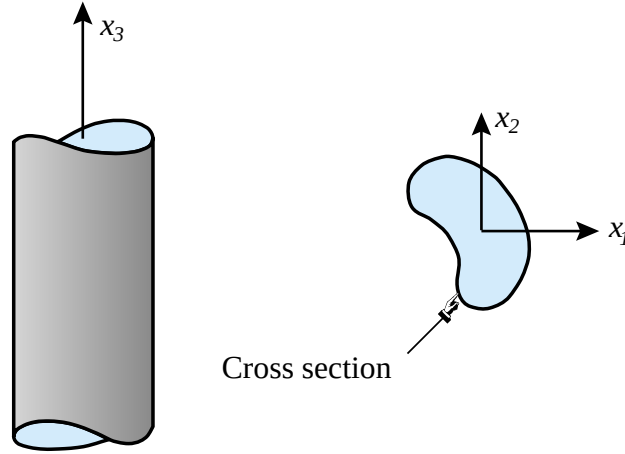


Figure 2.2. The Saint-Venant torsion problem for Exercise 2.4.

where θ the torsion angle of rotation (about x_3) per unit length. The function ϕ must be constant over the boundary B : $\phi = C$. In the case of singly-connected cross-section domains (solid bars) this constant can be chosen arbitrarily and for convenience may be taken as zero.

The applied torque is given by $M_t = 2 \int_A \phi \, dA = GJ\theta$, where J is the torsional rigidity. The shear stresses are

$$\sigma_{13} = \frac{\partial \phi}{\partial x_2}, \quad \sigma_{23} = -\frac{\partial \phi}{\partial x_1}. \quad (\text{E2.2})$$

The shear stresses satisfy the equilibrium equations

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0. \quad (\text{E2.3})$$

Draw an expanded Tonti diagram for this problem, in which ϕ is the primary variable, the shear stresses are taken as flux variables, the gradient of ϕ is the intermediate variable, and angle θ is the source. Where would you place the specified-moment condition in the diagram? (Hint: connect it to the source).

EXERCISE 2.5 [A:15] Draw the expanded Tonti diagram for potential flow (§2.4). Identify governing equations along the strong links in direct form. Where in the diagram would you specify a prescribed velocity boundary condition?

EXERCISE 2.6 [A:15] Draw the expanded Tonti diagram for electrostatics (§2.5). Identify governing equations along the strong links in direct form.

3

Weak and Variational Forms of the Poisson Equation

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§3.1. Introduction

The chapter explains the construction of Weak and Variational Forms for the Poisson equation introduced in the previous Chapter. The Tonti diagram is used to visualize links that are weakened in the construction process. Two examples are worked out. One leads to the *primal functional*, the other to one of several possible *mixed functionals*.

§3.2. The Poisson Equation

For convenience we repeat here the split governing equations for the Strong Form of the Poisson equation studied in §2.2. The problem domain is depicted in Figure 3.1. We employ a generic notation of u for the primary variable, k for the constitutive coefficient, s for the source, $\mathbf{g} = \nabla u$ for the gradient of u , and $\mathbf{q} = \rho \mathbf{g}$ for the flux function.¹

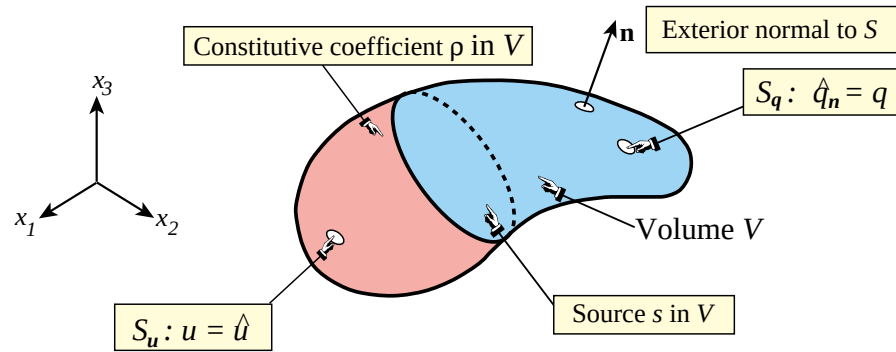


FIGURE 3.1. The problem domain for the Poisson equation, using generic notation.

Field equations:

KE:	$\nabla u = \mathbf{g}$	in V ,
CE:	$\rho \mathbf{g} = \mathbf{q}$	in V ,
BE:	$\nabla \cdot \mathbf{q} = s$	in V .

(3.1)

Classical boundary conditions:

PBC:	$u = \hat{u}$	on S_u ,
FBC:	$\mathbf{q} \cdot \mathbf{n} = \mathbf{q}^T \mathbf{n} = q_n = \hat{q}_n$,	on S_q .

(3.2)

Elimination of $\mathbf{g}$ and $\mathbf{q}$ yields the standard Poisson partial differential equation $\nabla \cdot (\rho \nabla u) = s$. If ρ is constant over V , this reduces to $\rho \nabla^2 u = s$, and if $s = 0$, to the Laplace equation $\nabla^2 u = 0$.

The Tonti diagram of the Strong Form is shown in Figure 3.2.

From the Strong Form one can proceed to several Weak Forms (WF) by selectively “weakening” strong connections. Two choices are worked out in the following sections: the Primal functional and a Hellinger-Reissner-like mixed functional.

¹ For the thermal conduction problem, u , ρ and s become T , $-k$ and $-h$, respectively. In fluid problems ρ is the mass density.

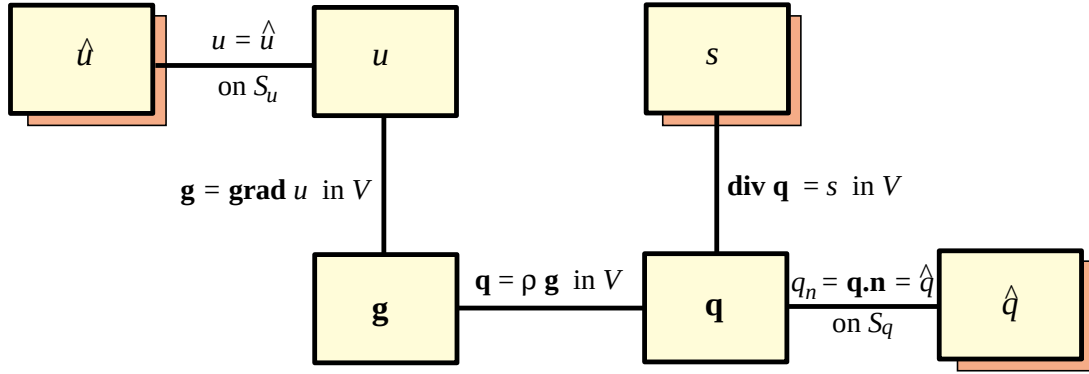


FIGURE 3.2. The Tonti diagram for the Strong Form of the generic Poisson equation.

§3.3. The Primal Functional

The *Primal functional* is the most practically important one as a basis for FEM models of the Poisson equation. It is the analog of the Total Potential Energy of elasticity.

§3.3.1. Weighted Residual Form

Two of the strong links of Figure 3.2 are weakened, as shown in Figure 3.3:

1. The Balance Equation (BE) $\nabla \cdot \mathbf{q} = s$ in V . The residual is $r_{BE} = \nabla \cdot \mathbf{q} - s$.
2. The Flux Boundary Condition (FBC) $\mathbf{q} \cdot \mathbf{n} = q_n = \hat{q}$ on S_q . The residual is $r_{FBC} = \mathbf{q} \cdot \mathbf{n} - \hat{q}$.

The weighted residual statement is

$$R_{BE} = \int_V (\nabla \cdot \mathbf{q} - s) w_{BE} dV = 0, \quad R_{FBC} = \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q}) w_{FBC} dS = 0. \quad (3.3)$$

Here w_{BE} and w_{FBC} denote the weighting functions applied to the residuals r_{BE} and r_{FBC} , respectively. Both are scalar functions.

§3.3.2. Master and Slave Fields

The statement (3.3) is not very useful by itself. The field $\mathbf{q}$ is “floating.” The weights are unknown. Looks like trying to find a black cat in a dark cellar at midnight.

We circumvent the first uncertainty by linking $\mathbf{q}$ to the primary variable u . How? By traversing the strong links: $u \rightarrow \mathbf{g} \rightarrow \mathbf{q}$. A special notation, illustrated in Figure 3.4, is introduced to remind us that the auxiliary fields $\mathbf{g}$ and $\mathbf{q}$ are “strongly anchored” to u :

$$\mathbf{g}^u = \nabla u, \quad \mathbf{q}^u = \rho \mathbf{g}^u = k \nabla u. \quad (3.4)$$

The notation separates the unknown fields u , $\mathbf{g}$ and $\mathbf{q}$ into two groups. u is the *master field*, also called *primary*, *varied* or *parent* field. It is the only one to be varied in the sense of Variational Calculus. The other two: $\mathbf{g}^u$ and $\mathbf{q}^u$ are the *slave* fields, also called *secondary*, *derived* or *sibling*

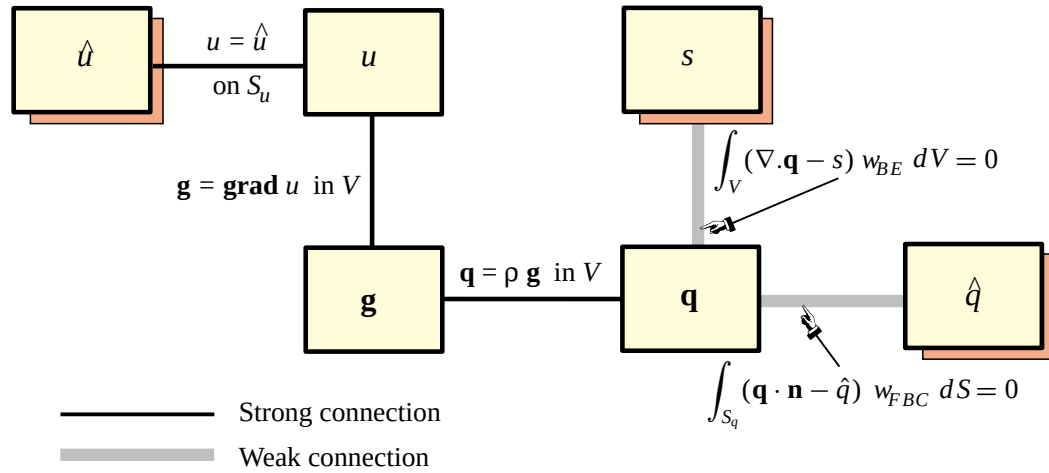


FIGURE 3.3. A Weak Form of the Poisson problem, in which the BE and FBC links have been weakened.

fields. The superscript, in this case $(\cdot)^u$, indicates ownership or “line of descent.” Slave fields *must be connected to their master through strong links*.

We now rewrite the integrated residuals (3.3) using the master field:

$$\begin{aligned} R_{BE} &= \int_V (\nabla \cdot \mathbf{q}^u - s) w_{BE} dV = \int_V (\nabla \cdot \rho \nabla u - s) w_{BE} dV \\ R_{FBC} &= \int_{S_q} (\mathbf{q}^u \cdot \mathbf{n} - \hat{q}) w_{FBC} dS = \int_{S_q} (\rho \nabla u) \cdot \mathbf{n} - \hat{q}) w_{FBC} dS. \end{aligned} \quad (3.5)$$

Now we have a better handle on the residuals since those depend only on u , whereas the weights are still at our disposal. These equations can be used to generate numerical methods upon selection of the weight functions, using the Method of Weighted Residuals. For example if $w_{BE} = w_{FBC} = 1$ one obtains the so-called subdomain method, one of whose variants (in fluid applications) is the Fluid Volume Method. Other possibility is to take weights to be the same as the corresponding residuals, which leads to the two-century-old method of Least Squares.

But for the Poisson equation a Variational Form exists. Why not try for the best?

§3.3.3. Going for the Gold

Start from (3.5) as the point of departure. Replace w_{BE} and w_{FBC} by the variations $-\delta u$ and δu of the primary variable u .² Also rename residual R as $\delta \Pi$ to emphasize that this will be hopefully the variation of a functional Π as yet unknown:

$$\delta \Pi_{BE} = \int_V (-\nabla \cdot \rho \nabla u + s) \delta u dV, \quad \delta \Pi_{FBC} = \int_{S_q} (\rho \nabla u \cdot \mathbf{n} - \hat{q}) \delta u dS. \quad (3.6)$$

² The minus sign in the substitution $w_{BE} \rightarrow -\delta u$ is inconsequential; it just gives a nicer fit with the divergence theorem transformation derived in §3.5.

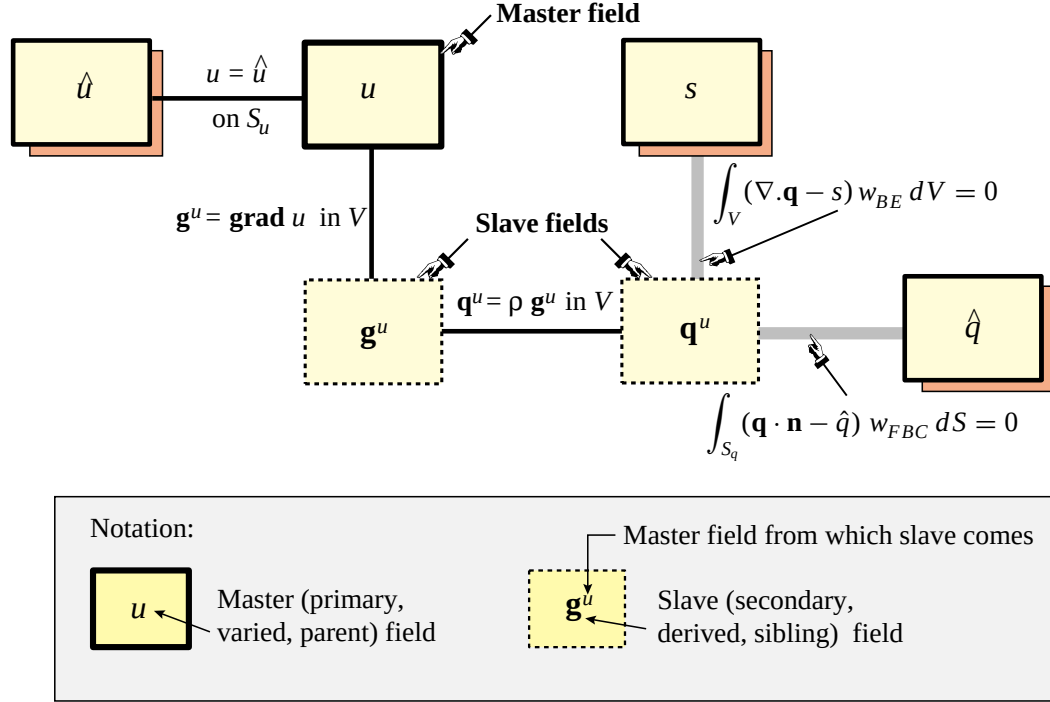


FIGURE 3.4. Figure 3.4. Rehash of the previous figure, in which the gradient $\mathbf{g}$ and flux $\mathbf{q}$, relabeled $\mathbf{g}^u$ and $\mathbf{q}^u$, are designated as *slave fields*. Both derive from the *master field* u .

The variation $\delta\Pi$ will be a combination of these two, for example $\delta\Pi_{BE} + \delta\Pi_{FBC}$. Signs could be adjusted so that $\delta\Pi$ is an exact variation, as worked out below.

The next operation is technical. We must reduce the order of the derivatives appearing in the volume integral of $\delta\Pi_{BE}$ from two to one using the form of the Gauss divergence theorem worked out in detail in §3.5.³ The useful transformation is:

$$-\int_V \nabla \cdot (\rho \nabla u) \delta u dV = \int_V \rho \nabla u \cdot \delta \nabla u dV - \int_S \rho \nabla u \cdot \mathbf{n} \delta u dS. \quad (3.7)$$

By definition $u = \hat{u}$ over S_u so $\delta u = 0$ there, and the surface integral over S reduces to one over S_q :

$$-\int_V \nabla \cdot (\rho \nabla u) \delta u dV = \int_V \rho \nabla u \cdot \delta \nabla u dV - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS. \quad (3.8)$$

Inserting this into the first of (3.6) yields

$$\delta\Pi_{BE} = \int_V (\rho \nabla u \cdot \delta \nabla u + s \delta u) dV - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS. \quad (3.9)$$

The combination $\delta\Pi_{BE} + \delta\Pi_{FBC}$ conveniently cancels out the integral of $k \nabla u \cdot \mathbf{n} \delta u$ over S_q . The

³ Presented there for convenience. The theorem can be found in any Advanced Calculus textbook.

variation symbol δ can be then pulled in front of the integrals:

$$\begin{aligned}\delta\Pi &= \delta\Pi_{BE} + \delta\Pi_{FBC} = \int_V (\rho \nabla u \cdot \delta \nabla u + s \delta u) dV - \int_{S_q} \hat{q} \delta u dS \\ &= \delta \int_V \frac{1}{2} \rho \nabla u \cdot \nabla u dV + \delta \int_V s u dV - \delta \int_{S_q} \hat{q} u dS.\end{aligned}\quad (3.10)$$

Consequently the required Primal functional Π , subscripted by “TPE” to emphasize its similarity to the Total Potential Energy functional of elasticity, is

$$\begin{aligned}\Pi_{\text{TPE}}[u] &= \frac{1}{2} \int_V k \nabla u \cdot \nabla u dV + \int_V s u dV - \int_{S_q} \hat{q} u dS \\ &= \frac{1}{2} \int_V (\mathbf{q}^u)^T \mathbf{g}^u dV + \int_V s u dV - \int_{S_q} \hat{q} u dS\end{aligned}\quad (3.11)$$

The bracket argument of Π is often used in Variational Calculus books to denote what is the primary variable. Using full component notation,

$$\Pi_{\text{TPE}}[u] = \frac{1}{2} \int_V \rho \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 + \left(\frac{\partial u}{\partial x_3} \right)^2 \right] dV + \int_V s u dV - \int_{S_q} \hat{q} u dS. \quad (3.12)$$

The variational principle is

$$\delta\Pi_{\text{TPE}} = 0. \quad (3.13)$$

where the variation is taken with respect to u . Equations (3.12) and (3.13) represent the Primal Variational Form of the Poisson equation.

Remark 3.1. If we work out the Euler-Lagrange (EL) equations of (3.12)-(3.13) using the rules of Variational Calculus, we obtain $\nabla \cdot \rho \nabla u = s$ in V as EL equation, and $\rho \nabla u = \hat{q}$ on S_q as natural boundary condition. These are precisely the equations that were weakened in §3.3.1. See Figure 3.4. The remaining equations, which pertain to the strong links, are assumed to hold *a priori*.

This happening is not accidental, but can be presented as general rule:

The variational principle only reproduces the weak links

as EL equations and natural boundary conditions, respectively. The rule is discussed in more detail in chapters dealing with hybrid principles.

§3.3.4. Work Pairings

The primal functional (3.11) or (3.12) displays the following “variable pairings” in the volume integrals: $s u$ and $\mathbf{q} \cdot \mathbf{g}$. In the surface integral over S_q we find $\hat{q} u$. These products can be interpreted physically as work or energy of the kind determined by the application being modeled.⁴

⁴ For example, thermal energy in the thermal conduction problem.

The physical units of the variables and data must reflect that fact. The groupings are known as *work pairings* or *energy conjugates*.⁵ For volume integrals, the rule is: work pairings relate the *left and right boxes drawn at the same level* in the field-equations portion of the Tonti diagram.

In variational statements, such as $\delta\Pi = 0$, we find pairings such as $s \delta u$ and $(\nabla \cdot \mathbf{q}) \delta u$ in V and $q_n \delta u$ on S_q . These pairings provide guidelines on how to replace weight functions by variations with the correct physical dimensions. The rules are particularly important when constructing multifield and mixed functionals, as done next.

§3.4. A Mixed Functional of HR Type

Some definitions to start. A *multifield functional* is one that has more than one master field. It is *single-field* otherwise; for example the functional (3.12) derived above. A multifield functional is called *mixed* when the multiple master fields are internal.

There are several motivations for constructing mixed functionals. One reason related to numerical methods is to try for *balanced* approximations. Generally the master field is well approximated by a FEM discretization, whereas associated slave fields, such as gradients and fluxes in the $\Pi_{\text{TPE}}[u]$ functional, may be comparatively inaccurate. This is because differentiation, represented here by operations such as $\mathbf{g}^u = \nabla u$, amplifies errors. A functional in which u and $\mathbf{q}$ are master fields would be more balanced in that regard. A more general motivation for mixed functionals is the development of hybrid functionals, a topic covered latered in this course.⁶

The prototype of mixed functionals is the famous Hellinger-Reissner (HR) functional of linear elasticity, in which both displacements and stresses are independently varied. We proceed to derive now the analogous functional for the Poisson equation.

§3.4.1. The Weak Form

The development of the single-field functional in §3.3 started from the identification of weak links, followed by picking a master field. In the case of a multifield functional, the process should be reversed because “multiple slaves” appear, which forces us to draw more boxes.

The first decision is: *select the masters*. Here we pick u and $\mathbf{q}$. Then weaken selected links. The appropriate choices for an HR-like principle are shown in Figure 3.5. Three links are weakened. They are BE, FBC, and the connection between the slave gradients $\mathbf{g}^u$ and $\mathbf{g}^q$, denoted as GG. Links PBC, KE and CE are kept strong.

The weak link residuals are $r_{BE} = \nabla \cdot k \nabla u - s$, $r_{FBC} = \rho \nabla u - \hat{q}$, and $r_{GG} = \mathbf{g}^u - \mathbf{g}^q = \nabla u - \rho^{-1} \mathbf{q}$, respectively. The weighted residuals, expressed in terms of the masters, are

$$\begin{aligned} R_{BE} &= \int_V (\nabla \cdot \mathbf{q} - s) w_{BE} dV, & R_{FBC} &= \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q}) w_{FBC} dS \\ R_{GG} &= \int_V (\mathbf{g}^u - \mathbf{g}^q) w_{GG} dV = \int_V (\nabla u - \rho^{-1} \mathbf{q}) w_{GG} dV. \end{aligned} \quad (3.14)$$

⁵ In mathematical-oriented treatments they receive names such as *bilinear pairs*, *inner product pairs* or *bilinear concomitants*.

⁶ Still another motivation in FEM applications is the reduction of interelement continuity requirements in problems of beams, plates and shells.

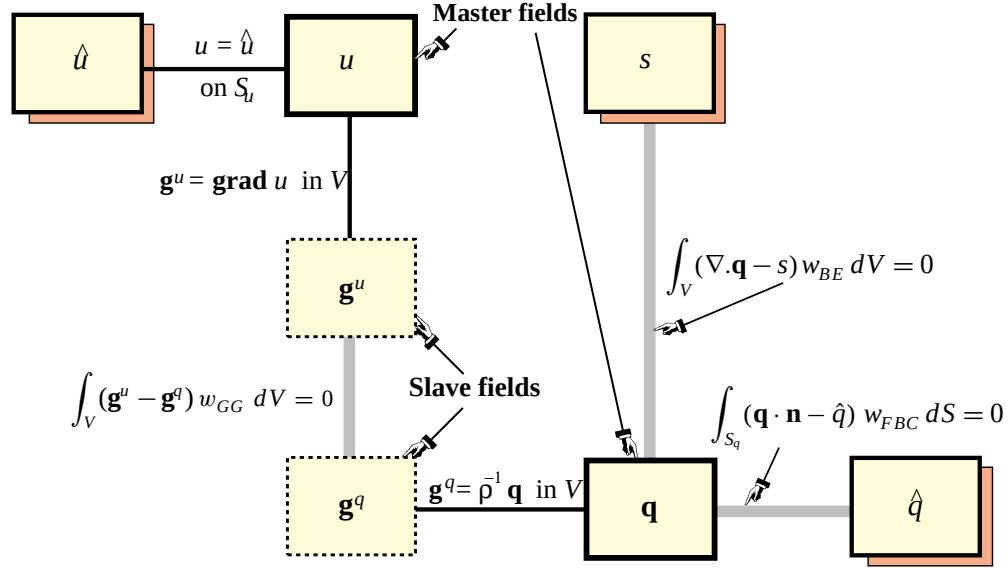


FIGURE 3.5. The Weak Form that leads to a mixed VF of HR type.

§3.4.2. Variational Statement

Equations (3.14) can be converted into a variational statement by replacing

$$w_{BE} \rightarrow -\delta u, \quad w_{FBC} \rightarrow \delta u, \quad w_{GG} \rightarrow \delta \mathbf{q}. \quad (3.15)$$

Why these choices? The key rule is: work pairing. As explained in §3.3.4, fields such as $\nabla \cdot \mathbf{q}$ and s should be paired with δu in V so that their product, once integrated to build a functional, represents work or energy. Consequently, w_{BE} must be $\pm \delta u$, and similarly for the other weight functions. The choice of the right sign is not that crucial, since signs can be tweaked to get cancellations on total-variations later.

Upon substitution, the R 's are renamed as variation components $\delta \Pi_{BE}$, $\delta \Pi_{FBC}$ and $\delta \Pi_{GG}$ of an alleged functional $\Pi[u, \mathbf{q}]$:

$$\begin{aligned} \delta \Pi_{BE} &= \int_V (-\nabla \cdot \mathbf{q} + s) \delta u \, dV = \int_V (\mathbf{q} \cdot \nabla u + s \delta u) \, dV - \int_{S_q} \mathbf{q} \cdot \mathbf{n} \delta u \, dS, \\ \delta \Pi_{FBC} &= \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q}) \delta u \, dS, \quad \delta \Pi_{GG} = \int_V (\nabla u - \rho^{-1} \mathbf{q}) \delta \mathbf{q} \, dV. \end{aligned} \quad (3.16)$$

The foregoing transformation of $\delta \Pi_{BE}$ comes from applying the divergence theorem to $\nabla \cdot \mathbf{q} \delta u$, as worked out at the end of §3.5: $-\int_V \nabla \cdot \mathbf{q} \delta u \, dV = \int_V \mathbf{q} \cdot \nabla u \, dV - \int_{S_q} \mathbf{q} \cdot \mathbf{n} \delta u \, dS$.

§3.4.3. The Variational Form

Adding the three weak link contributions gives

$$\begin{aligned} \delta \Pi &= \delta \Pi_{BE} + \delta \Pi_{FBC} + \delta \Pi_{GG} \\ &= \int_V [(\mathbf{q} \cdot \nabla u + (\nabla u - \rho^{-1} \mathbf{q}) \delta \mathbf{q} + s \delta u)] \, dV - \int_{S_q} \hat{q} \delta u \, dS. \end{aligned} \quad (3.17)$$

This is the variation of the mixed functional

$$\begin{aligned}\Pi_{\text{HR}}[u, \mathbf{q}] &= \int_V (\mathbf{q} \cdot \nabla u - \frac{1}{2} \rho^{-1} \mathbf{q} \cdot \mathbf{q}) dV + \int_V s u dV - \int_{S_q} \hat{q} u dS \\ &= \int_V \left[q_1 \frac{\partial u}{\partial x_1} + q_2 \frac{\partial u}{\partial x_2} + q_3 \frac{\partial u}{\partial x_3} - \frac{1}{2k} (q_1^2 + q_2^2 + q_3^2) \right] dV \\ &\quad + \int_V s u dV - \int_{S_q} \hat{q} u dS.\end{aligned}\tag{3.18}$$

in which the first term, which characterizes internal energy, is rewritten in full component notation in the second line of (3.18).

The variational principle is

$$\delta \Pi_{\text{HR}} = 0.\tag{3.19}$$

where the variations are taken with respect to u and $\mathbf{q}$.

§3.5. Overview of the Divergence Theorem

Specialized forms of Gauss' divergence theorem have been used on the way to the VF. In this section we summarize some useful forms in 3D. Assume that $\mathbf{a}$ is a differentiable 3-vector field in V . Begin from the canonical form of the theorem, which says that the vector divergence over a volume is equal to the vector flux over the surface:

$$\int_V \nabla \cdot \mathbf{a} dV = \int_S \mathbf{a} \cdot \mathbf{n} dS\tag{3.20}$$

Here $\nabla \cdot \mathbf{a}$ as usual denotes the vector divergence $\mathbf{div} \mathbf{a} = \partial a_1 / \partial x_1 + \partial a_2 / \partial x_2 + \partial a_3 / \partial x_3$.

Plug in $\mathbf{a} = \phi \mathbf{b}$, where ϕ is a scalar function and $\mathbf{b}$ a 3-vector field, both being differentiable:

$$\int_V (\phi \nabla \cdot \mathbf{b} + \nabla \phi \cdot \mathbf{b}) dV = \int_S \phi \mathbf{b} \cdot \mathbf{n} dS.\tag{3.21}$$

Next, if $\mathbf{b}$ is a gradient vector of the form $\mathbf{b} = \alpha \nabla \psi$, where α and ψ are scalar functions, the second being twice differentiable, (3.21) becomes

$$\int_V (\phi \nabla \cdot (\alpha \nabla \psi) + \nabla \phi \cdot (\alpha \nabla \psi)) dV = \int_S \phi \alpha \nabla \psi \cdot \mathbf{n} dS.\tag{3.22}$$

This form can be applied to the Poisson equation $\nabla \cdot (k \nabla u) = s$ by substituting $\phi \rightarrow -\delta u$, (the minus sign gives a nicer formula below) $\psi \rightarrow u$ and $\alpha \rightarrow \rho$ to get

$$-\int_V (\delta u \nabla \cdot (\rho \nabla u) + \nabla \delta u \cdot (\rho \nabla u)) dV = -\int_S \delta u \rho \nabla u \cdot \mathbf{n} dS.\tag{3.23}$$

Rearranging terms, separating the surface integral in two portions, and noting that $\delta u = 0$ on S_u (because $u = \hat{u}$ there) gives

$$\begin{aligned}-\int_V \nabla \cdot (\rho \nabla u) \delta u dV &= \int_V \rho \nabla u \cdot \nabla \delta u dV - \int_S \rho \nabla u \cdot \mathbf{n} \delta u dS \\ &= \int_V \rho \nabla u \cdot \nabla \delta u dV - \int_{S_u} \rho \nabla u \cdot \mathbf{n} \delta u dS - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS \\ &= \int_V \rho \nabla u \cdot \delta \nabla u dV - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS.\end{aligned}\tag{3.24}$$

Note that δ and ∇ commute: $\nabla \delta u = \delta \nabla u$, a fact used in the last equation. This relation is used for the development of the primal functional in §3.3.

Another useful formula for the mixed functional development in §3.4 is obtained by applying the divergence theorem to $\nabla \cdot \mathbf{q} \delta u$:

$$\int_V \nabla \cdot \mathbf{q} \delta u \, dV + \int_V \mathbf{q} \cdot \nabla \delta u \, dV = \int_S \mathbf{q} \cdot \mathbf{n} \delta u \, dS = \int_{S_q} \mathbf{q} \cdot \mathbf{n} \delta u \, dS. \quad (3.25)$$

Again δ and ∇ can be switched in the second term.

§3.6. Textbooks & Monographs on Variational Methods

There is a very large number of books that focus on variational methods and their applications to engineering and physics. Such material may be also found in supplementary form in books with primary focus on more general subjects, such as mathematical physics, or special ones such as finite elements.

The following list was prepared in 1993 for the Variational Methods in Mechanics (VMM) course, but is also relevant to AFEM. It was updated in 1999 and 2003 with references to Internet bookstores.⁷ It collects only books that the writer has examined, at least superficially.

§3.6.1. Textbooks Used in VMM

I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963. Used in 1991-93 as textbook for the standard variational calculus portion of VMM. Well written (Silverman's translation from the Russian is excellent), compact, modern, rigorous, notation occasionally fuzzy, many exercises of varying difficulty, good general-reference book to keep. Ages well. Technically still the best book on classical variational calculus, by a mile. Recently (2000) reprinted by Dover and thus inexpensive (\$9.95 new at Amazon).

B. D. Vujanovic and S. E. Jones, *Variational Methods in Nonconservative Phenomena*, Academic Press, 1989. As of this writing the only textbook that treats new, nonstandard techniques for the title subject, such as the method of vanishing parameters, time-dependent Lagrangians, and noncommutative variations. Exposition uneven, with flashes of brilliance followed by unending pedestrian examples. Out of print. Worth buying on the Internet despite cost (over \$120) if material covered can help your doctoral work.

C. Lanczos, *The Variational Principles of Mechanics*, Dover, 4th edition, 1970. (First edition 1949). Used in VMM courses 1991-93 for reading assignments. A classic. Beautifully written, respectful of history, sometimes down to a Scientific American style but never Popular Mechanics. Strength is in classical and relativistic particle and field mechanics. The continuum mechanics part (added in latter editions) is weak. Not good as literature guide. Inexpensive (about \$15); can be found in new-book bookstores since Dover periodically reprints it. Also easily purchased on the Internet as used book.

§3.6.2. A Potpourri of References

The following are listed by (first) author's alphabetic order.

M. Becker, *The Principles and Applications of Variational Methods*, MIT Press, Cambridge, 1964. A reprinted thesis. Focus on least-squares weighted residual methods applied to nuclear fuel problems.

⁷ Many of the books listed here are out of print. The advent of the Internet has meant that it is easier to surf for used books across the world without moving from your desk. There is a fast search engine for comparing prices at URL <http://www3.addall.com>: go to the "search for used books" link. Amazon.com has also a search engine, which is badly organized, confusing and full of unnecessary hype, but links to online reviews.

C. Caratheodory, *Calculus of Variations and Partial Differential Equations of the First Order*, Chelsea Pub. Co., 1982 (reprint of the original German edition, 1935). Still in print. A historically important work by a renowned mathematician. Good source for original topical articles in the XIX and early-XX Century. Hopelessly outdated for any other use. Written in a flat, boring style.

R. Courant and D. Hilbert, *Methods of Mathematical Physics*, 2 vols, Interscience Pubs, 1962. Periodically reprinted. Although a universally touted classical reference, it is now antiquated in style (first editions were written in the 1920s). Still useful as reference material and source to developments in the early half of the XX century.

H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, 1962. Periodically reprinted but also easy to find in the Internet as used book. Contains one chapter (14) on standard variational calculus, which gives a nice and quick introduction to the subject. If you have only a couple of hours to learn SVC in 20 pages, this is the book. Although certainly old, it feels more modern than some overhyped “classics”.

B. M. Finlayson, *The Methods of Weighted Residuals and Variational Principles*, Academic Press, 1972. Poorly written, disorganized and unfocused but contains material not available elsewhere in book form, especially in Chapters 9 and 10. Focus on chemical engineering problems hinders those interested in other applications. Good guide to literature before 1970. Out of print, not easily found as used book.

C. A. J. Fletcher, *Computational Galerkin Methods*, Springer-Verlag, 1984. A good exposition of the applications of Galerkin techniques to certain classes of problems in fluid dynamics. Has little on variational methods *per se*.

A. R. Forsyth, *Calculus of Variations*, Dover, 1960. Another oldie (Euler would feel at home with it) but more advanced than Fox and Weinstock. Out of print.

C. Fox, *An Introduction to the Calculus of Variations*, Oxford, 1963, in Dover since 1987. Periodically reprinted, easily found as used book, inexpensive. Readable but uneven and not well organized. Inexplicable omissions (for example, natural boundary conditions) and antiquated terminology. Good coverage of maxmin conditions, conjugate points and transversality conditions, but old-fashioned terminology hinders value.

P. Hammond, *Energy Methods in Electromagnetics*, Clarendon Press, Oxford, 1981. Best textbook coverage of the title subject.

H. L. Langhaar, *Energy Methods in Applied Mechanics*, McGraw-Hill, 1960. The most readable “old fashioned” explanation of the classical variational principles of structural mechanics. Beautiful treatment of virtual work. Out of print. Can be bought on the Internet for \$25 to \$60, depending on condition.

C. W. Misner, K. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco, 1973. Although devoted to general relativity and cosmology, a fun book to peruse through. At 8” x 10” x 2.5” and 1279 pages, it can be used to improve your upper body strength too. Still in print, lists for \$84 new. Has nice chapters on use of Hamilton-like variational principles. Here is an online review posted in Amazon.com:

“Yes, it’s so massive you can measure its gravitational field. Yes, people refer to it as “the phone book.” But all joking aside, as an undergraduate who is very curious about general relativity, I must say that this textbook has done more for me than any other. I’ve gotten occasional help from other books (Wald, Weinberg, etc.) but this is the one that I really LEARN from. There’s more physical insight in this book than any I’ve yet seen, and the reading is truly enjoyable. One great thing is the treatment of tensors. I knew next to nothing about tensors coming into the book, but the book assumes very little initial knowledge and teaches you the needed math as you go along. This book is truly a model for anyone who wants to write a textbook. Nothing I’ve seen even comes close.”

P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, 2 vols, McGraw-Hill, 1953. Same comments as for Courant-Hilbert. As the title says, it is oriented to physics, not engineering. Good treatment of adjoint and “mirror” systems. Out of print; can be bought on the Internet but the set is very expensive.

J. T. Oden and J. N. Reddy, *Variational Methods in Theoretical Mechanics*, Springer-Verlag, 1982. An advanced

monograph that contains material not readily available elsewhere in book form, such as Tonti diagrams and canonical functionals. Heavily theoretical, with abundant abstract math flourishes. No worked out problems or exercises. Selection of applications follows authors' interest. Good but not exhaustive reference source. Fairly inexpensive (about \$25) when it came out in paperback. Out of print, difficult to find as used book.

J. N. Reddy, *Energy and Variational Methods in Applied Mechanics*, Wiley, 1986. In print. This was selected as textbook for the first offering of the VMM course in 1987.

R. Santilli, *Foundations of Theoretical Mechanics I*, Springer-Verlag, Berlin, 1978. One of the few books that concentrate on the Inverse Problem of Lagrangian Mechanics. Unfortunately it is poorly written, disorganized, egocentric, wordy and repetitious, with many typos. For the patient specialist only.

M. J. Sewell, *Maximum and Minimum Principles*, Cambridge, 1987. Well written (author is British), fun to read, with a very good selection of examples and exercises in mechanics. [Sewell is an "applied mathematician" in the best British tradition of natural philosophy, a disciple of the great plasticity "guru" Rodney Hill]. Good and up-to-date bibliography. Main drawback is an inexplicable reluctance to use the standard variational calculus, which leaves many parts "dangling" and the reader wondering. Recommended despite that deficiency. Fairly inexpensive in paperback. Out of print.

M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden-Day, 1964. An advanced monograph highly touted when it appeared since it was one of the first books covering nonlinear variational operators and the Newton-Kantorovich solution method. Requires good command of functional analysis, else forget it. Translated from Russian, but the job was not well done. Out of print.

K. Washizu, *Variational Methods in Elasticity and Plasticity*, Pergamon Press, 1972 (2nd expanded edition 1981). With an encyclopædic coverage of the title subject, this is a good reference monograph. Drawbacks are the flat exposition style (there is no unifying, lifting theme a la Lanczos) and the cost (since it is out of print — Washizu passed away in 1985 — used copies, if found on the Internet, go for over \$150).

R. Weinstock, *Calculus of Variations, with Applications to Physics and Engineering*, McGraw-Hill, 1952. In Dover edition since 1974. Inexpensive and very old fashioned.

W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory*, Dover, 1968. A potpourri of history, philosophy and mathematical physics brewed in a small teacup (only 200 pages). In depth treatment of Hamilton's principle, which is fundamental in quantum physics. Two chapters on quantum mechanics and one on fluid mechanics including He superconductivity. Occasionally sloppy and effusive but worth the modest admission price (\$7 to \$10 used; can be bought new at Amazon for \$15.).

§3.6.3. Variational Methods as Supplementary Material

In addition to the foregoing, many books in mechanics give "recipe" introductions to variational methods. One of the best is the textbook, unfortunately out of print,⁸ by Fung:

Y. C. Fung, *Foundations of Solid Mechanics*, Prentice-Hall, 1965, Chapters 10ff.

Gurtin's article in the Encyclopedia of Physics gives a terse but rigorous coverage of the classical principles of linear elasticity, including the famous Gurtin convolution-type principles for initial-value problems in dynamics:

M. Gurtin, The Linear Theory of Elasticity, in *Encyclopedia of Physics* VIa, Vol II, ed. by C. Truesdell, Springer-Verlag, Berlin, 1972, pp. 1–295; reprinted as *Mechanics of Solids* Vol II, Springer-Verlag, 1984.

⁸ Much of Fung's material was recently "modernized" as a reasonably priced new book co-authored by Y. C. Fung and P. Tong, *Classic and Computational Solid Mechanics*, World Scientific Pub. Co., 2001. This has been used as text for Mechanics of Aerospace Structures (in Aero) and Mechanics of Solids (in ME). According to the instructors, student reaction has been negative. Shows how easy is to mung a good oldie.

Finite element books always provide some coverage ranging from superficial to adequate through pedantic to insufferable. The 4th edition of Zienkiewicz-Taylor is substantially improved, going from superficial (with noticeable mistakes in the previous 3 editions) to adequate, thanks to Bob Taylor's contribution:

O. C. Zienkiewicz and R. E. Taylor, *The Finite Element Method*, Vol. I, 4th ed. McGraw-Hill, New York, 1989.⁹

The coverage in the popular Cook-Malkus-Plesha's textbook is elementary but appropriate for the intended audience:

R. D. Cook, D. S. Malkus and M. E. Plesha, *Concepts and Application of Finite Element Methods*, 3rd ed., Wiley, New York, 1989. (More recent editions have appeared but have not improved on the 3rd.)

The most readable FEM-oriented treatment from a mathematical standpoint is still the excellent monograph by Strang and Fix:

G. Strang and G. Fix, *An Analysis of the Finite Element Method*. Prentice-Hall, 1973. Out of print. Available on the Internet as used book in range \$100-\$250.

Engineers should avoid any "FEM math" book other than Strang-Fix unless they are comfortable with advanced functional analysis.

⁹ A 5th edition in several volumes has appeared in the late 1990s.

Homework Exercises for Chapter 3

Weak and Variational Forms of the Poisson Equation

For an explanation of the Exercise ratings (given in brackets at the start of each one) see page 2 of the Homework Guidelines posted on the course web site.

EXERCISE 3.1 [A:5] Convert the functional Π_{TPE} derived in §3.3 into the corresponding functional of the heat conduction equation treated in §2.3. Hint: cf. footnote 1 of this Chapter, and be careful with signs.

EXERCISE 3.2 [A:20] Consider the generic Poisson problem of §3.2. Suppose that the boundary S splits into three parts: S_u , S_q and S_r so $S : S_u \cup S_q \cup S_r$. The BCs on S_u and S_q are the classical PBC and FBC: $u = \hat{u}$ and $q_n = \hat{q}$, respectively. On S_r the boundary condition is

$$q_n = \chi(u - u_0) \quad \text{on } S_r, \quad (\text{E3.1})$$

where u_0 and χ are given; both may be functions of position on S_r . This is called a *Robin boundary condition* or RBC. For the heat conduction problem, (E3.1) models a *convection* boundary condition: a moving fluid contacting the body on S_r dissipates or conveys heat.¹⁰

- (a) Show that expanding Π_{TPE} with a surface term $\pm \frac{1}{2} \int_{S_r} \chi(u - u_0)^2 dS$ accounts for this boundary condition. Find out which sign fits.
- (b) Account for the RBC (E3.1) in the extended SF Tonti diagram.¹¹

EXERCISE 3.3 [A:20] Consider the thermal conduction problem with *absolute temperature* T (in Kelvin) as primary variable. Suppose that the boundary S splits into three parts: S_T , S_q and S_r so $S : S_T \cup S_q \cup S_r$. The BCs on S_T and S_q are the classical PBC and FBC: $T = \hat{T}$ and $q_n = \hat{q}$, respectively. On S_r we have a *radiation* boundary condition:

$$q_n = \sigma(T^4 - T_r^4) \quad \text{on } S_r, \quad (\text{E3.2})$$

where T_r is a given reference temperature (for orbiting space structures, $T_r \approx 3^\circ\text{K}$) and σ is a given coefficient that characterizes radiational heat emission per unit of area.

Show that (E3.2) can be exactly linearized as $q_n = h_r(T - T_r)$, where h_r is a function of σ , T and T_r . Find h_r and explain how this trick can make (E3.2) fit into the Robin BC treated in the previous Exercise.¹²

EXERCISE 3.4 [A:20] Show that the Π_{HR} mixed functional derived in §3.4 can be expanded with a term $\pm \int_{S_u} q_n(u - \hat{u}) dS$ to account for weakening the PBC link: $u = \hat{u}$ on S_u (find which sign fits).

¹⁰ More precisely, (E3.1) is a linearization of an actual convection condition. If the flow is turbulent (e.g. the Earth atmosphere) the actual condition is nonlinear.

¹¹ The answer is not unique; try something. Whatever you come up with, it will mess up the neat form of the diagram.

¹² The resulting functional is called a *Restricted Variational Form* because h_r must be kept fixed during variation. This form can be used as a basis for numerically solving this highly nonlinear problem, which is important in Aerospace, Environmental and Mechanical Engineering.

4

The Bernoulli-Euler Beam

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§4.1. Introduction

From the Poisson equation we move to elasticity and structural mechanics. Rather than tackling the full 3D problem first this Chapter illustrates, in a tutorial style, the derivation of Variational Forms for a one-dimensional model: the Bernoulli-Euler beam.

Despite the restriction to 1D, the mathematics offers a new and challenging ingredient: the handling of functionals with second space derivatives of displacements. Physically these are curvatures of deflected shapes. In structural mechanics curvatures appear in problems involving beams, plates and shells. In fluid mechanics second derivatives appear in slow viscous flows.

§4.2. The Beam Model

The beam under consideration extends from $x = 0$ to $x = L$ and has a bending rigidity EI , which may be a function of x . See Figure 4.1(a). The transverse load $q(x)$ is given in units of force per length. The unknown fields are the transverse displacement $w(x)$, rotation $\theta(x)$, curvature $\kappa(x)$, bending moment $M(x)$ and transverse shear $V(x)$. Positive sign conventions for M and V are illustrated in Figure 4.1(b).

Boundary conditions are applied only at A and B . For definiteness the end conditions shown in Figure 4.1(c) will be used.

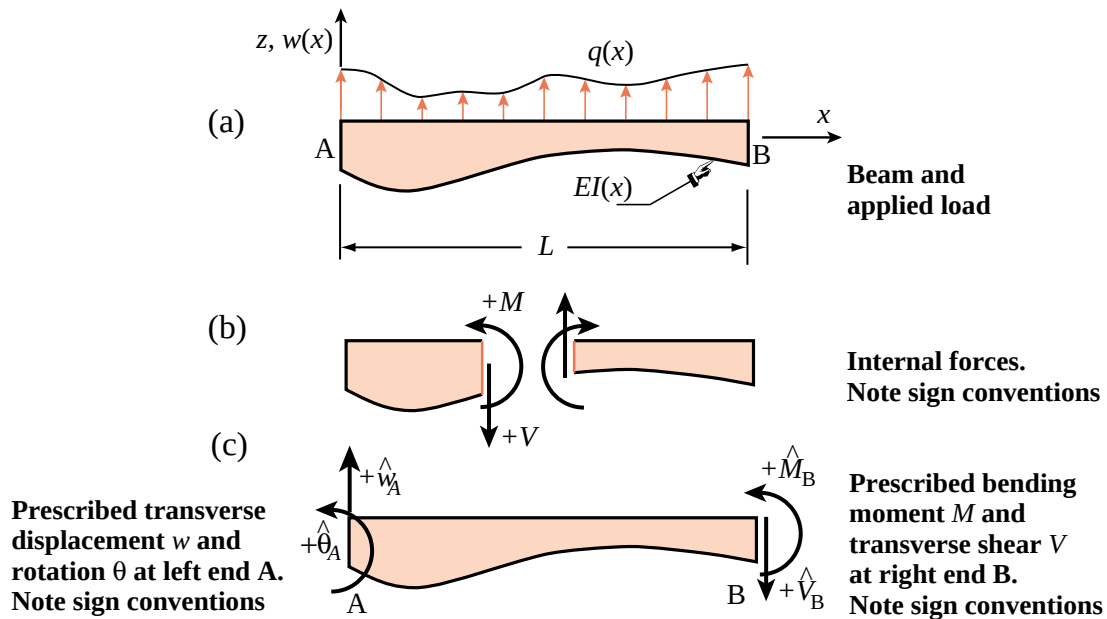


FIGURE 4.1. The Bernoulli-Euler beam model: (a) beam and transverse load; (b) positive convention for moment and shear; (c) boundary conditions.

§4.2.1. Field Equations

In what follows a prime denotes derivative with respect to x . The field equations over $0 \leq x \leq L$ are as follows.

(KE) Kinematic equations:

$$\theta = \frac{dw}{dx} = w', \quad \kappa = \frac{d^2w}{dx^2} = w'' = \theta', \quad (4.1)$$

where θ is the rotation of a cross section and κ the curvature of the deflected longitudinal axis. Relations (4.1) express the kinematics of an Bernoulli-Euler beam: plane sections remain plane and normal to the deflected neutral axis.

(CE) Constitutive equation:

$$M = EI\kappa, \quad (4.2)$$

where I is the second moment of inertia of the cross section with respect to y (the neutral axis). This moment-curvature relation is a consequence of assuming a linear distribution of strains and stresses across the cross section. It is derived in elementary courses of Mechanics of Materials.

(BE) Balance (equilibrium) equations:

$$V = \frac{dM}{dx} = M', \quad \frac{dV}{dx} - q = V' - q = M'' - q = 0. \quad (4.3)$$

Equations (4.3) are established by elementary means in Mechanics of Materials courses.¹

§4.2.2. Boundary Conditions

For the sake of specificity, the boundary conditions assumed for the example beam are of primary-variable (PBC) type on the left and of flux type (FBC) on the right:

$$\begin{array}{ll} (PBC) & \text{at } A (x = 0) : \quad w = \hat{w}_A, \quad \theta = \hat{\theta}_A, \\ (FBC) & \text{at } B (x = L) : \quad M = \hat{M}_B, \quad V = \hat{V}_B, \end{array} \quad (4.4)$$

in which $\hat{w}_A$, $\hat{\theta}_A$, $\hat{M}_B$ and $\hat{V}_B$ are prescribed. If $\hat{w}_A = \hat{\theta}_A = 0$, these conditions physically correspond to a cantilever (fixed-free) beam. We will let $\hat{w}_A$ and $\hat{\theta}_A$ be arbitrary, however, to further illuminate their role in the functionals.

Remark 4.1. Note that a positive $\hat{V}_B$ acts downward (in the $-z$ direction) as can be seen from Figure 4.1(b), so it disagrees with the positive deflection $+w$. On the other hand a positive $\hat{M}_B$ acts counterclockwise, which agrees with the positive rotation $+\theta$.

¹ In those courses, however, $+q(x)$ is sometimes taken to act downward, leading to $V' + q = 0$ and $M'' + q = 0$.

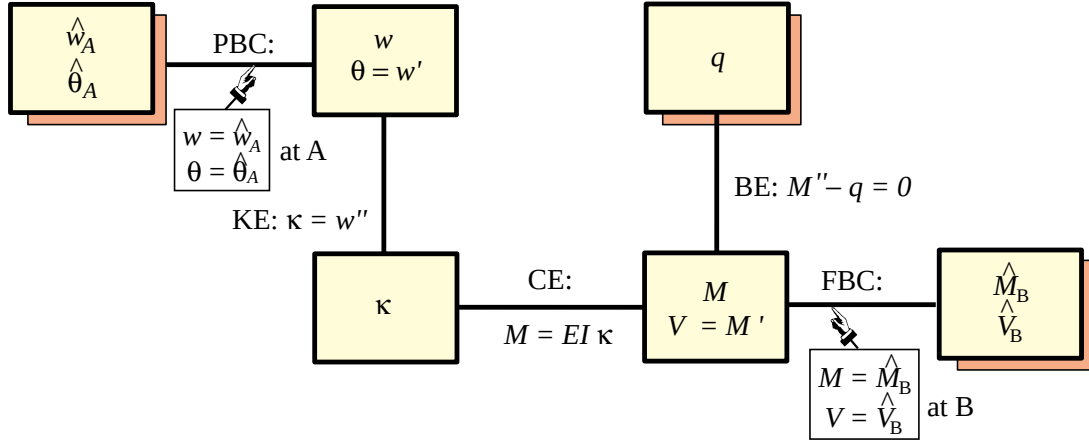


FIGURE 4.2. Tonti diagram of Strong Form of Bernoulli-Euler beam model.

§4.2.3. The SF Tonti Diagram

The Strong Form Tonti diagram for the Bernoulli-Euler model of Figure 4.1 is drawn in Figure 4.2. The diagram lists the three field equations (KE, CE, BE) and the boundary conditions (PBC, FBC). The latter are chosen in the very specific manner indicated above to simplify the boundary terms.²

Note than in this beam model the rotation $\theta = w'$ and the transverse shear $V = M'$ play the role of auxiliary variables that *are not constitutively related*. The only constitutive equation is the moment-curvature equation $M = EI\kappa$. The reason for the presence of such auxiliary variables is their direct appearance in boundary conditions.³

§4.3. The TPE (Primal) Functional

Select w as only master field. Weaken the BE and FBC connections to get the Weak Form (WF) diagram of Figure 4.4 as departure point. Choose the weighting functions on the weak links BE, FBC on M and FBC on V to be δw , $\delta\theta^w$ and δw , respectively. The weak links are combined as follows:

$$\delta\Pi = \int_0^L \left[(M^w)'' - q \right] \delta w \, dx + (M^w - \hat{M}) \delta\theta^w \Big|_B - (V^w - \hat{V}) \delta w \Big|_B = 0. \quad (4.5)$$

Why the different signs for the moment and shear boundary terms? If confused, read Remark 4.1.

² In fact $2^4 = 16$ boundary condition combinations are mathematically possible. Some of these correspond to physically realizable support conditions, for example simply supports, whereas others do not.

³ In the Timoshenko beam model, which accounts for transverse shear energy, θ appears in the constitutive equations.

Next, integrate $\int_0^L (M^w)'' \delta w dx$ twice by parts:

$$\begin{aligned}
 \int_0^L (M^w)'' \delta w dx &= - \int_0^L (M^w)' \delta w' dx + \left[(M^w)' \delta w \right]_A^B \\
 &= \int_0^L M^w \delta w'' dx + \left[(M^w)' \delta w \right]_A^B - \left[M^w \delta w' \right]_A^B \\
 &= \int_0^L M^w \delta \kappa^w dx + V^w \delta w \Big|_B - M^w \delta \theta^w \Big|_B.
 \end{aligned} \tag{4.6}$$

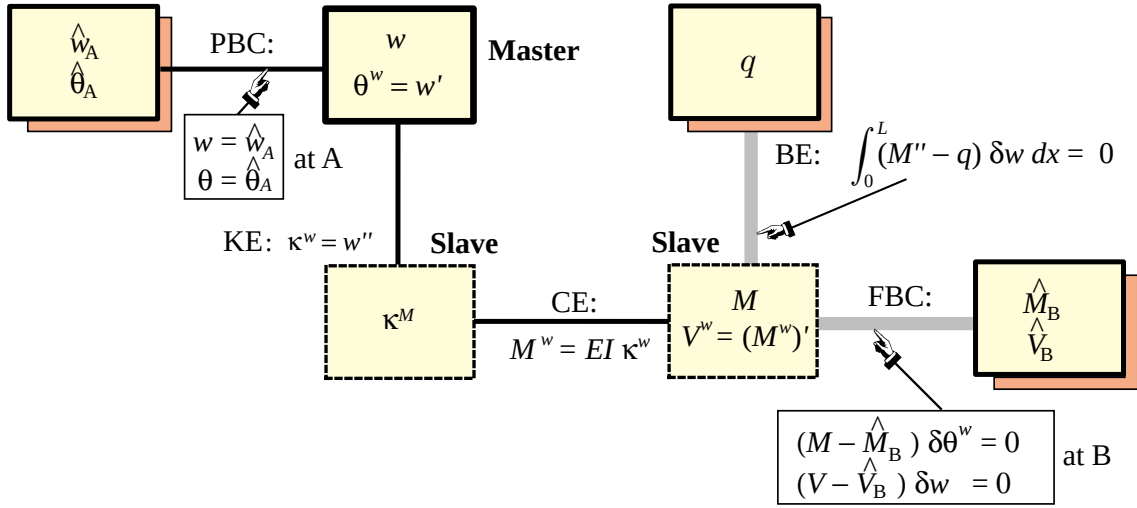


FIGURE 4.3. Weak Form used as departure point to derive the TPE functional for the Bernoulli-Euler beam.

The disappearance of boundary terms at A in the last equation results from $\delta w_A = 0$, $\delta w'_A = \delta \theta^w_A = 0$ on account of the strong PBC connection at $x = 0$. Inserting (4.6) into (4.5) gives

$$\begin{aligned}
 \delta \Pi &= \int_0^L (M^w \delta \kappa^w - q \delta w) dx - \hat{M} \delta \theta^w \Big|_B + \hat{V} \delta w \Big|_B \\
 &= \int_0^L (EI w'' \delta w'' - q \delta w) dx - \hat{M} \delta w' \Big|_B + \hat{V} \delta w \Big|_B.
 \end{aligned} \tag{4.7}$$

This is the first variation of the functional

$$\Pi_{\text{TPE}}[w] = \frac{1}{2} \int_0^L EI (w'')^2 dx - \int_0^L q w dx - \hat{M}_B w'_B + \hat{V}_B w_B. \tag{4.8}$$

This is called the Total Potential Energy (TPE) functional of the Bernoulli-Euler beam. It was used in the Introduction to Finite Element Methods course to derive the well known Hermitian beam element in Chapter 12. For many developments it is customarily split into two terms

$$\Pi[w] = U[w] - W[w], \tag{4.9}$$

in which

$$U[w] = \frac{1}{2} \int_0^L EI(w'')^2 dx, \quad W[w] = \int_0^L qw dx + \hat{M}_B w'_B - \hat{V}_B w_B. \quad (4.10)$$

Here U is the internal energy (strain energy) of the beam due to bending deformations (bending moments working on curvatures), whereas W gathers the other terms that collectively represent the *external work* of the applied loads.⁴

Remark 4.2. Using integration by parts one can show that if $\delta\Pi = 0$,

$$U = \frac{1}{2}W. \quad (4.11)$$

In other words: at equilibrium the *internal energy is half the external work*. This property is valid for any linear elastic continuum. (It is called Clapeyron's theorem in the literature of Structural Mechanics.) It has a simple geometric interpretation for structures with finite number of degrees of freedom.

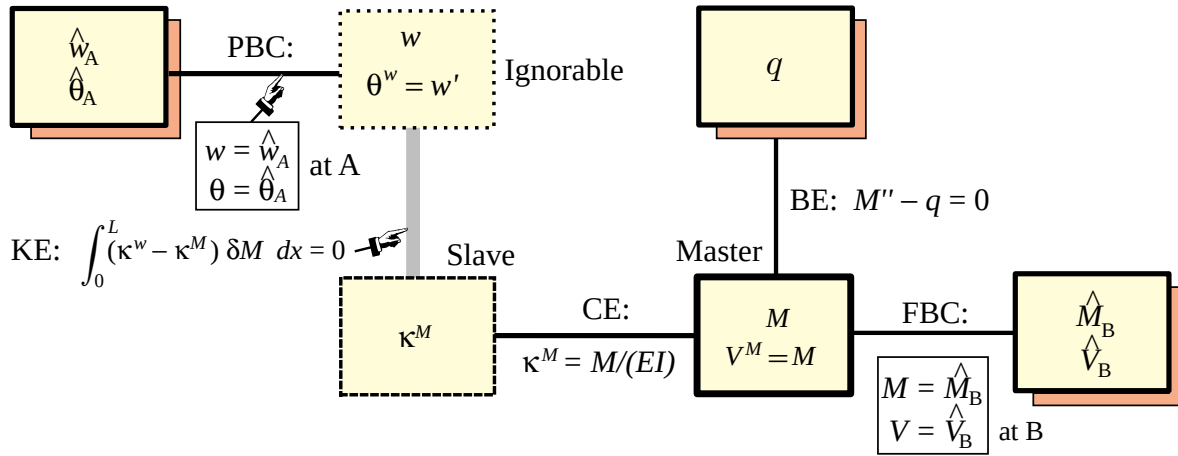


FIGURE 4.4. Weak form used as departure point to derive the TCPE functional for the Bernoulli-Euler beam.

§4.4. The TCPE (Dual) Functional

The Total Complementary Potential Energy (TCPE) functional is mathematically the dual of the primal (TPE) functional.

Select the bending moment M as the only master field. Make KE weak to get the Weak Form diagram displayed in Figure 4.4. Choose the weighting function on the weakened KE to be δM . The only contribution to the variation of the functional is

$$\delta\Pi = \int_0^L (\kappa^M - \kappa^w) \delta M dx = \int_0^L (\kappa^M - w'') \delta M dx = 0. \quad (4.12)$$

⁴ Recall that work and energy have opposite signs, since energy is the capacity to produce work. It is customary to write $\Pi = U - W$ instead of the equivalent $\Pi = U + V$, where $V = -W$ is the external work potential. This notational device also frees the symbol V to be used for transverse shear in beams and voltage in electromagnetics.

Integrate $\int w'' \delta M dx$ by parts twice:

$$\begin{aligned} \int_0^L w'' \delta M dx &= - \int_0^L w' \delta M' dx + \left[w' \delta M \right]_A^B \\ &= \int_0^L w \delta M'' dx + \left[w' \delta M \right]_A^B - \left[w \delta M' \right]_A^B \\ &= \int_0^L w \delta M'' dx - \hat{\theta} \delta M \Big|_A + \hat{w} \delta V^M \Big|_A. \end{aligned} \quad (4.13)$$

The disappearance of the boundary terms at B results from enforcing strongly the free-end boundary conditions $M = \hat{M}_B$ and $V = \hat{V}_B$, whence the variations $\delta M_B = 0$, $\delta V^M = \delta M'_B = 0$. Because of the strong BE connection, $M'' - q = 0$, $\delta M''$ vanishes identically in $0 \leq x \leq L$. Consequently

$$\int_0^L w'' \delta M dx = -\hat{\theta} \delta M \Big|_A + \hat{w} \delta V^M \Big|_A. \quad (4.14)$$

Replacing into (4.12) yields

$$\delta \Pi = \int \kappa^M \delta M dx + \hat{\theta} \delta M \Big|_A - \hat{w} \delta V^M \Big|_A = 0. \quad (4.15)$$

This is the first variation of the functional

$$\Pi[M] = \frac{1}{2} \int_0^L \kappa^M M dx + M \hat{\theta}_A - V^M \hat{w}_A, \quad (4.16)$$

and since $\kappa^M = M/EI$, we finally get

$$\boxed{\Pi_{\text{TCPE}}[M] = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx + M \hat{\theta}_A - V^M \hat{w}_A.} \quad (4.17)$$

This is the TCPE functional for the Bernoulli-Euler beam model. As in the case of the TPE, it is customarily split as

$$\Pi[M] = U^*[M] - W^*[M], \quad (4.18)$$

where

$$U^* = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx, \quad W^* = -M \hat{\theta}_A + V^M \hat{w}_A. \quad (4.19)$$

Here U^* is the internal *complementary energy* stored in the beam by virtue of its deformation, and W^* is the *external complementary energy* that collects the work of the prescribed end displacements and rotations.

Note that only M (and its slaves), $\hat{\theta}_A$ and $\hat{w}_A$ remain in this functional. The transverse displacement $w(x)$ is gone and consequently is labeled as *ignorable* in Figure 4.5. Through the integration by parts process the WF diagram of Figure 4.4. collapses to the one sketched in Figure 4.5. The reduction may be obtained by invoking the following two rules:

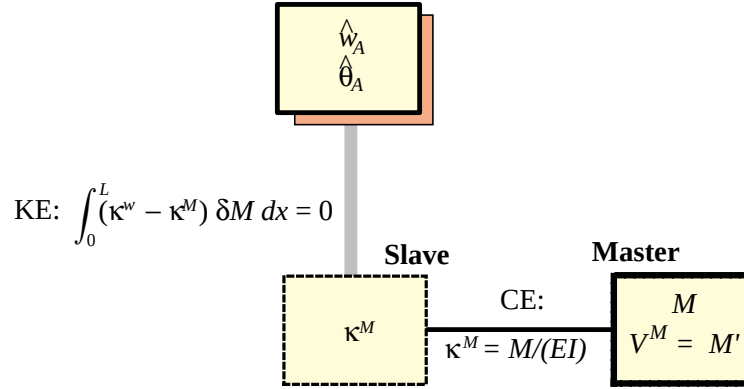


FIGURE 4.5. The collapsed WF diagram for the TCPE functional, showing only “leftovers” boxes.

- (1) The “ignorable box” w, θ of Figure 4.4 may be replaced by the data box $\hat{w}_A, \hat{\theta}_A$ because only the boundary values of those quantities survive.
- (2) The data boxes q and $\hat{M}, \hat{V}$ of Figure 4.4 may be removed because they are strongly connected to the varied field M .

The collapsed WF diagram of Figure 4.5 displays the five quantities $(M, V^M, \kappa^M, \hat{w}_A, \hat{\theta}_A)$ that survive in the TCPE functional.

Remark 4.3. One can easily show that for the actual solution of the beam problem, $U^* = U$, a property valid for any linear elastic continuum. Furthermore $U^* = \frac{1}{2} W^*$.

§4.5. The Hellinger-Reissner Functional

The TPE and TCPE functionals are *single-field*, because there is only one master field that is varied: displacements in the former and moments in the latter. We next illustrate the derivation of a two-field mixed functional, identified as the Hellinger-Reissner (HR) functional $\Pi_{\text{HR}}[w, M]$, for the Bernoulli-Euler beam. Here both displacements w and moments M are picked as master fields and thus are independently varied.

The point of departure is the WF diagram of Figure 4.6. As illustrated, three links: KE, BE and FBC, have been weakened. The master (varied) fields are w and M . It is necessary to distinguish between displacement-derived curvatures $\kappa^w = w''$ and moment-derived curvatures $\kappa^M = M/EI$, as shown in the figure. The two curvature boxes are weakly connected, expressing that the equality $w'' = M/EI$ is not enforced strongly.

The mathematical expression of the WF, having chosen weights $\delta M, \delta w, \delta w' = \delta \theta^w$ and $-\delta w$ for the weak connections KE, BE, moment M in FBC and shear V in FBC, respectively, is

$$\delta \Pi[w, M] = \int_0^L (\kappa^w - \kappa^M) \delta M dx + \int_0^L (M'' - q) \delta w dx + \left| (M - \hat{M}) \delta \theta^w \right|_B - \left| (V - \hat{V}) \delta w \right|_B. \quad (4.20)$$

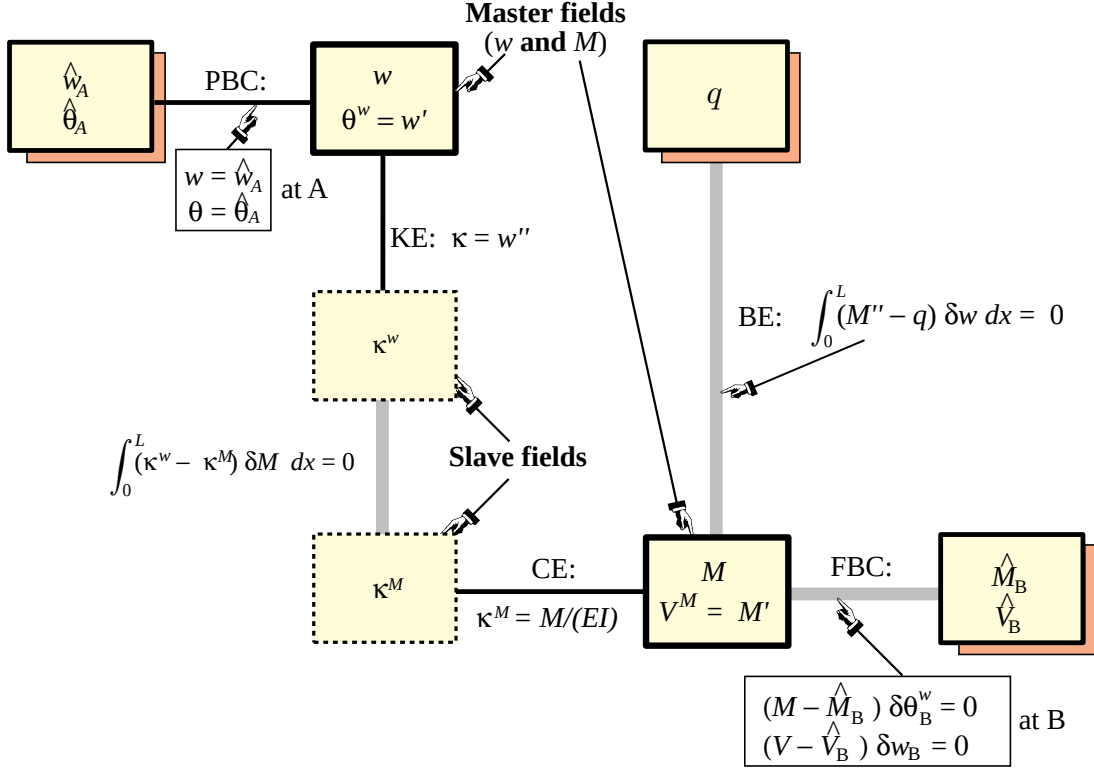


FIGURE 4.6. The WF Tonti diagram used as departure point for deriving the Hellinger-Reissner (HR) functional.

Integrating $\int_0^L M'' \delta w$ twice by parts as in the TPE derivation, inserting in (4.20), and enforcing the strong PBCs at A, yields

$$\delta \Pi[w, M] = \int_0^L \left[(w'' - \kappa^M) \delta M + M \delta w'' \right] dx - \int_0^L q \delta w dx + \hat{V} \delta w \Big|_B - \hat{M} \delta \theta^w \Big|_B, \quad (4.21)$$

This is the first variation of the Hellinger-Reissner (HR) functional

$$\Pi_{\text{HR}}[w, M] = \int_0^L \left(M w'' - \frac{M^2}{2EI} - q w \right) dx + \hat{V}_B w_B - \hat{M}_B \theta_B^w. \quad (4.22)$$

Again this can be split as $\Pi_{\text{HR}} = U - W$, in which

$$U[w, M] = \int_0^L \left(M w'' - \frac{M^2}{2EI} \right) dx, \quad W[w] = \int_0^L q w dx - \hat{V}_B w_B + \hat{M}_B \theta_B^w, \quad (4.23)$$

represent internal energy and external work, respectively.

Remark 4.4. If the primal boundary conditions (PBC) at A are weakened, the functional (4.22) gains two extra boundary terms.

Remark 4.5. The $Mw'' = M\kappa^w$ term in (4.23) may be transformed by applying integration by parts once:

$$\int_0^L Mw'' dx = Mw' \Big|_0^L - \int_0^L M'w' dx = \hat{M}_B \theta_B^w - M_A \hat{\theta}_A^w - \int_0^L M'w' dx \quad (4.24)$$

to get an alternative form of the HR equation with “balanced derivatives” in w and M . Such transformations are common in the finite element applications of mixed functionals. The objective is to exert control over interelement continuity conditions.

Homework Exercises for Chapter 4

The Bernoulli- Euler Beam

EXERCISE 4.1 [A:25] An assumed-curvature mixed functional. The WF diagram of a two-field displacement-curvature functional $\Pi(w, \kappa)$ for the Bernoulli-Euler beam is shown in Figure E4.1.

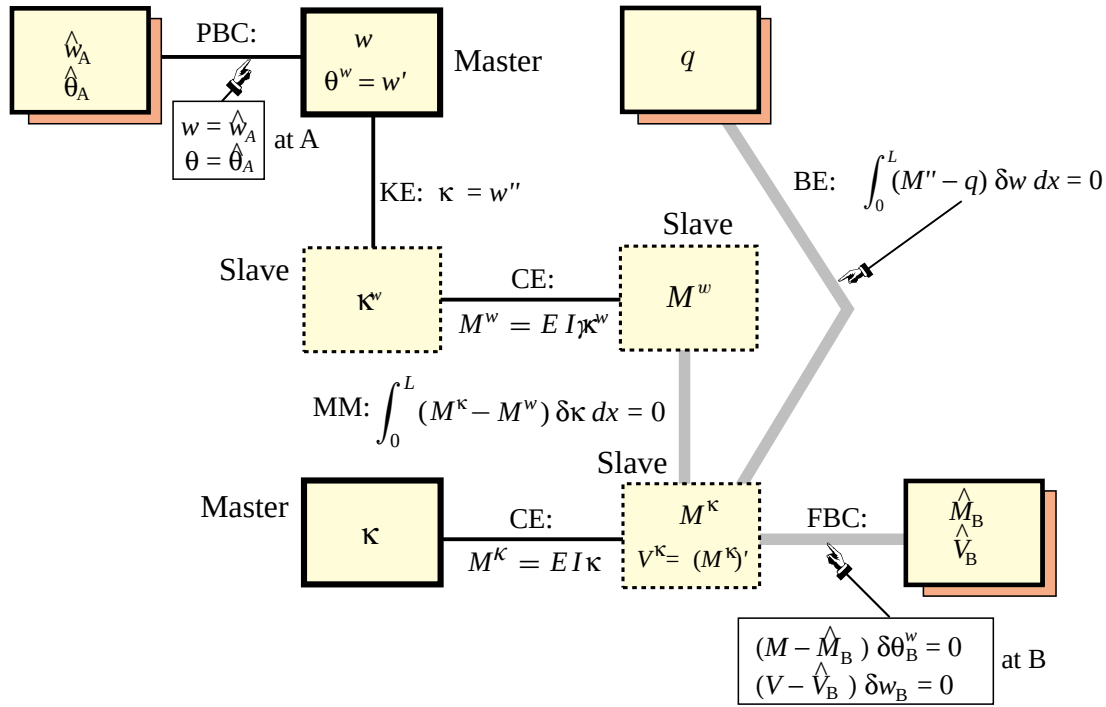


FIGURE E4.1. The WF Tonti diagram used as departure point for deriving the Hellinger-Reissner (HR) functional.

Starting from this form, derive the functional

$$\Pi[w, \kappa] = \int_0^L (M^w - \frac{1}{2} M^\kappa) \kappa \, dx - \int_0^L q w \, dx - \hat{M} \theta^w \Big|_B - \hat{V} w \Big|_B. \quad (\text{E4.1})$$

Which extra term appears if link PBC is weakened?

Hints: The integral $\int_0^L (M^w \delta \kappa + M^\kappa \delta w'') \, dx$ is the variation of $\int_0^L M^w \kappa \, dx$ (work it out, it is a bit tricky) whereas the integral $\int_0^L -M^\kappa \delta \kappa \, dx$ is the variation of $-\frac{1}{2} \int_0^L M^\kappa \kappa \, dx$.

EXERCISE 4.2 [A:15] Prove the property stated in Remark 4.2.

EXERCISE 4.3 [A:30] The most general mixed functional in elasticity is called the Veubeke-Hu-Washizu or VHW functional. The three internal fields: displacements w , curvatures κ and moments M , are selected as masters and independently varied to get

$$\Pi[w, \kappa, M] = \int_0^L \left[\frac{1}{2} M^\kappa \kappa + M(\kappa^w - \kappa) \right] dx - \int_0^L q w dx - \hat{M} \theta^w \Big|_B - \hat{V} w \Big|_B. \quad (\text{E4.2})$$

Derive this functional starting from the WF diagram shown in Figure E4.2.

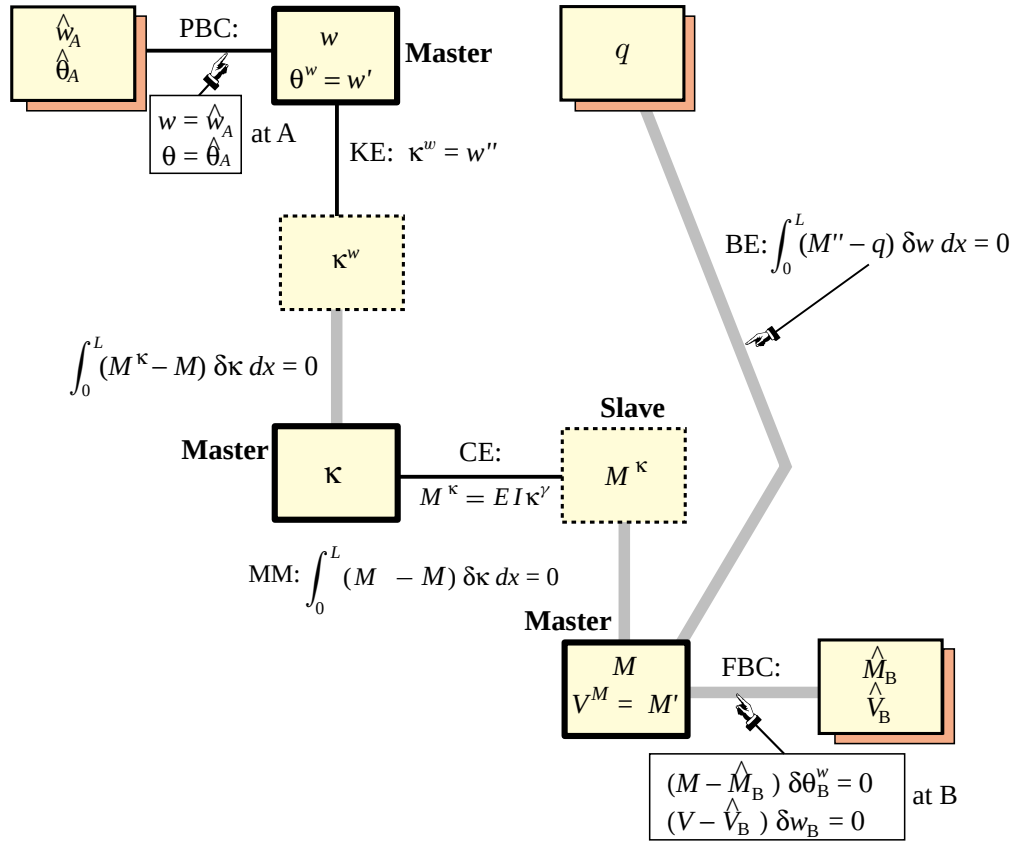


FIGURE E4.2. Starting WF diagram to derive the three-master-field Veubeke-Hu-Washizu mixed functional, which is the topic of Exercise 4.3.

EXERCISE 4.4 [A:35] (Advanced, research paper level) Suppose that on the beam of Figure 4.1, loaded by $q(x)$, one applies an *additional* concentrated load P at an arbitrary cross section $x = x_P$. The additional transverse displacement under that load is w_P . The additional deflection elsewhere is $w_P \phi(x)$, where $\phi(x)$ is called an *influence function*, whose value at $x = x_P$ is 1. For simplicity assume that the end forces at B vanish: $\hat{M}_B = \hat{V}_B = 0$. The TPE functional can be viewed as function of two arguments:

$$\Pi[w, w_P] = \frac{1}{2} \int_0^L EI (w'' + w_P \phi'')^2 dx - \int_0^L [q + P \delta(x_P)] (w + w_P \phi) dx, \quad (\text{E4.3})$$

where $w = w(x)$ denotes here the deflection for $P = 0$ and $\Delta(x_P)$ is Dirac's delta function.⁵ Show that if the

⁵ This is denoted by $\Delta(\cdot)$ instead of the usual $\delta(\cdot)$ to avoid confusion with the variation symbol.

beam is in equilibrium (that is, $\delta\Pi = 0$):

$$P = \frac{\partial U[w, w_P]}{\partial w_P}. \quad (\text{E4.4})$$

This is called *Castigliano's theorem on forces* (also Castigliano's first theorem).⁶ In words: the partial derivative of the internal (strain) energy expressed in terms of the beam deflections with respect to the displacement under a concentrated force gives the value of that force.⁷

⁶ Some mathematical facility with integration by parts and delta functions is needed to prove this, but it is an excellent exercise for advanced math exams.

⁷ This energy theorem can be generalized to arbitrary elastic bodies (not just beams) but requires fancy mathematics. It also applies to concentrated couples by replacing "displacement of the load" by "rotation of the couple." This result is often used in Structural Mechanics to calculate reaction forces at supports. Castigliano's energy theorem on deflections (also called Castigliano's second theorem), which is $w_Q = \partial U^* / \partial Q$ in which U^* is the internal complementary energy, is the one normally taught in undergraduate courses for Structures. Textbooks normally prove these theorems only for systems with finite number of degrees of freedom. "Proofs" for arbitrary continua are usually faulty because singular integrals are not properly handled.

5

Three-Dimensional Linear Elastostatics

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§5.1. Introduction

We move now from the easy ride of Poisson problems and Bernoulli-Euler beams to the tougher road of elasticity in three dimensions. This Chapter summarizes the governing equations of linear elastostatics. Various notational systems are covered in sufficient detail to help readers with the literature of the subject, which is enormous and spans over two centuries. The governing equations are displayed in a Strong Form Tonti diagram.

The classical single-field variational principle of Total Potential Energy is derived in this Chapter as prelude to mixed and hybrid variational principles, which are presented in the next two Chapters.¹

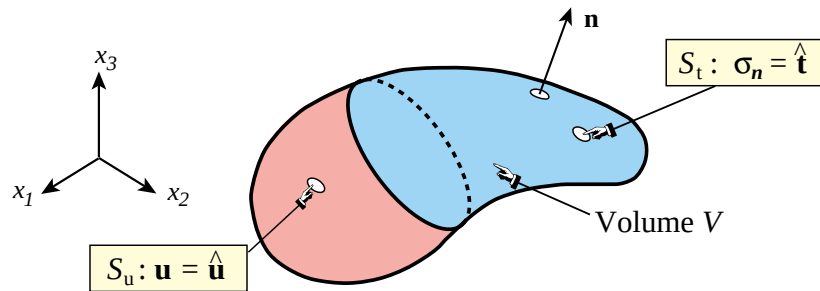


FIGURE 5.1. A linear-elastic body of volume V in static equilibrium. The body surface $S : S_t \cup S_u$ is split into S_t , on which surface tractions are prescribed, and S_u , on which surface displacements are prescribed.

§5.2. The Governing Equations

Consider a linearly elastic body of volume V , which is bounded by surface S , as shown in Figure 5.1. The body is referred to a three dimensional, rectangular, right-handed Cartesian coordinate system $x_i \equiv \{x_1, x_2, x_3\}$. The body is in *static* equilibrium under the action of body forces b_i in V , prescribed surface tractions $\hat{t}_i$ on S_t and prescribed displacements $\hat{u}_i$ on S_u , where $S_t \cup S_u \equiv S$ are two complementary portions of the boundary S . This separation of boundary conditions and source data is displayed in more detail in Figure 5.2.

The three unknown internal fields are *displacements* u_i , *strains* $e_{ij} = e_{ji}$ and *stresses* $\sigma_{ij} = \sigma_{ji}$. All of them are defined in V . In the absence of internal interfaces the three fields may be assumed to be continuous and piecewise differentiable.² At internal interfaces (for example a change in material) certain strain and stress components may jump, but such “jump conditions” are ignored in the present treatment.

The three known or data fields are the body forces b_i , prescribed surface tractions $\hat{t}_i$ and prescribed displacements $\hat{u}_i$. These are given in V , on S_t , and on S_u , respectively.

The equations that link the various volume fields are called the *field equations* of elasticity. Those linking volume fields (evaluated at the surface) and prescribed surface fields are called *boundary*

¹ The material in this and next two chapters is mostly taken from the Variational Methods in Mechanics course complemented with additional material on problem-solving.

² See, e.g., M. Gurtin, The Linear Theory of Elasticity, in *Encyclopedia of Physics* VIa, Vol II, ed. by C. Truesdell, Springer-Verlag, Berlin, 1972, pp. 1–295; reprinted as *Mechanics of Solids* Vol II, Springer-Verlag, 1984.

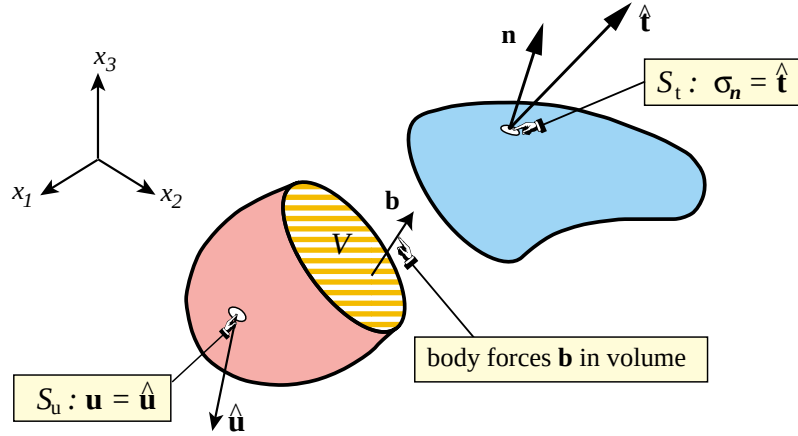


FIGURE 5.2. Showing in more detail the separation of the surface S into two complementary regions S_t and S_u .

conditions. The ensemble of field equations and boundary conditions represent the *governing equations* of elastostatics.

Remark 5.1. The field equations are generally partial differential equations (PDEs) although for elasticity the constitutive equations become algebraic. The classical boundary conditions are algebraic relations.

Remark 5.2. The separation of S into traction-specified S_t and displacement-specified S_u may be more complex than the simple surface partition of the Poisson problem. This is because $\hat{t}_i$ and $\hat{u}_i$ are now vectors with several components. These may be specified at the same surface point in various combinations. This happens in many practical problems. For example, one may consider a portion of S where a pressure force is applied whereas the tangential displacement components are zero. Or a bridge roller support: the displacement normal to the rollers is precluded (a displacement condition) but the tangential displacements are free (a traction condition). This mixture of force and displacement conditions over the same surface element would complicate the notation considerably. We shall use the “union of” notation $S \equiv S_t \cup S_u$ for notational simplicity but the presence of such complications should be kept in mind.

§5.2.1. Direct Tensor Notation

In the foregoing description we have used the so-called *component notation* or *indicial notation* for fields. More precisely, the notation appropriate to rectangular Cartesian coordinates. In this notation, writing u_i is equivalent to writing the three components u_1, u_2, u_3 of the displacement field $\mathbf{u}$. We now review the so-called *direct tensor notation* or *compact tensor notation*.

Scalars, which are zero-dimensional tensors, are represented by non-boldface Roman or Greek symbols. Example: ρ for mass density and g for the acceleration of gravity.

Vectors, which are one-dimensional tensors, are represented by **boldface** symbols. These will be usually lowercase letters unless common usage dictates the use of uppercase symbols.³ For

³ This happens in electromagnetics: tradition since Maxwell has kept field vectors such as $\mathbf{E}$ (electric field) and $\mathbf{B}$ (magnetic field) in uppercase.

example:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \quad (5.1)$$

identify the vectors of displacements, body forces and surface tractions, respectively.

Two-dimensional tensors are represented by **underlined boldface** lowercase symbols. These will usually be lowercase Roman or Greek letters. For example

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ & e_{22} & e_{23} \\ \text{symm} & & e_{33} \end{bmatrix} \equiv e_{ij}, \quad \underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \text{symm} & & \sigma_{33} \end{bmatrix} \equiv \sigma_{ij}, \quad (5.2)$$

denote the strain and stress tensors, respectively. The *transpose* of a second order tensor, denoted by $(\cdot)^T$ is obtained by switching the two indices. A tensor is *symmetric* if it equates its transpose. Both the stress and strain tensors are symmetric: $\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{\sigma}}^T$ or $\sigma_{ij} = \sigma_{ji}$. Likewise $\underline{\mathbf{e}} = \underline{\mathbf{e}}^T$ or $e_{ij} = e_{ji}$.

Two product operations may be defined between second-order tensors. The *scalar product* or *inner product* is a scalar, which in terms of components is defined as⁴

$$\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}} = \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} e_{ij} = \sigma_{ij} e_{ij}. \quad (5.3)$$

With $\underline{\boldsymbol{\sigma}}$ and $\underline{\mathbf{e}}$ as stress and strain tensors, respectively, $\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}}$ is twice the strain energy density $\mathcal{U}$.

The *tensor product* or *open product* of two second order tensors is a second-order tensor defined by the composition rule:

$$\text{if } \underline{\mathbf{p}} = \underline{\boldsymbol{\sigma}} \cdot \underline{\mathbf{e}}, \quad \text{then } p_{ij} = \sum_{k=1}^3 \sigma_{ik} e_{kj} = \sigma_{ik} e_{kj}. \quad (5.4)$$

This is exactly the same rule as the matrix product. For matrices the dot is omitted. Some authors also omit the dot for tensors.

Four-dimensional tensors are represented by **underlined boldface uppercase** symbols. In elasticity the tensor of elastic moduli provides the most important example:

$$\underline{\mathbf{E}} \equiv E_{ijkl}, \quad (5.5)$$

The components of $\underline{\mathbf{E}}$ form a $3 \times 3 \times 3 \times 3$ hypercube with $3^4 = 81$ components, so the whole thing cannot be displayed so compactly as (5.2).

Operators that map vectors to vectors are usually represented by boldface uppercase symbols. An ubiquitous operator is nabla: ∇ , which should be boldface except that the symbol is not available in bold. Applied to a scalar function, say ϕ , it produces its gradient:

$$\nabla \phi = \mathbf{grad} \phi \equiv \phi_{,i} = \frac{\partial \phi}{\partial x_i} = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{bmatrix}. \quad (5.6)$$

⁴ Some textbooks use the notation $\underline{\boldsymbol{\sigma}} \cdot \underline{\mathbf{e}}$ for the scalar $\sigma_{ij} e_{ji}$, but this is unnecessary as it is easily expressed in terms of $:$ by transposing the second tensor.

Applying nabla to a vector via the dot product yields the divergence of the vector:

$$\nabla \cdot \mathbf{u} = \mathbf{div} \mathbf{u} \equiv u_{i,i} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \quad (5.7)$$

Applying nabla to a second order tensor yields the divergence of a tensor, which is a vector. For example:

$$\nabla \cdot \underline{\sigma} \equiv \mathbf{div} \sigma = \sigma_{ij,j} = \sum_{j=1}^3 \frac{\partial \sigma_{ij}}{\partial x_j} = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} \quad (5.8)$$

Applying ∇ to a vector via the cross product yields the curl or spin operator. This operator is not needed in classical elasticity but it appears in applications that deal with rotational fields such as fluid dynamics with vorticity, or corotational structural dynamics.

§5.2.2. Matrix Notation

Matrix notation is a modification of direct tensor notation in which everything is placed in matrix form, with some trickery used if need be. The main advantages of matrix notation are historical compatibility with finite element formulations, and ready computer implementation in symbolic or numeric form.⁵

The representation of scalars, which may be viewed as 1×1 matrices, does not change. Neither does the representation of vectors because vectors are column (or row) matrices.

Two-dimensional *symmetric* tensors are converted to one-dimensional arrays that list only the independent components (six in three dimensions, three in two dimensions). Component order is a matter of convention, but usually the diagonal components are listed first followed by the off-diagonal components. A factor of 2 may be applied to the latter, as the strain vector example below shows. The tensor is then represented as if were an actual vector, that is by non-underlined **boldface** lowercase Roman or Greek letters.

For the strain and stress tensors this “vectorization” process produces the 6-vectors

$$\underline{\mathbf{e}} \equiv \mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix}, \quad \underline{\sigma} \equiv \sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}, \quad (5.9)$$

Note that off-diagonal (shearing) components of the strain vector are scaled by 2, but that no such scaling applies to the off-diagonal (shear) stress components. The idea behind the scaling is to

⁵ Particularly in high level languages such as *Matlab*, *Mathematica* or *Maple*, which directly support matrix operators.

maintain inner product equivalence so that for example, the strain energy density is simply

$$\begin{aligned}\mathcal{U} &= \frac{1}{2} \underline{\sigma} : \underline{\mathbf{e}} = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} e_{ij} = \frac{1}{2} \sigma_{ij} e_{ij} = \frac{1}{2} \underline{\sigma}^T \underline{\mathbf{e}} \\ &= \frac{1}{2} (\sigma_{11} e_{11} + \sigma_{22} e_{22} + \sigma_{33} e_{33} + 2\sigma_{31} e_{31} + 2\sigma_{23} e_{23} + 2\sigma_{12} e_{12}).\end{aligned}\quad (5.10)$$

Four-dimensional tensors are mapped to square matrices and denoted by matrix symbols, that is, **non-underlined boldface uppercase** Roman or Greek letters. Indices are appropriately collapsed to reflect symmetries and maintain product equivalence. Rather than stating boring rules, the example of the elastic moduli tensor is given to illustrate the mapping technique.

The stress-strain relation for linear elasticity in component notation is $\sigma_{ij} = E_{ijkl} e_{kl}$, and in compact tensor form $\underline{\sigma} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}$. We would of course like to have $\underline{\sigma} = \underline{\mathbf{E}} \underline{\mathbf{e}}$ in matrix notation. This can be done by defining the 6×6 elastic modulus matrix

$$\underline{\mathbf{E}} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ \text{symm} & & & & & E_{66} \end{bmatrix} \quad (5.11)$$

The components E_{pq} of $\underline{\mathbf{E}}$ are related to the components E_{ijkl} of $\underline{\mathbf{E}}$ through an appropriate mapping that preserves the product relation. For example: $\sigma_{11} = E_{1111}e_{11} + E_{1122}e_{22} + E_{1133}e_{33} + E_{1112}e_{12} + E_{1121}e_{21} + E_{1113}e_{13} + E_{1131}e_{31} + E_{1123}e_{23} + E_{1132}e_{32}$ maps to $\sigma_{11} = E_{11}e_{11} + E_{12}e_{22} + E_{13}e_{33} + E_{14}2e_{23} + E_{15}2e_{31} + E_{16}2e_{12}$, whence $E_{11} = E_{1111}$, $E_{14} = E_{1123} + E_{1132}$, etc.

Finally, operators that can be put in vector form are usually represented by a vector symbol **boldface lowercase** whereas operators that can be put in matrix form are usually represented as matrices. Here is an example:

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underline{\mathbf{D}} \underline{\mathbf{u}}. \quad (5.12)$$

Operator $\underline{\mathbf{D}}$ is called the *symmetric gradient* in the continuum mechanics literature.⁶ In the matrix notation defined above it is written as a 6×3 matrix. In direct tensor notation $\underline{\mathbf{D}} = \frac{1}{2}(\nabla + \nabla^T)$ is the tensor that maps $\underline{\mathbf{u}}$ to $\underline{\mathbf{e}}$, and we write $\underline{\mathbf{e}} = \underline{\mathbf{D}} \cdot \underline{\mathbf{u}}$. For the indicial form see below.

⁶ Some books use variants of ∇ , such as $\bar{\nabla}$, $\hat{\nabla}$, ∇_S or ∇^S , for this operator.

§5.3. The Field Equations

§5.3.1. The Strain-Displacement Equations

The strain-displacement equations, also called the kinematic equations (KE) or deformation equations, yield the strain field given the displacement field. For linear elasticity the infinitesimal strain tensor e_{ij} is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (5.13)$$

where a comma denotes differentiation with respect to the space variable whose index follows.

In compact tensor notation, with $\underline{\mathbf{D}}$ as the symmetric gradient operator,

$$\underline{\mathbf{e}} = \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \underline{\mathbf{D}} \cdot \mathbf{u}. \quad (5.14)$$

The matrix form is $\mathbf{e} = \mathbf{D}\mathbf{u}$. The full form is given in (5.12).

The inverse problem: given a strain field find the displacements, is not generally soluble unless the strain components satisfy the *strain compatibility conditions*. These are complicated second-order partial differential equations given in any book on elasticity. This inverse problem will not be considered here.

§5.3.2. Constitutive Equations

The constitutive equations connect the stress and strain fields in V . These equations are intended to model the behavior of materials as continuum media. Generally they are partial differential equations (PDEs) or even integrodifferential equations in space and time. For linear elasticity, however, a considerable simplification occurs because the relation becomes algebraic, linear and homogeneous. For this case the stress-strain relations may be written in component notation as

$$\sigma_{ij} = E_{ijkl} e_{kl} \quad \text{in } V. \quad (5.15)$$

The E_{ijkl} are called *elastic moduli*. They are the components of a fourth order tensor $\mathbf{E}$ called the *elasticity tensor*. The elastic moduli satisfy generally the following symmetries

$$E_{ijkl} = E_{jikl} = E_{ijlk}, \quad (5.16)$$

which reduce their number from $3^4 = 729$ to $6^2 = 36$. Furthermore, if the body admits a strain energy (that is, the material is not only elastic but hyperelastic) the elastic moduli satisfy additional symmetries:

$$E_{ijkl} = E_{klij}, \quad (5.17)$$

which reduce their number to 21. Further symmetries occur if the material is orthotropic or isotropic. In the latter case the elastic moduli may be expressed as function of only two independent material constants, for example Young's modulus E and Poisson ratio ν .

In compact tensor notation:

$$\underline{\sigma} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}. \quad (5.18)$$

In matrix form:

$$\boldsymbol{\sigma} = \mathbf{E} \mathbf{e}, \quad (5.19)$$

where $\mathbf{E}$ is the 6×6 matrix given in (5.11).

If the elasticity tensor is invertible, the relation that connects strains to stresses is written

$$e_{ij} = C_{ijkl} \sigma_{kl} \quad \text{in } V. \quad (5.20)$$

The C_{ijkl} are called *elastic compliances*. They are also the components of a fourth order tensor called the *compliance tensor*, which satisfies the same symmetries as $\mathbf{E}$. In compact tensor notation

$$\underline{\mathbf{e}} = \underline{\mathbf{C}} \cdot \underline{\sigma} = \underline{\mathbf{E}}^{-1} \cdot \underline{\sigma}, \quad (5.21)$$

and in matrix form: $\mathbf{e} = \mathbf{C} \boldsymbol{\sigma} = \mathbf{E}^{-1} \boldsymbol{\sigma}$.

§5.3.3. Internal Equilibrium Equations

The internal equilibrium equations of elastostatics are

$$\sigma_{ij,j} + b_i = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \text{in } V. \quad (5.22)$$

These follow from the linear momentum balance equations derived in books on continuum mechanics.

The compact tensor notation is

$$\nabla \cdot \underline{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V. \quad (5.23)$$

The matrix form is

$$\mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V. \quad (5.24)$$

Here $\mathbf{D}^T$ is the transpose of the symmetric gradient operator (5.12).

§5.4. The Boundary Conditions

§5.4.1. Surface Compatibility Equations

The surface compatibility equations, also called *displacement boundary conditions*, are

$$\boxed{u_i = \hat{u}_i \quad \text{on } S_u,} \quad (5.25)$$

or in direct notation (both tensor and matrix)

$$\boxed{\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } S_u.} \quad (5.26)$$

The physical meaning is that the displacement components at points of S_t must match the prescribed values.

§5.4.2. Surface Equilibrium Equations

The surface equilibrium equations, also called *stress boundary conditions*, or *traction boundary conditions*, are

$$\boxed{\sigma_{ij} n_j = \hat{t}_i \quad \text{on } S_t,} \quad (5.27)$$

where n_j are the components of the external unit normal $\mathbf{n}$ at points of S_t where tractions are specified; see Figure 5.2. Note that

$$\sigma_{ni} = \sigma_{ij} n_j = t_i \quad (5.28)$$

are the components of the *internal traction vector* $\mathbf{t} \equiv \sigma_n$. The physical interpretation of the stress boundary condition is that the internal traction vector must equal the prescribed traction vector. Or: the net flux $t_i - \hat{t}_i$ on S_t vanishes, component by component. In compact tensor form

$$\boxed{\mathbf{t} = \sigma_n = \underline{\sigma} \cdot \mathbf{n} = \hat{\mathbf{t}}.} \quad (5.29)$$

Stating (5.27) in a matrix form that uses the stress vector σ defined in (5.9) requires some care. It would be incorrect to write either $\mathbf{t} = \sigma^T \mathbf{n}$ or $\mathbf{t} = \mathbf{n}^T \sigma$ because σ is 6×1 and $\mathbf{n}$ is 3×1 . Not only are these vectors non-conforming but their inner product is a scalar. The proper matrix form is a bit contrived:

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} n_1 & 0 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & 0 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{P}_n \sigma, \quad (5.30)$$

where $\mathbf{P}_n$ is the 3×6 normal-projection matrix shown above.

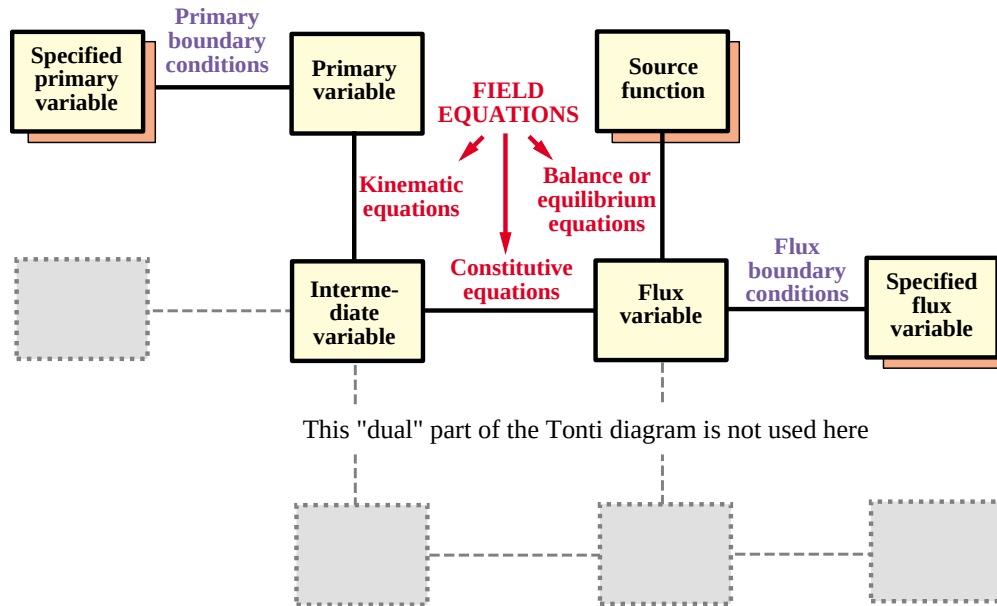


FIGURE 5.3. The general configuration of the Tonti diagram. Upper portion reproduced from Chapter 2. The diagram portion shown in dashed lines, which represents the so-called dual or potential equations, is not used in this book.

§5.5. Tonti Diagrams

The Tonti diagram was introduced in Chapter 2 to represent the field equations of a mathematical model in graphical form. The general configuration of the expanded form of that diagram (“expanded” means that it shows boundary conditions) is repeated in Figure 5.3 for convenience. This diagram lists generic names for the “box occupants” and the connecting links.

Boxes and box-connectors drawn in solid lines are said to constitute the *primal* formulation of the governing equations. Dashed-lines boxes and connectors shown in the bottom pertain to the so-called *dual* formulation in terms of potentials, which will not be used in this book.⁷

Figure 5.4 shows the primal formulation of the linear elasticity problem represented as a Tonti diagram. For this particular problem the displacements are the primary (or primal) variables, the strains the intermediate variables, and the stresses the flux variables. The source variables are the body forces. The prescribed configuration variables are prescribed displacements on S_t and the prescribed flux variables are the surface tractions on S_t .

Tables 5.1 and 5.2 lists the generic names for the components of the Tonti diagram, as well as those specific for the elasticity problem. Table 5.3 summarizes the governing equations of linear elastostatics written down in three notational schemes.

⁷ In the dual formulation the intermediate and flux variable exchange roles, so that boundary conditions of flux type are linked to the intermediate variable of the primal formulation. In this way it is possible, for instance, to specify strain boundary conditions: just to for the dual formulation.

Table 5.1 Abbreviations for Tonti Diagram Box Contents

<i>Acronym</i>	<i>Meaning</i>	<i>Alternate names in literature</i>
PV	Primary variable	Primal variable, configuration variable, “across” variable
IV	Intermediate variable	First intermediate variable, auxiliary variable
FV	Flux variable	Second intermediate variable, “through” variable
SV	Source variable	Internal force variable, production variable
PPV	Prescribed primary variable	
PFV	Prescribed flux variable	

Table 5.2 Abbreviations for Tonti Diagram Box Connectors

<i>Acronym</i>	<i>Generic name</i>	<i>Name(s) given in the elasticity problem</i>
KE	Kinematic equations	Strain-displacement equations
CE	Constitutive equations	Stress-strain equations, material equations
BE	Balance equations	Internal equilibrium equations
PBC	Primary boundary conditions	Displacement BCs
FBC	Flux boundary conditions	Stress BCs, traction BCs

Table 5.3 Summary of Elastostatic Governing Equations

<i>Acr</i>	<i>Valid</i>	<i>Compact or direct tensor form</i>	<i>Matrix form</i>	<i>Component (indicial) form</i>
KE	in V	$\underline{\mathbf{e}} = \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \underline{\mathbf{D}} \cdot \mathbf{u}$	$\mathbf{e} = \mathbf{D} \mathbf{u}$	$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
CE	in V	$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}$	$\boldsymbol{\sigma} = \mathbf{E} \mathbf{e}$	$\sigma_{ij} = E_{ijkl} e_{kl}$
BE	in V	$\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0}$	$\mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$	$\sigma_{ij,j} + b_i = 0$
PBC	on S_u	$\mathbf{u} = \hat{\mathbf{u}}$	$\mathbf{u} = \hat{\mathbf{u}}$	$u_i = \hat{u}_i$
FBC	on S_t	$\underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$	$\mathbf{P}_n \boldsymbol{\sigma} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$	$\sigma_{ij} n_j = \sigma_{ni} = t_i = \hat{t}_i$

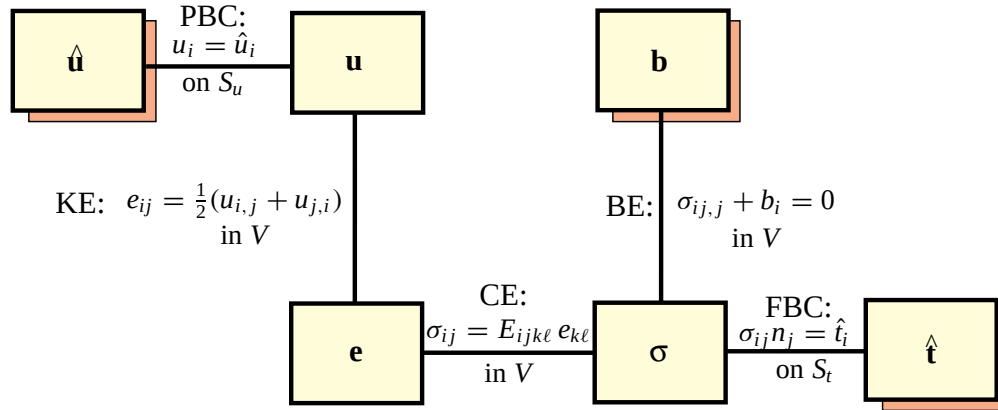


FIGURE 5.4. Tonti diagram for linear elastostatics. Governing equations are expressed along links in indicial form.

§5.6. Other Notational Conventions

To facilitate comparison with older textbooks and papers, the governing equations are restated below in two more alternative forms: in “grad/div” notation, and in full form.

§5.6.1. Grad-Div Direct Tensor Notation

This is a variation of the “nabla” direct tensor notation. Symbols **grad** and **div** are used instead of ∇ and $\nabla \cdot$ for gradient and divergence, respectively, and **symm grad** means the symmetric gradient operator $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$. The notation is slightly more readable but takes more room.

KE:	$\underline{\mathbf{e}} = \text{symm grad } \mathbf{u}$	in V ,	(5.31)
CE:	$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \underline{\mathbf{e}}$	in V ,	
BE:	$\text{div } \underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0}$	in V ,	
PBC:	$\mathbf{u} = \hat{\mathbf{u}}$	on S_u ,	
FBC:	$\underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$,	on S_t .	

§5.6.2. Full Notation

In the full-form notation everything is spelled out. No ambiguities of interpretation can arise; consequently this works well as a notation of last resort, and also as a “comparison template” against one can check out the meaning of more compact notations. It is also useful for programming in low-order languages.

The full form has, however, two major problems. First, it can become quite voluminous when higher order tensors are involved. Notice that most of the equations below are truncated because there is no space to state them fully. Second, compactness encourages visualization of essentials: long-windedness can obscure the forest with too many trees.

KE:	$e_{11} = \frac{\partial u_1}{\partial x_1}, \quad e_{12} = e_{21} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right), \quad \dots$	in V ,	(5.32)
CE:	$\sigma_{11} = E_{1111}e_{11} + E_{1112}e_{12} + \dots$ (7 more terms), $\sigma_{12} = \dots$	in V ,	
BE:	$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0, \dots$	in V ,	
PBC:	$u_1 = \hat{u}_1, \quad u_2 = \hat{u}_2, \quad u_3 = \hat{u}_3$	on S_u ,	
FBC:	$\sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = \hat{t}_1, \dots$	on S_t .	

§5.7. Solving Elastostatic Problems

By *solving an elastostatic problem* it is meant to find the displacement, strain and stress fields that satisfy all governing equations; that is, the field equations and the boundary conditions.

Under mild assumptions of primary interest to mathematicians, the elastostatic problem has one and only one solution. There are, however, practical problems where the solution is not unique. Two instances:

1. “*Free Floating*” Structures. The displacement field is not unique but strains and stresses are. Example are aircraft structures in flight and space structures in orbit.
2. *Incompressible Materials*. The mean (hydrostatic) stress field is not determined from the displacements and strains. Determination of the hydrostatic stress field depends on the stress boundary conditions, and these may be insufficient in some cases.

An analytical solution of the elastostatic problem is only possible for very simple cases. Most practical problems demand a numerical solution. Numerical methods require a *discretization* process through which an approximate solution with a finite number of degrees of freedom is constructed.

§5.7.1. Discretization Methods in Computational Mechanics

Discretization methods of highest importance in mechanics can be grouped into three classes: finite difference, finite element, and boundary methods.

Finite Difference Method (FDM). The governing differential equations are replaced by difference expressions based on the field values at nodes of a finite difference grid. Although FDM remains important in fluid mechanics and in dynamic problems for the time dimension, it has been largely superseded by the finite element method in a structural mechanics in general and elastostatics in particular.

Finite Element Method (FEM). This is the most important “volume integral” method. One or more of the governing equations are recast to hold in some *average* sense over subdomains of simple geometry. This recasting is often done in terms of *variational forms* if variational principles can be readily constructed, as is the case in elastostatics. The procedure for constructing the simplest class of these principles is outlined in the next section.

Boundary Methods. Under certain conditions the field equations between volume fields can be eliminated in favor of *boundary unknowns*. This dimensionality reduction process leads to integro-differential equations taken over the boundary S . Discretization of these equations through finite element or collocation techniques leads to the so-called *boundary element methods* (BEM).

Further discussion on the role of these methods within the process of simulating of structural systems was offered in Chapter 1.

§5.8. Constructing Variational Forms

Finite element methods for the elasticity problem are based on *Variational Forms*, or VFs, of the foregoing Strong Form (SF) equations. Although the SF is unique, there are many VFs.⁸ As explained in Chapters 2–4, the search for a VF begins by selecting one or more master fields, and weakening one or more links. This process produces a set of equations called the Weak Form, or WF, which may be viewed as an midway stop between the SF and the VF.

The end result of the process is the construction of a *functional* Π that contains integrals of the known and unknown fields. Associated with the functional is a *variational principle*: setting the first variation $\delta\Pi$ to zero recovers the strong form of the weakened field equations as Euler-Lagrange equations, and the strong form of the weakened BCs as natural boundary conditions.

Here is a summary of the VF construction steps: (1) choose the master(s), (2) weaken selected links, (3) work out the (total) variation of the alleged functional, (4) construct the functional. These four steps are elaborated below keeping the elasticity equations in mind. There are then illustrated with the construction of the single-field primal functional, called the Total Potential Energy.

§5.8.1. Step 1: Choose Master Field(s)

One or more of the unknown internal fields

$$u_i, \quad e_{ij}, \quad \sigma_{ij}, \quad (5.33)$$

are chosen as masters. A master (also called *primary*, *varied* or *parent*) field is one that is subjected to the δ -variation process of the calculus of variations. Fields that are not masters, *i.e.* not subject to variation, are called *slave*, *secondary* or *derived*. The *owner* (also called *parent* or *source*) of a slave field is the master from which it comes from.

If only one master field is chosen, the resulting variational principle (obtained after going through Steps 2, 3 and 4) is called *single-field*, and *multifield* otherwise.

A known or data field (for example: body forces or surface tractions in elastostatics) cannot be a master field because it is not subject to variation, and is not a secondary field because it does not derive from others. Hence we see that *fields can only be of three types*: master, slave, or data.

§5.8.2. Step 2: Choose Weak Connections

Given a master field, consider the equations that link it to other known and unknown fields. These are called the *connections* of that field. Classify these connections into two types:

Strong connection. The connecting relation is enforced *point by point* in its original form. For example if the connection is a PDE or an algebraic equation we use it as such. Also called *a priori* enforcement. When applied to a boundary condition, a strong connection is also referred to as an *essential constraint* or *essential B.C.*

⁸ There is in fact an infinite number, parametrizable by a finite number of parameters, as shown in: C. A. Felippa, A survey of parametrized variational principles and applications to computational mechanics, *Comp. Meths. Appl. Mech. Engrg.*, **113**, 109–139, 1994. Most books give the impression, however, that there is only a finite number.

Weak connection. The connection relationship is enforced only in an *average* or *mean* sense through the use of a weight or test function, or of a distributed Lagrange multiplier. Also called *a-posteriori* enforcement. When applied to a boundary condition, a weak connection is also referred to as a *natural constraint* or *natural B.C.*

A general rule to keep in mind is that *a slave field must be reachable from its owner through strong connections.*

If there is more than one master field (*i.e.* we are constructing a multifield principle), the foregoing definitions *must be applied to each master field in turn.* In other words, we must consider the connections that “emanate” from each of the master fields. The end result is that the same field may appear more than once. For example in elasticity the strain field $\mathbf{e}$ may appear up to three times: (1) as a master field, (2) as a slave field derived from displacements, and (3) as a slave field derived from stresses. These complications cannot occur with single-field principles.

Remark 5.3. There is usually limited freedom as regards the choice of strong vs. weak connections. The key test comes when one tries to form the total variation in Step 3. If this happens to be the exact variation of a functional, the choice is admissible. Else is back to the drawing board.

§5.8.3. Step 3: Construct a First Variation

Once all choices of Steps 1 and 2 have been made, the remaining manipulations are technical in nature, and essentially consist of applying the tools and techniques of vector, tensor and variational calculus: Lagrange multipliers, integration by parts, homogenization of variations, surface integral splitting, and so on. Since the number of operational combinations is huge, the techniques are best illustrated through specific examples.

The end result of these gyrations should be a variational statement

$$\boxed{\delta \Pi = 0,} \quad (5.34)$$

where the symbol δ here embodies variations with respect to *all master fields.*

§5.8.4. Step 4: Functionalize

With luck, the variational statement (5.34) will be recognized as the *exact variation* of a functional Π , whence the variational statement becomes a true variational principle. If so, Π represents the Variational Form we were looking for, and the search is successful.

We now illustrate the foregoing steps with the detailed derivation of the most important single-field VF in elastostatics: the principle of Total Potential Energy or TPE.

§5.9. Derivation of Total Potential Energy Principle

§5.9.1. A Long Journey Starts with the First Step

The departure point for deriving the classical TPE principle is the WF diagrammed in Figure 5.5. Such modifications are briefly explained in the figure label and in the text below. The displacement field u_i is the only master. The strain and stress fields are slaves. The slave-provenance notation

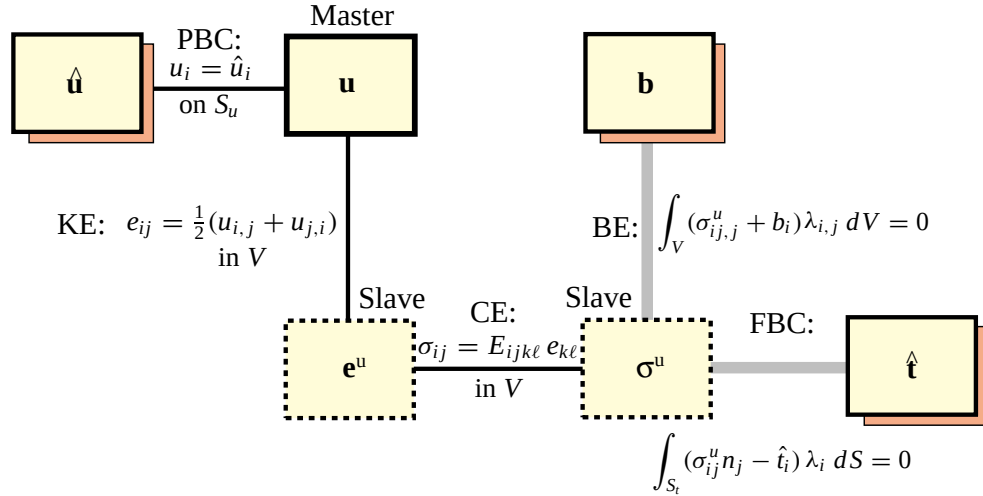


FIGURE 5.5. The WF used as departure point for deriving the TPE functional of linear elastostatics.

introduced in Chapter 3 is used: the owner of a slave field is marked by a superscript. For example, $\mathbf{e}^u = \mathbf{D} \mathbf{u}$ means “ $\mathbf{e}^u$ is owned by $\mathbf{u}$ ” through the strong KE link.

The strong connections are the kinematic equations KE (in elasticity the strain-displacement equations), the constitutive equations CE, and the primary boundary conditions PBC (in elasticity the displacement boundary conditions). These are depicted in Figure 5.5 as solid box-connecting lines:

$$\text{Strong : } e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } V, \quad \sigma_{ij} = E_{ijkl} e_{kl} \text{ in } V, \quad u_i = \hat{u}_i \text{ on } S_u. \quad (5.35)$$

The *weak* connections are the balance equations BE (in elasticity the stress equilibrium equations), and the flux boundary conditions FBC (in elasticity the traction boundary conditions), These are shown in Figure 5.5 as shaded lines:

$$\text{Weak: } \sigma_{ij,j} + b_i = 0 \text{ in } V, \quad \sigma_{ij} n_j = \hat{t}_i \text{ on } S_t. \quad (5.36)$$

§5.9.2. Lagrangian Glue

Now we get down to the business of variational calculus. A slight notational variation of the *residual weighting* technique of previous Chapters is used. The notation has certain interpretation advantages that will become apparent later when dealing with hybrid principles.

To treat BE as a weak connection, take the first of (5.36), replace σ_{ij} by the slave σ_{ij}^u , multiply by a piecewise differentiable 3-vector field λ_i and integrate over V :

$$\int_V (\sigma_{ij,j}^u + b_i) \lambda_i dV = 0. \quad (5.37)$$

Apply the divergence theorem to the first term in (5.37):

$$\int_V \sigma_{ij,j}^u \lambda_i dV = - \int_V \sigma_{ij}^u \lambda_{i,j} dV + \int_S \sigma_{ij}^u n_j \lambda_i dS. \quad (5.38)$$

For a symmetric stress tensor $\sigma_{ij}^u = \sigma_{ji}^u$ this formula may be transformed⁹ to

$$\int_V \sigma_{ij,j}^u \lambda_i dV = - \int_V \sigma_{ij}^u \frac{1}{2} (\lambda_{i,j} + \lambda_{j,i}) dV + \int_S \sigma_{ij}^u n_j \lambda_i dS. \quad (5.39)$$

Assignment of meaning of internal energy to the second term in (5.39) suggests identifying λ_i with the variation of the displacement field u_i (a “lucky guess” that can be proved rigorously *a posteriori*):

$$\int_V \sigma_{ij,j}^u \delta u_i dV = - \int_V \sigma_{ij}^u \delta e_{ij}^u dV + \int_S \sigma_{ij}^u n_j \delta u_i dS, \quad (5.40)$$

in which the strain-variation symbol means

$$\delta e_{ij}^u = \frac{1}{2} (\delta u_{i,j} + \delta u_{j,i}) \text{ in } V, \quad (5.41)$$

because of the strong connection $e_{ij}^u = \frac{1}{2} (u_{i,j} + u_{j,i})$, which if varied with respect to u_i yields (5.41).

Remark 5.4. Although the essence of the treatment of weak conditions is ultimately the same, there is far from universal agreement on terminology in the literature. The foregoing scheme is known as the *Lagrange multiplier* treatment. It closely follows Fraeijs de Veubeke (a major contributor to variational mechanics). The technique was originally introduced by Friedrichs (a disciple of Courant and Hilbert) in a mathematical context.

Other authors, primarily in fluid mechanics, favor *weight functions* (as in previous Chapters) or *test functions*. If the WF is directly discretized, as often done in fluid mechanics, the former technique leads to weighted-residual subdomain methods (for example the Fluid Volume Method) whereas test functions lead to Galerkin and Petrov-Galerkin methods. Some authors, such as Lanczos,¹⁰ multiply directly equilibrium residuals by displacement variations, which are called then *virtual displacements*. Some music but different lyrics.

§5.9.3. Constructing the First-Variation Pieces

Substituting (5.40) into (5.37), with $\lambda_i \rightarrow \delta u_i$, we obtain

$$\int_V \sigma_{ij}^u \delta e_{ij}^u dV - \int_V b_i \delta u_i dV - \int_S \sigma_{ij}^u n_j \delta u_i dS = 0. \quad (5.42)$$

The surface integral may be split as follows:

$$\int_S \sigma_{ij}^u n_j \delta u_i dS = \int_{S_t} \sigma_{ij}^u n_j \delta u_i dS + \int_{S_u} \sigma_{ij}^u n_j \delta \hat{u}_i dS = \int_{S_t} \sigma_{ij}^u n_j \delta u_i dS. \quad (5.43)$$

where the substitution $\delta u_i = \delta \hat{u}_i$ on S_u results from the strong connection $u_i = \hat{u}_i$ on S_u . But $\delta \hat{u}_i = 0$ because prescribed (data) fields are not subject to variation, and the S_u integral drops out. Treating the FBC weak connection with δu_i as 3-vector weight function we obtain

$$\int_{S_t} (\sigma_{ij}^u n_j - \hat{t}_i) \delta u_i dS = 0, \quad \text{whence} \quad \int_{S_t} \sigma_{ij}^u n_j \delta u_i dS = \int_{S_t} \hat{t}_i \delta u_i dS. \quad (5.44)$$

⁹ This transformation is stated in §5.5 of Sewell's book: M. J. Sewell, *Maximum and Minimum Principles*, Cambridge, 1987. It may also be verified directly using indicial calculus, as in Exercise 5.4.

¹⁰ C. Lanczos, *The Variational Principles of Mechanics*, Dover, 4th edition, 1970.

§5.9.4. A Happy Ending

Substituting (5.43) and the second of (5.44) into (5.42), we obtain the final form of the variation in the master field u_i , which we write (hopefully) as the variation of a functional Π_{TPE} :

$$\delta \Pi_{\text{TPE}} = \int_V \sigma_{ij}^u \delta e_{ij}^u dV - \int_V b_i \delta u_i dV - \int_{S_t} \hat{t}_i \delta u_i dS = 0. \quad (5.45)$$

And indeed (5.45) can be recognized¹¹ as the exact variation, with respect to u_i , of

$$\Pi_{\text{TPE}}[u_i] = \frac{1}{2} \int_V \sigma_{ij}^u e_{ij}^u dV - \int_V b_i u_i dV - \int_{S_t} \hat{t}_i u_i dS. \quad (5.46)$$

This Π_{TPE} is called the *total potential energy* functional. It is often written as the difference of the strain energy and the external work functionals:

$$\begin{aligned} \Pi_{\text{TPE}} &= U_{\text{TPE}} - W_{\text{TPE}}, & \text{in which} \\ U_{\text{TPE}} &= \frac{1}{2} \int_V \sigma_{ij}^u e_{ij}^u dV, & W_{\text{TPE}} = \int_V b_i u_i dV + \int_{S_t} \hat{t}_i u_i dS. \end{aligned} \quad (5.47)$$

Consequently (5.45) is a true variational principle and not just a variational statement.

The physical interpretation is well known: $\frac{1}{2} \sigma_{ij}^u e_{ij}^u$ is the *strain energy density* $\mathcal{U}$ in terms of displacements. Integrating this density over the volume V gives the total strain energy stored in the body. In elasticity this is the only stored energy, and consequently it is also the *internal energy* U . Likewise, $b_i u_i$ is the *external work density* of the body forces, whereas $\hat{t}_i u_i$ is the *external work density* of the applied surface tractions. Integrating these densities over V and S_t , respectively, and adding gives the total *external work potential* W .

Remark 5.5. What we have just gone through is called the Inverse Problem of Variational Calculus: given the governing equations (field equations and boundary conditions), find the functional(s) that have those governing equations as Euler-Lagrange equations and natural boundary conditions, respectively.

The Direct Problem of Variational Calculus is the reverse one: given a functional such as (5.46), show that the vanishing of its variation is equivalent to the governing equations. This problem is normally the first one tackled in Variational Calculus instruction in math support courses.¹² The Direct Problem is done by carrying out the foregoing steps in reverse order: get the variation (5.45), integrate by parts as appropriate to homogenize variations, and use the strong connections to finally arrive at

$$\delta \Pi_{\text{TPE}} = \int_V (\sigma_{ij,j}^u + b_i) \delta u_i dV + \int_{S_t} (\sigma_{ij} n_j - \hat{t}_i) \delta u_i dS. \quad (5.48)$$

Using the fundamental lemma of variational calculus¹³ one then shows that $\delta \Pi_{\text{TPE}} = 0$ yields the weak connections (5.36) as Euler-Lagrange equations and natural boundary conditions, respectively.

¹¹ See Exercise 5.5 for the variation of the strain energy term.

¹² For example, Aerospace Math.

¹³ Ch. 1, §3 of I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963, reprinted by Dover, 2000.

§5.10. The Tensor Divergence Theorem and the PVW

Recall from §3.6 the canonical form of the theorem, which says that the vector divergence of a vector $\mathbf{a}$ over a volume is equal to the vector flux over the surface:

$$\int_V \nabla \cdot \mathbf{a} \, dV = \int_S \mathbf{a} \cdot \mathbf{n} \, dS. \quad (5.49)$$

Take $\mathbf{a} = \boldsymbol{\sigma} \cdot \mathbf{u}$, where $\boldsymbol{\sigma} = [\sigma_{ij}]$ is a symmetric stress tensor and $\mathbf{u} = [u_i]$ a displacement vector:

$$\int_V (\boldsymbol{\sigma} : \nabla \mathbf{u} + \nabla \boldsymbol{\sigma} \cdot \mathbf{u}) \, dV = \int_S \boldsymbol{\sigma} \cdot \mathbf{u} \cdot \mathbf{n} \, dS. \quad (5.50)$$

Here $\nabla \mathbf{u} = [\partial u_i / \partial x_j]$ is an unsymmetric tensor called the deformation gradient. Its transpose is $\mathbf{u}^T \nabla^T = [\partial u_j / \partial x_i]$. Now $\boldsymbol{\sigma} : \nabla \mathbf{u} = (\boldsymbol{\sigma} : \nabla \mathbf{u})^T = \boldsymbol{\sigma} : \mathbf{u}^T \nabla^T = \boldsymbol{\sigma} : \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \boldsymbol{\sigma} : \mathbf{D} \cdot \mathbf{u}$, where $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$. Hence

$$\int_V \boldsymbol{\sigma} : \mathbf{D} \cdot \mathbf{u} \, dV = - \int_V \nabla \boldsymbol{\sigma} \cdot \mathbf{u} \, dV + \int_S \boldsymbol{\sigma} \cdot \mathbf{u} \cdot \mathbf{n} \, dS. \quad (5.51)$$

In indicial notation this is

$$\int_V \sigma_{ij} \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) dV = - \int_V \frac{\partial \sigma_{ij}}{\partial x_i} u_j \, dV + \int_S \sigma_{ij} u_j n_i \, dS. \quad (5.52)$$

Recognizing that $e_{ij}^u = \frac{1}{2}(\partial u_j / \partial x_i + \partial u_i / \partial x_j)$ we finally arrive at

$$\int_V \sigma_{ij} e_{ij}^u \, dV = - \int_V \frac{\partial \sigma_{ij}}{\partial x_i} u_j \, dV + \int_S \sigma_{ij} u_j n_i \, dS. \quad (5.53)$$

Taking the variation of this equation with respect to the displacements while keeping σ_{ij} fixed yields the Principle of Virtual Work (PVW):

$$\int_V \sigma_{ij} \delta e_{ij}^u \, dV = - \int_V \frac{\partial \sigma_{ij}}{\partial x_i} \delta u_j \, dV + \int_S \sigma_{ij} \delta u_j n_i \, dS. \quad (5.54)$$

So far σ_{ij} and e_{ij}^u are disconnected in (5.54) because no constitutive assumption has been stated in this derivation. Consequently the PVW is valid for arbitrary materials (for example, in plasticity), which underscores its generality. Setting $\sigma_{ij} = \sigma_{ij}^u$ provides the form used in §5.9.2.

Homework Exercises for Chapter 5
Three-Dimensional Linear Elastostatics

EXERCISE 5.1 [A:10] Specialize the elasticity problem to a bar directed along x_1 . Write down the field equations in indicial, tensor and matrix form.

EXERCISE 5.2 [A:10] Justify the matrix form (5.30).

EXERCISE 5.3 [A:20] Suppose that the displacement $\hat{\mathbf{u}}_P$ at an *internal* point $P(\mathbf{x}_P)$ is known. How can that condition be accommodated as a boundary condition on S_u ? Hint: draw a little sphere of radius ϵ about P , then ... oops I almost told the story.

EXERCISE 5.4 [A:20] Justify passing from (5.38) to (5.39) by proving that if σ_{ij} is symmetric, that is, $\sigma_{ij} = \sigma_{ji}$, then $\sigma_{ij}\lambda_{i,j} = \sigma_{ij} \frac{1}{2}(\lambda_{i,j} + \lambda_{j,i})$. Hint: one (elegant) way is to split $\lambda_{i,j} + \lambda_{j,i}$ into symmetric and antisymmetric parts; other approaches are possible.

EXERCISE 5.5 [A:15] Prove that $\delta(\frac{1}{2}\sigma_{ij}^u e_{ij}^u) = \sigma_{ij}^u \delta e_{ij}^u$, where the variation δ is taken with respect to displacements u_i .

6

The HR Variational Principle of Elastostatics

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§6.1. Introduction

In Chapter 3, a *multifield* variational principle was defined as one that has more than one master field. That is, more than one unknown field is subject to independent variations. The present Chapter begins the study of such functionals within the context of elastostatics. Following a classification of the so-called canonical functionals, the Hellinger-Reissner (HR) mixed functional is derived.

The HR principle is applied to the derivation of a couple of 1D elements in the text, and others are provided in the Exercises.

§6.1.1. Mixed Versus Hybrid

The terminology pertaining to multifield functionals is not uniform across applied mechanics and FEM literature. Sometimes all multifield principles are called mixed; sometimes this term is restricted to specific cases. This book takes a middle ground:

A *Mixed principle* is one where all master fields are internal fields (volume fields in 3D).

A *Hybrid principle* is one where master fields are of different dimensionality. For example one internal volume field and one surface field.

Hybrid principles will be studied in Chapter 8 and 9. They are intrinsically important for FEM discretizations but have only a limited role outside of FEM.

§6.1.2. The Canonical Functionals

If hybrid functionals are excluded, three unknown internal fields of linear elastostatics are candidates for master fields to be varied: displacements u_i , strains e_{ij} , and stresses σ_{ij} . Seven combinations, listed in Table 6.1, may be chosen as masters. These are called the *canonical* functionals of elasticity.

Table 6.1 The Seven Canonical Functionals of Linear Elastostatics

#	Type	Master fields	Name
(I)	Single-field	Displacements	Total Potential Energy (TPE)
(II)	Single-field	Stresses	Total Complementary Potential Energy (TCPE)
(III)	Single-field	Strains	No name
(IV)	Mixed 2 field	Displacements & stresses	Hellinger-Reissner (HR)
(V)	Mixed 2-field	Displacements & strains	No agreed upon name
(VI)	Mixed 2-field	Strains & stresses	No name
(VII)	Mixed 3-field	Displacements, stresses & strains	Veubeke-Hu-Washizu (VHW)

Four of the canonical functionals: (I), (II), (IV) and (VII), have identifiable names. From the standpoint of finite element development those four, plus (V), are most important although they are not equal in importance. By far (I) and (IV) have been the most seminal, distantly followed by (II), (V) and (VII). Functionals (III) and (VI) are mathematical curiosities.

The construction of mixed functionals involves more expertise than single-field ones. And their FEM implementation requires more care and patience.¹

¹ Strang's famous dictum is "mixed elements lead to mixed results." In other words: more master fields are not necessarily better than one. Some general guides as to when mixed functionals pay off will appear as byproduct of examples.

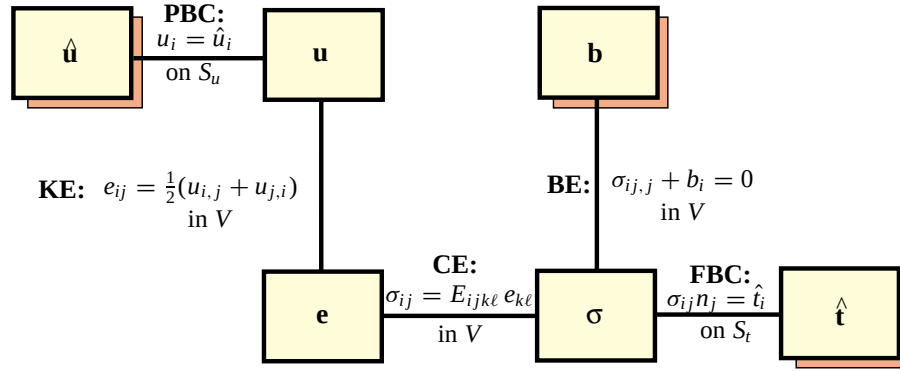


FIGURE 6.1. The Strong Form Tonti diagram for linear elastostatics, reproduced for convenience.

For convenience the Strong Form Tonti diagram of linear elastostatics is shown in Figure 6.1.

§6.2. The Hellinger-Reissner (HR) Principle

§6.2.1. Assumptions

The Hellinger-Reissner (HR) canonical functional of linear elasticity allows displacements and stresses to be varied separately. This establishes the master fields. Two slave strain fields appear, one coming from displacements and one from stresses:

$$e_{ij}^u = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad e_{ij}^\sigma = C_{ijkl} \sigma_{kl} \quad (6.1)$$

Here C_{ijkl} are the entries of the compliance tensor or strain-stress tensor $\underline{\mathbf{C}}$, which is the inverse of $\underline{\mathbf{E}}$. In matrix form this is $\mathbf{e}^\sigma = \mathbf{C}\sigma$, where $\mathbf{C} = \mathbf{E}^{-1}$ is a 6×6 matrix of elastic compliances.

At the exact solution of the elasticity problem, the two strain fields coalesce point by point. But when these fields are obtained by an approximation procedure such as FEM, strains recovered from displacements and strains computed from stresses will not generally agree.

Three weak links appear: BE and FBC (as in the Total Potential Energy principle derived in the previous Chapter), plus the link between the two slave strain fields, which is identified as EE. Figure 6.2 depicts the resulting Weak Form.

Remark 6.1. The weak connection between $\mathbf{e}^u$ and $\mathbf{e}^\sigma$ could have been substituted by a weak connection between σ^u and σ . The results would be the same because the constitutive equation links are strong. The choice of $\mathbf{e}^u$ and $\mathbf{e}^\sigma$ simplifies slightly the derivations below.

§6.2.2. The Weak Equations

We follow the weighting residual technique used in Chapter 3 for the TPE derivation. Take the residuals of the three weak connections shown in Figure 6.2, multiply them by Lagrange multiplier fields and integrate over the respective domains:

$$\int_V (e_{ij}^u - e_{ij}^\sigma) w_{ij} dV + \int_V (\sigma_{ij,j} + b_i) w_i^* dV + \int_S (\sigma_{ij} n_j - \hat{t}_i) w_i^{**} dS = 0 \quad (6.2)$$

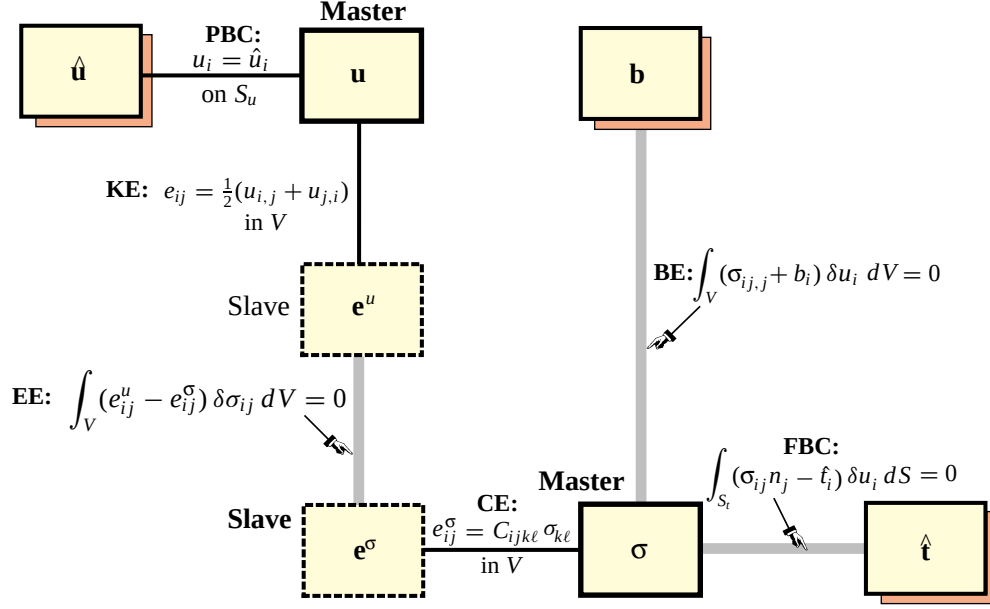


FIGURE 6.2. The starting Weak Form for derivation of the HR principle.

For conformity, w_{ij} must be a second order tensor, whereas w_i^* and w_i^{**} are 3-vectors. These weights must be expressed as variations of either master: either displacements u_i or stresses σ_{ij} , based on work pairing considerations. The residuals of KE are volume forces integrated over V , and those of FBC are surface forces integrated over S . Hence w_i^* and w_i^{**} must be displacement variations to obtain energy density. The residuals of EE are strains integrated over V ; consequently w_{ij} must be stress variations. Based on these considerations we set $w_{ij} = \delta\sigma_{ij}$, $w_i^* = -\delta u_i$, $w_i^{**} = \delta u_i$, where the minus sign in the second one is chosen to anticipate eventual cancellation in the surface integrals. Adding the weak link contributions gives

$$\int_V (e_{ij}^u - e_{ij}^\sigma) \delta\sigma_{ij} dV - \int_V (\sigma_{ij,j} + b_i) \delta u_i dV + \int_S (\sigma_{ij} n_j - \hat{t}_i) \delta u_i dS = 0. \quad (6.3)$$

Next, integrate the $\sigma_{ij,j} \delta u_i$ term by parts to eliminate the stress derivatives, split the surface integral into $S_u \cup S_t$, and enforce the strong link $u_i = \hat{u}_i$ over S_u :

$$\begin{aligned} - \int_V \sigma_{ij,j} \delta u_i dV &= \int_V \sigma_{ij} \delta e_{ij}^u dV - \int_S \sigma_{ij} n_j \delta u_i dS \\ &= \int_V \sigma_{ij} \delta e_{ij}^u dV - \int_{S_u} \sigma_{ij} n_j \delta \hat{u}_i dS - \int_{S_t} \sigma_{ij} n_j \delta u_i dS \\ &= \int_V \sigma_{ij} \delta e_{ij}^u dV - \int_{S_t} \sigma_{ij} n_j \delta u_i dS. \end{aligned} \quad (6.4)$$

in which δe_{ij}^u means the variation of $\delta \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$, as in §5.9.2.

Upon simplification of the cancelling terms $\sigma_{ij} n_j \delta u_i$ on S_t we end up with the following variational statement, written hopefully as the exact variation of a functional Π_{HR} :

$$\delta \Pi_{\text{HR}} = \int_V [(e_{ij}^u - e_{ij}^\sigma) \delta\sigma_{ij} + \sigma_{ij} \delta e_{ij}^u - b_i \delta u_i] dV - \int_{S_t} \hat{t}_i \delta u_i dS. \quad (6.5)$$

§6.2.3. The Variational Form

And indeed (6.5) is the exact variation of

$$\Pi_{\text{HR}}[u_i, \sigma_{ij}] = \int_V (\sigma_{ij} e_{ij}^u - \frac{1}{2} \sigma_{ij} C_{ijkl} \sigma_{kl} - b_i u_i) dV - \int_{S_t} \hat{t}_i u_i dS. \quad (6.6)$$

This is called the *Hellinger-Reissner functional*, abbreviated HR.² It is often stated in the literature as

$$\Pi_{\text{HR}}[u_i, \sigma_{ij}] = \int_V [-\mathcal{U}^*(\sigma_{ij}) + \sigma_{ij} \frac{1}{2} (u_{i,j} + u_{j,i}) - b_i u_i] dV - \int_{S_t} \hat{t}_i u_i dS, \quad (6.7)$$

in which

$$\mathcal{U}^*(\sigma_{ij}) = \frac{1}{2} \sigma_{ij} C_{ijkl} \sigma_{kl} = \frac{1}{2} \sigma_{ij} e_{ij}^\sigma, \quad (6.8)$$

is the *complementary energy density* in terms of the master stress field.

In FEM work the functional is usually written in the split form

$$\begin{aligned} \Pi_{\text{HR}} &= U_{\text{HR}} - W_{\text{HR}}, \quad \text{in which} \\ U_{\text{HR}} &= \int_V (\sigma_{ij} e_{ij}^u - \frac{1}{2} \sigma_{ij} C_{ijkl} \sigma_{kl}) dV, \quad W_{\text{HR}} = \int_V b_i u_i dV + \int_{S_t} \hat{t}_i u_i dS. \end{aligned} \quad (6.9)$$

The HR principle states that stationarity of the total variation

$$\delta \Pi_{\text{HR}} = 0 \quad (6.10)$$

provides the KE and EE strong links as Euler-Lagrange equations, whereas the FBC strong link appears as a natural boundary condition.

Remark 6.2. To verify the assertion about (6.5) being the first variation of Π_{HR} , note that

$$\delta(\sigma_{ij} e_{ij}^u) = e_{ij}^u \delta \sigma_{ij} + \sigma_{ij} \delta e_{ij}^u, \quad \delta(\frac{1}{2} \sigma_{ij} C_{ijkl} \sigma_{kl}) = C_{ijkl} \sigma_{kl} \delta \sigma_{ij} = e_{ij}^\sigma \delta \sigma_{ij}. \quad (6.11)$$

§6.2.4. Variational Indices and FEM Continuity Requirements

For a single-field functional, the *variational index* of its primary variable is the highest derivative m of that field that appears in the variational principle. The connection between variational index and required continuity in FEM shape functions was presented (as recipe) in the introductory FEM course (IFEM). That course considered only the single-field TPE functional, in which the primary variable, and only master, is the displacement field. It was stated that displacement shape functions

² The basic idea was contained in the work of Hellinger: E. Hellinger, Die allgemeine Ansätze der Mechanik der Kontinua, Encyklopædia der Mathematische Wissenschaften, Vol 4⁴, ed. by F. Klein and C. Müller, Teubner, Leipzig, 1914. As a proven theorem for the traction specified problem (no PBC) it was first given by Prange: G. Prange, Der Variations- und Minimalprinzip der Statik der Baukonstruktionen, Habilitationsschrift, Tech. Univ. Hanover, 1916. As a complete theorem containing both PBC and FBC it was given much later by Reissner: E. Reissner, On a variational theorem in elasticity, *J. Math. Phys.*, **29**, 90–95, 1950.

must be C^{m-1} continuous between elements and C^m inside. For the bar and plane stress problem covered in IFEM, $m = 1$, whereas for the Bernoulli-Euler beam $m = 2$.

In multifield functionals the variational index concept applies to *each varied field*. Thus there are as many variational indices as master fields. In the HR functional (6.9) of 3D elasticity, the variational index m_u of the displacements is 1, because first order derivatives appear in the slave strains e_{ij}^u . The variational index m_σ of the stresses is 0 because no stress derivatives appear. The required continuity of FEM shape functions for displacements and stresses is dictated by these indices. More precisely, if Π_{HR} is used as source functional for element derivation:

1. Displacement shape functions must be C^0 (continuous) between elements and C^1 inside (continuous and differentiable).
2. Stress shape functions can be C^{-1} (discontinuous) between elements, and C^0 (continuous) inside.

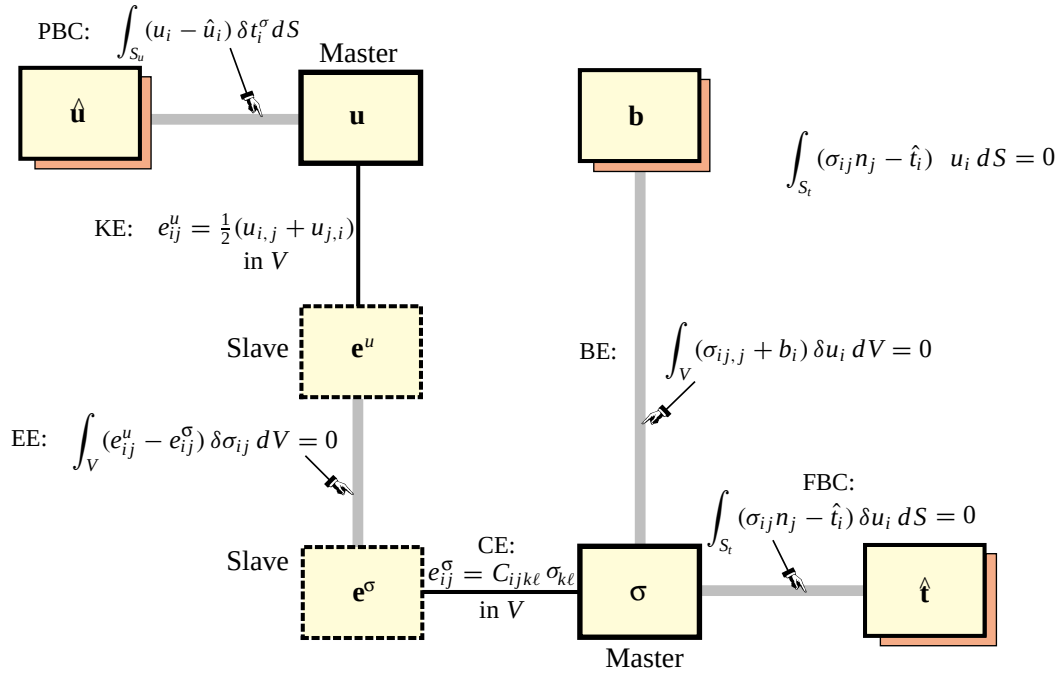


Figure 6.3. WF diagram for displacement-BC-generalized HR, in which PBC is weakened.

§6.2.5. Displacement-BC Generalized HR

If the PBC link (displacement BCs) between u_i and $\hat{u}_i$ is weakened as illustrated in Figure 6.3, the functional Π_{HR} generalizes to

$$\Pi_{\text{HR}}^g = \Pi_{\text{HR}} - \int_{S_u} \sigma_{ij} n_j (u_i - \hat{u}_i) dS = \Pi_{\text{HR}} - \int_{S_u} t_i^\sigma (u_i - \hat{u}_i) dS. \quad (6.12)$$

in which $\sigma_{ij} n_j = t_i^\sigma$ is the surface traction associated with the master stress field.

§6.3. Application Example 1: Tapered Bar Element

In this section the use of the HR functional to construct a very simple finite element is illustrated. Consider a tapered bar made up of isotropic elastic material, as depicted in Figure 6.4(a). The $x_1 \equiv x$ axis is placed along the longitudinal direction. The bar cross section area A varies linearly between the end node areas A_1 and A_2 . The element has length L and constant elastic modulus E . Body forces are ignored.

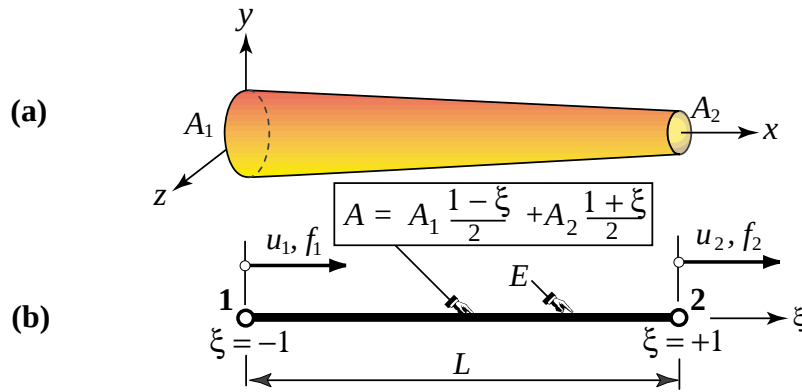


Figure 6.4. Two-node tapered bar element by HR: (a) shows the bar as a 3D object and (b) as a FEM model.

The reduction of the HR functional (6.6) to the bar case furnishes an instructive example of the derivation of a *structural* model based on stress resultants and Mechanics of Materials approximations.

In the theory of bars, the only nonzero stress is $\sigma_{11} \equiv \sigma_{xx}$, which will be denoted by σ for simplicity. The only internal force is the bar axial force $N = A\sigma_{xx}$. The only displacement component that participates in the functional is the axial displacement u_x , which is only a function of x and will be simply denoted by $u(x)$. The value of the axial displacement at end sections 1 and 2 is denoted by u_1 and u_2 , respectively. The axial strain is $e_{11} \equiv e_{xx}$, which will be denoted by e . The strong links are $e^u = du(x)/dx = u'$, where primes denote derivatives with respect to x , and $e^\sigma = \sigma/E = N/(EA)$. We call $N^u = EA e^u = EA u'$, etc.

As for as boundary conditions, for a free (unconnected) element S_t embodies the whole surface of the bar. But according to bar theory the lateral surface is traction free and thus drops off from the surface integral. That leaves the two end sections, at which uniform longitudinal surface tractions $\hat{t}_x$ are prescribed whereas the other component vanishes. On assuming a uniform traction distribution over the end cross sections, we find that the node forces are $f_1 = -t_{x1}A_1$ at section 1 and $f_2 = t_{x2}A_2$ at section 2. (The negative sign in the first one arises because at section 1 the external normal points along $-x$.)

Plugging these relations into the HR functional (6.6) and integrating over the cross section gives

$$\Pi_{\text{HR}}[u, N] = \int_L \left(Nu' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2. \quad (6.13)$$

This is an example of a functional written in term of *stress resultants* rather than actual stresses. The theory of beams, plates and shells leads also to this kind of functionals.

§6.3.1. Formulation of the Tapered Bar Element

We now proceed to construct the two-node bar element (e) depicted in Figure 6.3(b), from the functional (6.13). Define ξ is a natural coordinate that varies from $\xi = -1$ at node 1 to $\xi = 1$ at node 2. Assumptions must be made on the variation of displacements and axial forces. Displacements are taken to vary linearly whereas the axial force will be assumed to be constant over the element:

$$u(x) \approx u_1^{(e)} \frac{1-\xi}{2} + u_2^{(e)} \frac{1+\xi}{2}, \quad N(x) \approx \bar{N}^{(e)} \quad (6.14)$$

These assumptions comply with the C^0 and C^{-1} continuity requirements for displacements and stresses, respectively, stated in §6.2.4. Inserting (6.13) and (6.14) into the functional (6.12) and carrying out the necessary integral over the element length yields³

$$\Pi_{\text{HR}}^{(e)} = \frac{1}{2} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix}^T \begin{bmatrix} -\frac{\gamma L}{EA_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} - \begin{bmatrix} 0 \\ f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}^T \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} \quad (6.15)$$

in which

$$A_m = \frac{1}{2}(A_1 + A_2), \quad \gamma = \frac{A_m}{A_2 - A_1} \log \frac{A_2}{A_1}. \quad (6.16)$$

Note that if the element is prismatic, $A_1 = A_2 = A_m$, and $\gamma = 1$ (take the limit of the Taylor series for γ).

For this discrete form of $\Pi_{\text{HR}}^{(e)}$, the Euler-Lagrange equations are simply the stationarity conditions

$$\frac{\partial \Pi_{\text{HR}}^{(e)}}{\partial \bar{N}^{(e)}} = \frac{\partial \Pi_{\text{HR}}^{(e)}}{\partial u_1^{(e)}} = \frac{\partial \Pi_{\text{HR}}^{(e)}}{\partial u_2^{(e)}} = 0, \quad (6.17)$$

which supply the finite element equations

$$\begin{bmatrix} -\frac{\gamma L}{EA_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} = \begin{bmatrix} 0 \\ f_1^{(e)} \\ f_2^{(e)} \end{bmatrix} \quad (6.18)$$

This is an example of a *mixed finite element*, where the qualifier “mixed” implies that approximations are made in more than one unknown internal quantity; here axial forces and axial displacements.

Because the axial-force degree of freedom $\bar{N}^{(e)}$ is not continuous across elements (recall that C^{-1} continuity for stress variables is allowed), it may be eliminated or “condensed out” at the element level. The static condensation process studied in IFEM yields

$$\frac{EA_m}{\gamma L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} = \begin{bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}, \quad (6.19)$$

or

$$\boxed{\mathbf{K}^{(e)} \mathbf{u}^{(e)} = \mathbf{f}^{(e)}}. \quad (6.20)$$

These are the *element stiffness equations*, obtained here through the HR principle. Had these equations been derived through the TPE principle, one would have obtained a similar expression except that $\gamma = 1$ for any end-area ratio. Thus if the element is prismatic ($A_1 = A_2 = A_m$) the HR and TPE functionals lead to the same element stiffness equations.

³ Derivation details are worked out in an Exercise.

Table 6.2 Results for one-element analysis of fixed-free tapered bar

Area ratio	u_2 from HR	u_2 from TPE	Exact u_2
$A_1/A_2 = 1$	$PL/(EA_m)$	$PL/(EA_m)$	$PL/(EA_m)$
$A_1/A_2 = 2$	$1.0397PL/(EA_m)$	$PL/(EA_m)$	$1.0397PL/(EA_m)$
$A_1/A_2 = 5$	$1.2071PL/(EA_m)$	$PL/(EA_m)$	$1.2071PL/(EA_m)$

§6.3.2. Numerical Example

To give a simple numerical example, suppose that the bar of Figure 6.2 is fixed at end 1 whereas end 2 is under a given axial force P . Results for sample end area ratios are given in Table 6.2. It can be seen that the HR formulation yields the exact displacement solution for all area ratios.

Also note that the discrepancy of the one-element TPE solution from the exact one grows as the area ratio deviates from one. The TPE elements underestimate the actual deflections, and are therefore on the stiff side. To improve the TPE results we need to divide the bar into more elements.

§6.3.3. The Bar Flexibility

From (6.19) we immediately obtain

$$u_2 - u_1 = \frac{\gamma L}{EA_m}(f_2 - f_1) = F(f_2 - f_1) \quad (6.21)$$

This called a *flexibility equation*. The number $F = \gamma L/(EA_m)$ is the flexibility coefficient or *influence coefficient*. For more complicated elements we would obtain a *flexibility matrix*. Relations such as (6.21) were commonly worked out in older books in matrix structural analysis. The reason is that flexibility equations are closely connected to classical static experiments in which a force is applied, and a displacement or elongation measured.

§6.4. Application Example 2: A Curved Cable Element

§6.4.1. Connector Elements

The HR functional is useful for deriving a class of elements known as *connector elements*.⁴ The concept is illustrated in Figure 6.5(a). The *connector nodes* are those through which the element links to other elements through the node displacements. These displacements are the *connector degrees of freedom*, or simple the *connectors*. The box models the intrinsic response of the element; if it is best described in terms of response to forces or stresses, as depicted in Figures 6.5(b,c), it is called a *flexibility box* or F-box.

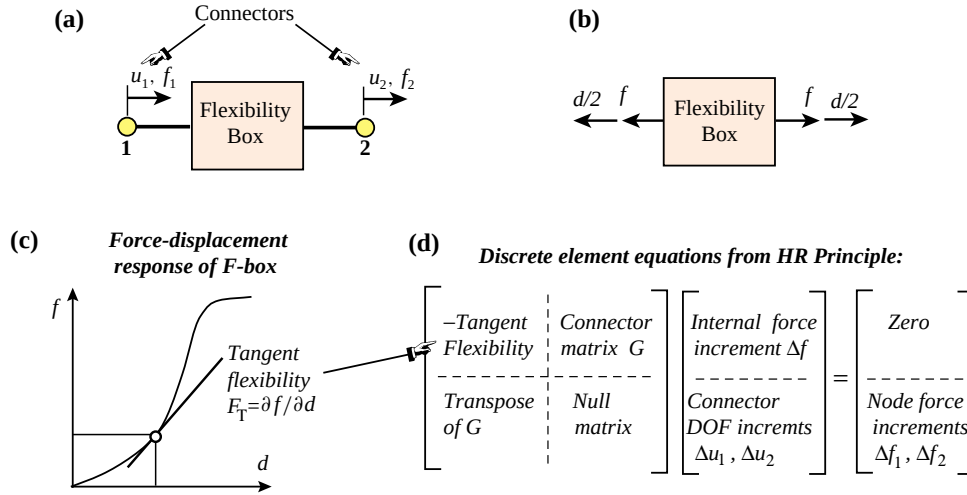


Figure 6.5. A connector element (sketch) developed with the help of the HR principle.

In many applications the box response is nonlinear. Examples are elements modelling contact, friction and joints. If this is the only place where nonlinear behavior occur, the flexibility element acts as a device to isolate local nonlinearities. This is an effective way to reuse linear FEM programs. Consider for simplicity a one-dimensional, 2 node flexibility element as the one sketched in Figure 6.5. The connector nodes are 1 and 2. The connector DOF are the axial displacements u_1 and u_2 . The relative displacement is $\Delta = u_2 - u_1$. The kernel behavior is described by the response to an axial force f , as pictured in Figure 6.5(c):

$$d = F(f). \quad (6.22)$$

The tangent flexibility is

$$F_T = \frac{\partial d}{\partial f} = \frac{\partial F(f)}{\partial f}. \quad (6.23)$$

Application of the HR principle leads to the tangent equation

$$\begin{bmatrix} -F_T & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta f \\ \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \quad (6.24)$$

⁴ Hybrid elements, covered in Sections 8ff, are also useful in this regard. Often the two approaches lead to identical results.

where Δ denote increments.⁵ Condensation of Δf as internal freedom gives the stiffness matrix

$$\begin{bmatrix} K_T & -K_T \\ -K_T & K_T \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix} \quad (6.25)$$

This result could also been obtained directly from physics, or from the displacement formulation. However, the HR approach remains unchanged when passing to 2 and 3 dimensions.

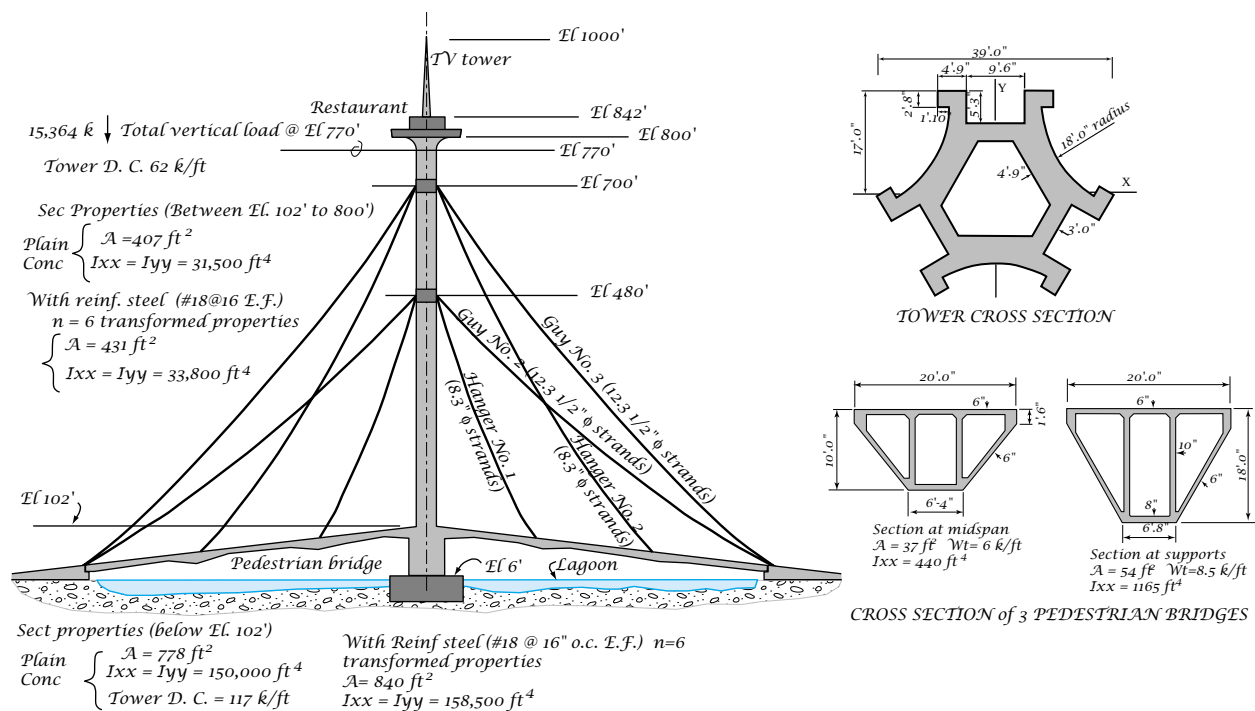


Figure 6.6. 1000-ft guyed tower studied in 1967 for the South Florida coast.

§6.4.2. A Curved Cable Element

As an application consider the development of a curved cable element used to model the guy and hanger members of the tower structure shown in Figure 6.6(a).⁶

Figure 6.7(a) shows a two-dimensional FEM model, with 62 nodes and 3 freedoms per node.⁷ To cut down the number of elements along the cable members, a curved cable element, pictured in

⁵ The first entry of the right hand side has been set to zero for simplicity. Generally it is not.

⁶ A 1000-ft guyed tower proposed for the South Florida coast by a group of rich Cuban expatriates and dubbed the "Tower of Freedom" as it was supposed to serve as a guide beacon for boats escaping Cuba with refugees. The preliminary design of Figure 6.6 was made by a well known structural engineering company and dated June 1967. Ray W. Clough and Joseph Penzien were consultants for the verification against hurricane winds. Analyzed using an ad-hoc 2D FEM code by Mike Shears and the writer, who was then a post-doc at UC Berkeley, July–September 1967. The project was canceled as too costly and plans for a 3D cable analysis code shelved.

⁷ The structure has 120° circular symmetry. Reduced to one plane of symmetry (plane of the paper) by appropriate projections of the right-side (windward) portion.

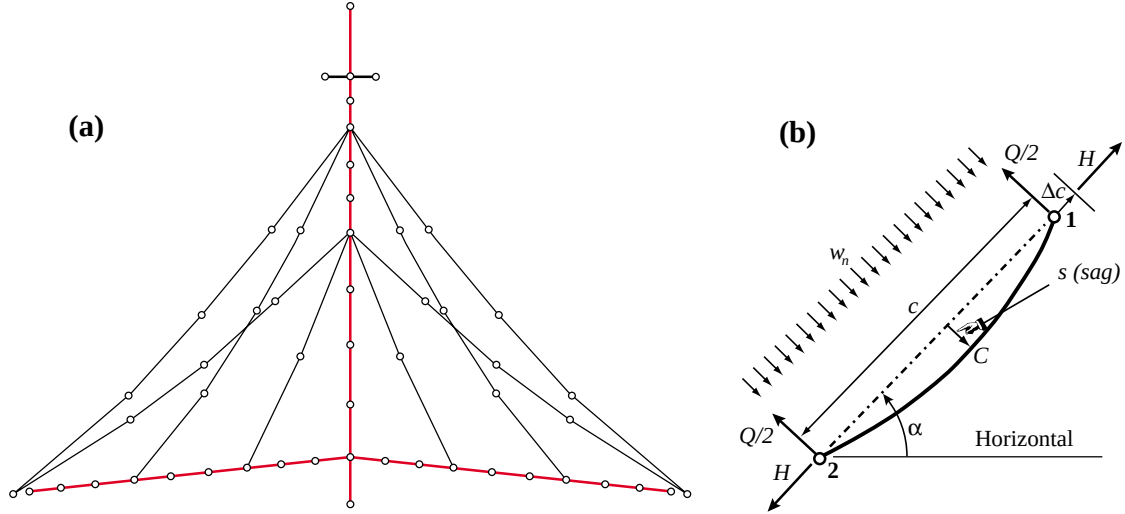


Figure 6.7. (a): 2D FEM model of guyed tower of Figure 6.6 for vibration and dynamic analysis under hurricane wind loads (1967); (b): curved cable element developed to model the guy and hanger cables with few elements along the length.

Figure 6.7(b) was constructed. The method that follows illustrates the application of the flexibility approach to connector elements.

The element has two nodes, 1 and 2. The distance 1–2 is the chord distance c . The actual length of the strained cable element is L , so $L \geq c$. The force H along the chord is called the *thrust*. H and the chord change Δc play the role of f and d , respectively, in the flexibility response sketched in Figure 6.5. The cable is subjected to a uniform transverse load w_n specified per unit of chord length. (The load is usually a combination of self-weight and wind.) The elastic rigidity of the cable is EA_0 , where E is the apparent elastic modulus (which depends on the fabrication of the cable) and A_0 the original structural area.

The following simplifying assumptions are made at the element level:

1. The sag is small compared to chord length: $s < c/10$, which characterizes a taut element.⁸
2. The load w_n is uniform. As a consequence, the transverse reaction loads at nodes are $Q/2$, with $Q = w_n c$. See Figure chapdot7(b).
3. $Q = w_n c$ is fixed even if c changes. This is exact for self weight, and approximately verified for wind loads.
4. The effect of tangential loads (along the chord) on the element deformation is neglected.
5. Hooke's law applies in the form $L - L_0 = H/(EA_0)$, where L_0 is the unstrained length of the element.

Under the foregoing assumptions, the cable deflection profile is parabolic, and we get

$$s = \frac{Qc}{8H}, \quad L = L_0 \left(1 + \frac{H}{EA_0} \right) = c + \frac{8s^2}{3c^2}, \quad \frac{c}{L_0} = \frac{1 + \frac{H}{EA_0}}{1 + \frac{Q^2}{24H^2}}. \quad (6.26)$$

⁸ If this property is not realized, the cable member should be divided into more elements. Dividing one element into two cuts c and s approximately by 2 and 4, respectively, so s/c is roughly halved.

The first equation comes from moment equilibrium at the sagged element midpoint C , the second from the shallow parabola-arclength formula, and the third one from eliminating the sag s between the first two. Differentiation gives the tangent flexibility

$$F_T = \frac{\partial c}{\partial H} = \frac{L_0}{EA_0} \frac{1}{1 + \frac{Q^2}{24H^2}} + \frac{\frac{Q^2 L_0}{12H^3} \left(1 + \frac{H}{EA_0}\right)}{\left(1 + \frac{Q^2}{24H^2}\right)^2} \quad (6.27)$$

For most structural cables, $H \ll EA_0$ and $(Q/H)^2 \ll 1$. Accordingly the above formula simplifies to

$$F_T = \frac{L_0}{EA_0} + \frac{Q^2 L_0}{12H^3}, \quad (6.28)$$

which was used in the 1967 dynamic analysis at Berkeley. If $Q \rightarrow 0$ or $H \rightarrow \infty$, (6.28) reduces to the flexibility $L_0/(EA_0)$ of a linear bar element, as can be expected. Replacing into (6.24) and condensing out ΔH gives the tangent local stiffness matrix of the cable element as (6.25), where u_1 and u_2 are axial displacements at nodes 1 and 2 along the chord, and $K_T = 1/F_T$. This matrix relation can be transformed to the global coordinate system in the usual manner.

Homework Exercises for Chapter 6
The HR Variational Principle of Elastostatics

EXERCISE 6.1 [A:10] Derive the Euler-Lagrange equations and natural BCs of Π_{HR} given in (6.-3).

EXERCISE 6.2 [A:20] As shown in Table 6.1, the tapered bar HR model derived in §6.3 gives the exact end displacement with one element. But the assumed linear-displacement variation does not agree with the displacement of the exact solution, which is nonlinear in x if $A_1 \neq A_2$. Explain this contradiction. *Hint*: a variational freak; integrate the Nu' term in (6.13) by parts and use the fact that the exact solution has constant N .

EXERCISE 6.3 [A:25] Construct the HR functional for the cable element treated in §6.4.2. *Hint*: construct the complementary energy function of the F-box.

EXERCISE 6.4 [A:15] Show that if the master σ_{ij} is replaced by the slave $\sigma_{ij}^u = E_{ijkl}e_{ij}^u$ with $e_{ij}^u = (u_{i,j} + u_{j,i})/2$, the HR functional reduces to the TPE functional.

EXERCISE 6.5 [A:25] Derive the Total Complementary Potential Energy (TCPE) functional of linear elastostatics

$$\Pi_{\text{TCPE}}[\sigma_{ij}] = -\frac{1}{2} \int_V \sigma_{ij} C_{ijkl} \sigma_{kl} dV + \int_{S_u} \sigma_{ij} n_j \hat{u}_i dS, \quad (\text{E6.1})$$

by choosing stresses as the only primary field. Choose BE, FBC, and the right-to-left constitutive equations [the strain-stress relations $e_{ij}^\sigma = C_{ijkl} \sigma_{kl}$] as strong connections whereas KE and PBC are weak connections. Notice that the displacement field u_i does not appear in this functional; only the prescribed displacements $\hat{u}_i$.

EXERCISE 6.6 [A:30] Derive the anonymous functional $\Pi_S[e_{ij}]$ based on strains as only primary field. Take all connections as strong except the constitutive equations. Note: this functional seems to appear in only one book,⁹ and therein only as a curiosity. It is weird looking because of its abnormal simplicity:

$$\Pi_S[e_{ij}] = \int_V (\sigma_{ij} - \frac{1}{2} E_{ijkl} e_{kl}) e_{ij} dV. \quad (\text{E6.2})$$

That's right, no boundary terms, and σ_{ij} is here a *data* field!

⁹ J. T. Oden and J. N. Reddy, *Variational Methods in Theoretical Mechanics*, Springer-Verlag, Berlin (1982).

Homework Exercises for Chapter 6 Solutions

EXERCISE 6.1

Not assigned.

EXERCISE 6.2

Integrate the Nu' term by parts:

$$\Pi_{HR} = \int_0^L \left(Nu' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2 = - \int_0^L \frac{N^2}{2EA} dx - \int_0^L u N' dx + Nu|_0^L - f_1 u_1 - f_2 u_2 \quad (\text{E6.3})$$

The exact solution is constant N , which is assumed in the element. Thus $N' = 0$ and the functional reduces to

$$\Pi_{HR} = - \int_0^L \frac{N^2}{2EA} dx + N(u_2 - u_1) - f_1 u_1 - f_2 u_2. \quad (\text{E6.4})$$

The internal $u(x)$ has disappeared, and the axial displacements only come in through their end values u_1 and u_2 . Therefore, it does not matter what is taken for the displacement field inside the element as long as a constant N is assumed. This is a variational freak since it applies only to that specific example problem.

EXERCISE 6.3

Not assigned.

EXERCISE 6.4 The weak links are (cf. Figure E6.1):

$$e_{ij}^u - e_{ij}^\sigma = 0 \quad \text{in } V, \quad \hat{u}_i - u_i = 0 \quad \text{on } S_u, \quad (\text{E6.5})$$

where $e_{ij}^u = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $e_{ij}^\sigma = C_{ijkl}\sigma_{kl}$. Multiply the residuals (E6.5) by $\delta\sigma_{ij}$ and $\delta t_i = \delta\sigma_{ij}n_j$ and integrate over V and S_u , respectively:

$$\int_V (e_{ij}^u - e_{ij}^\sigma) \delta\sigma_{ij} dV + \int_{S_u} (\hat{u}_i - u_i) \delta\sigma_{ij} n_j dS = 0. \quad (\text{E6.6})$$

Apply the divergence theorem to the first term on the left:

$$\int_V e_{ij}^u \delta\sigma_{ij} dV = \int_V \frac{1}{2}(u_{i,j} + u_{j,i}) \delta\sigma_{ij} dV = - \int_V u_i \delta\sigma_{ij,j} dV + \int_S u_i \delta\sigma_{ij} n_j dS, \quad (\text{E6.7})$$

in which the indicated term vanishes because the BE are strongly satisfied: $\delta(\sigma_{ij,j} + b_i) = \delta\sigma_{ij,j} = 0$ in V . Replacing into (E6.6) gives

$$\delta\Pi_{\text{TCPE}} = \int_V -C_{ijkl}\sigma_{kl} \delta\sigma_{ij} dV + \int_S u_i \delta\sigma_{ij} n_j dS + \int_{S_u} (\hat{u}_i - u_i) \delta\sigma_{ij} n_j dS = 0. \quad (\text{E6.8})$$

Split the S integral over S_t and S_u . Over S_t we can set $\delta\sigma_{ij}n_j = \delta(\sigma_{ij}n_j) = \delta\hat{t}_i = 0$ because the FBCs are strongly satisfied. The integrals of $u_i \delta\sigma_{ij} n_j$ over S_u cancel out and we are left with

$$\delta\Pi_{\text{TCPE}} = - \int_V C_{ijkl}\sigma_{kl} \delta\sigma_{ij} dV + \int_{S_u} \hat{u}_i \delta\sigma_{ij} n_j dS = 0. \quad (\text{E6.9})$$

This is the exact first variation of the complementary energy functional (E6.3).

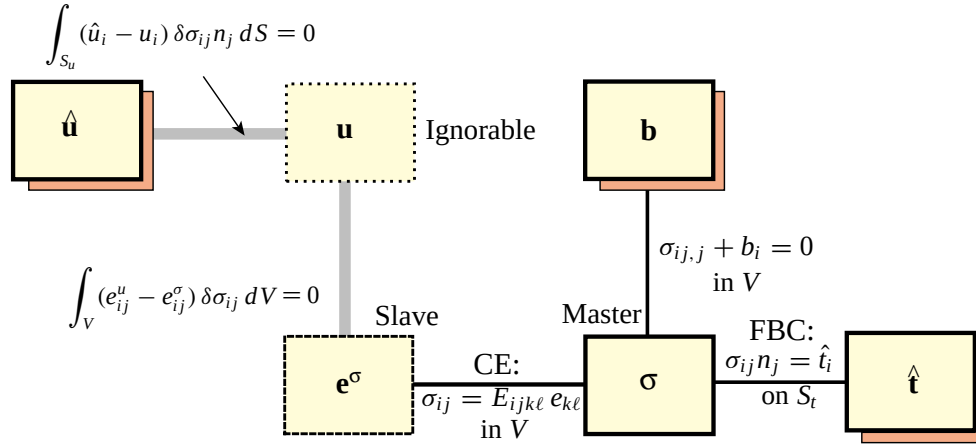


Figure E6.1. Departure Weak Form to derive the TCPE functional.

EXERCISE 6.5 The only weak connection is $\sigma_{ij} = E_{ijkl} e_{kl}$. We begin as above, trying

$$\delta \Pi_S = \int_V (\sigma_{ij} - E_{ijkl} e_{kl}) \delta e_{ij} dV, \quad (\text{E6.10})$$

where σ_{ij} must be viewed as a data field.¹⁰ This is the exact variation of

$$\Pi_S(e_{ij}) = \int_V (\sigma_{ij} - \frac{1}{2} E_{ijkl} e_{kl}) e_{ij} dV. \quad (\text{E6.11})$$

And this is the end. No further progress can be made. This is the only canonical functional of elasticity that contains no boundary integrals.

In the Exercise statement it was noted that functional Π_S has limited practical value. The reason is that all of the difficult field equations are taken as strong. The stress field must satisfy both equilibrium equations and stress BC point by point *a priori*, while the strain field must be compatible with a displacement field that satisfies the displacement BC. The only relaxation of the governing equations pertains to the constitutive equations.

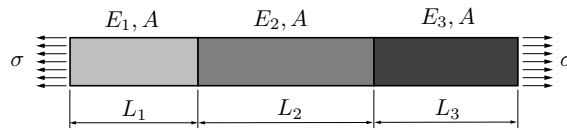


Figure E6.2. Application of the strain-only canonical functional to material homogenization.

Nevertheless the principle may be occasionally useful in material homogenization, as the following simple example illustrates. Consider a bar of uniform cross section A and total length $L = L_1 + L_2 + L_3$ made up of three materials with elastic moduli E_1 , E_2 and E_3 , respectively. Using the functional Π_S , find an average modulus E .

¹⁰ Why? Because the stresses must satisfy the volume equilibrium equations and the surface traction BC *a priori*. Thus the stress field must be known at every point in V .

To carry out the homogenization process, assume that the bar is under a constant axial stress field $\sigma = P/A$ (see Figure E6.2), which obviously satisfies all stress equilibrium equations and surface traction boundary conditions. The average strain $e = E^{-1}\sigma$ is taken as the only unknown to be varied in Π_S :

$$\Pi_S(e) = AL\sigma e - \frac{1}{2}A(E_1L_1 + E_2L_2 + E_3L_3)e^2. \quad (\text{E6.12})$$

The condition $\delta\Pi_S = 0$ gives $\partial\Pi_S/\partial e = 0$, from which

$$e = \frac{L}{E_1L_1 + E_2L_2 + E_3L_3}\sigma, \quad E = \frac{\sigma}{e} = \frac{E_1L_1 + E_2L_2 + E_3L_3}{L}. \quad (\text{E6.13})$$

This technique essentially amounts to equating the strain energies absorbed by the actual and homogenized (fictitious) bars. Note that the displacement field does not appear in this statically determinate problem.

7

The Three-Field Mixed Principle of Elastostatics

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§7.1. Introduction

This Chapter concludes the presentation of canonical variational principles of elastostatics by constructing the Veubeke-Hu-Washizu (VHW) principle. This is used for the development of a special beam element.

§7.2. The Veubeke-Hu-Washizu Principle

The Veubeke-Hu-Washizu (VHW) principle is the canonical principle of elasticity that allows simultaneous variation of displacements, strains and stresses.¹ The VHW principle is the most general *canonical* principle of elasticity. Contrary to what the literature states, however, this is not the most general variational principle. Within the framework of parametrized variational principles² the VHW principle appears as an instance.

§7.2.1. The Variational Statement

We derive here a slightly generalized version of the VHW principle, in which the displacement boundary condition link (the PBC link) is weakened. This functional will be identified as Π_{VHW}^g . The Weak Form used as departure point is shown in Figure 7.1.

Because we have picked three masters, in principle we will have three strain fields, one master: e_{ij} , and two slaves: e_{ij}^u and e_{ij}^σ . Similarly there are three stress fields, one master: σ_{ij} , and two slaves: σ_{ij}^u and σ_{ij}^e . The boxes of $\sigma_{ij}^u = E_{ijkl} e_{kl}^u$ and $e_{ij}^\sigma = C_{ijkl} \sigma_{kl}$ are not shown, however, in Figure 7.1 because those slave fields do not appear in the derivation below. There are five weak connections.³

To streamline the derivation we skip the preparatory steps in writing down residuals and Lagrange multiplier fields, and proceed directly to the variational statement in which weak connection residuals are work paired with appropriate variations of the master fields:

$$\begin{aligned} \delta \Pi_{\text{VHW}}^g = & \int_V (e_{ij}^u - e_{ij}) \delta \sigma_{ij} dV + \int_V (\sigma_{ij}^e - \sigma_{ij}) \delta e_{ij} dV - \int_V (\sigma_{ij,j} + b_i) \delta u_i dV \\ & + \int_{S_t} (\sigma_{ij} n_j - \hat{t}_i) \delta u_i dS - \int_{S_u} (u_i - \hat{u}_i) n_j \delta \sigma_{ij} dS. \end{aligned} \quad (7.1)$$

Treat $\sigma_{ij,j} \delta u_i$ with the divergence theorem to get rid of stress derivatives:

$$\begin{aligned} - \int_V \sigma_{ij,j} \delta u_i &= \int_V \sigma_{ij} \delta e_{ij}^u - \int_S \sigma_{ij} n_j \delta u_i dS \\ &= \int_V \sigma_{ij} \delta e_{ij}^u - \int_{S_u} \sigma_{ij} n_j \delta u_i dS - \int_{S_t} \sigma_{ij} n_j \delta u_i dS. \end{aligned} \quad (7.2)$$

¹ The VHW functional was published simultaneously in 1955 by H. Hu, "On some variational principles in the theory of elasticity and the theory of plasticity," *Sci. Sinica* (Peking) **4**, pp. 33–54, 1955, and K. Washizu, "On the variational principles of elasticity and plasticity," Rept 25-18, Massachusetts Institute of Technology, March 1955. However, four years earlier B. M. Fraeijs de Veubeke had published a version of the principle in 1951 that was overlooked: B. M. Fraeijs de Veubeke, Diffusion des inconnues hyperstatiques dans les voilures à longeron couplés, *Bull. Serv. Technique de L'Aéronautique* No. 24, Imprimerie Marcel Hayez, Bruxelles, 56pp., 1951.

² C. A. Felippa, A survey of parametrized variational principles and applications to computational mechanics, *Comp. Meths. Appl. Mech. Engrg.*, **113**, 109–139, 1994.

³ Other weak connections combinations between strain and stress boxes may be taken, leading to the same result.

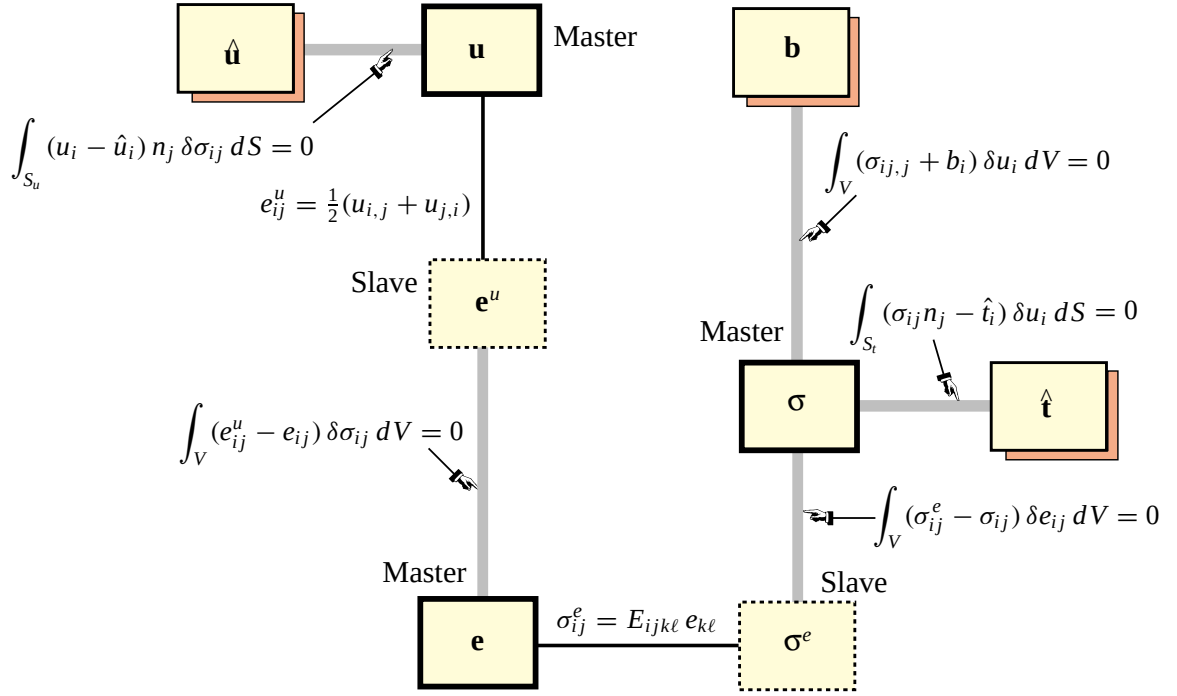


Figure 7.1. The Weak Form for derivation of the generalized VHW principle. Preliminary steps are skipped: the diagram shows the appropriate work pairings. The standard form of the principle is obtained if the PBC link is strong.

Substitute (7.2) into (7.1) and collect terms:

$$\begin{aligned} \delta \Pi_{\text{VHW}}^g = & \int_V \left[(e_{ij}^u - e_{ij}) \delta \sigma_{ij} + (\sigma_{ij}^e - \sigma_{ij}) \delta e_{ij} + \sigma_{ij} \delta e_{ij}^u - b_i \delta u_i \right] dV \\ & - \int_{S_t} \hat{t}_i \delta u_i dS - \int_{S_u} \left[(u_i - \hat{u}_i) n_j \delta \sigma_{ij} + \sigma_{ij} n_j \delta u_i \right] dS. \end{aligned} \quad (7.3)$$

§7.2.2. The Variational Form

Equation (7.3) can be recognized as the first variation of the functional

$$\begin{aligned} \Pi_{\text{VHW}}^g[u_i, \sigma_{ij}, e_{ij}] = & \int_V \left[\sigma_{ij} (e_{ij}^u - e_{ij}) + \mathcal{U}(e_{ij}) - b_i u_i \right] dV - \int_{S_t} \hat{t}_i u_i dS \\ & - \int_{S_u} (u_i - \hat{u}_i) \sigma_{ij} n_j dS. \end{aligned} \quad (7.4)$$

This is called here “generalized VHW.” In this form

$$\mathcal{U}(e_{ij}) = \frac{1}{2} e_{ij} E_{ijkl} e_{kl} = \frac{1}{2} \sigma_{ij}^e e_{ij}, \quad (7.5)$$

is the strain energy density in terms of the master (varied) strains. The VHW principle asserts that

$$\delta \Pi_{\text{VHW}}^g = 0 \quad (7.6)$$

in which the variation is taken simultaneously with respect to displacements, strains and stresses, yields all field equations of elasticity as its Euler-Lagrange equations, and all boundary conditions (displacements and tractions) as its natural boundary conditions.⁴

From Π_{VHW}^g we may derive other forms that also arise in the applications. For example, if one enforces *a priori* the displacement BCs $u_i = \hat{u}_i$ as a strong link, the integral over S_u drops out and (7.4) reduces to the standard form of the functional:

$$\Pi_{\text{VHW}}[u_i, \sigma_{ij}, e_{ij}] = \int_V [\sigma_{ij}(e_{ij}^u - e_{ij}) + \mathcal{U}(e_{ij}) - b_i u_i] dV - \int_{S_t} \hat{t}_i u_i dS. \quad (7.7)$$

In FEM work this functional, as was the case with TPE and HR, is often written in the “internal plus external” split form

$$\begin{aligned} \Pi_{\text{VHW}} &= U_{\text{VHW}} - W_{\text{VHW}}, \quad \text{in which} \\ U_{\text{VHW}} &= \int_V [\sigma_{ij}(e_{ij}^u - e_{ij}) + \mathcal{U}(e_{ij})] dV, \quad W_{\text{VHW}} = \int_V b_i u_i dV + \int_{S_t} \hat{t}_i u_i dS. \end{aligned} \quad (7.8)$$

Other reductions are the subject of Exercises.

Additional forms of the functionals (7.5) and (7.7) may be constructed through integration by parts of the $\sigma_{ij} e_{ij}^u dV$ term to get rid of displacement derivatives at the cost of introducing stress derivatives. This can be done using

$$\int_V \sigma_{ij} e_{ij}^u dV = - \int_V u_i \sigma_{ij,j} dV + \int_S u_i \sigma_{ij} n_j dS. \quad (7.9)$$

This is the subject of an Exercise.

§7.2.3. Continuity Requirements

Inspection of the functionals (7.5) and (7.7) shows that the variational index of the displacement field is $m_u = 1$ because first order displacement derivatives appear in the slave strain field e_{ij}^u . The variational indices m_σ and m_e of stresses and strains are zero because no derivatives of these two master fields appear.

From this characterization, it follows that when the VHW principle in the form (7.5) or (7.7) is used to derive finite elements, the assumed displacements should be C^0 interelement continuous, whereas assumed stresses and strains can be discontinuous between elements.

⁴ The generalized form (7.5) is actually that derived by Fraeijns de Veubeke in the cited 1951 reference. Hu and Washizu only derived the more restricted form (7.7).

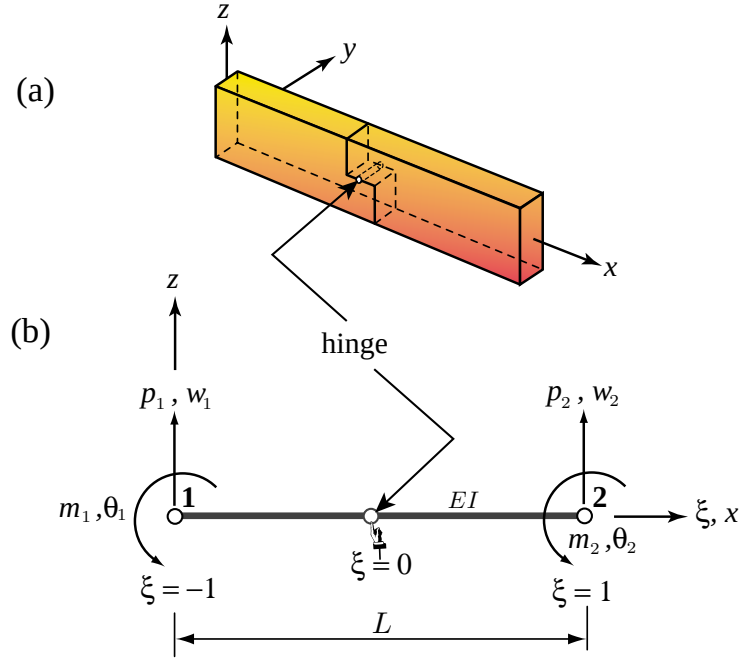


Figure 7.2. Hinged plane beam discretization by the VHW principle: (a) hinged beam, (b) two-node finite element model.

§7.3. VHW Application: A Hinged Plane Beam Element

§7.3.1. Element Description

The use of the VHW functional to derive a specialized beam element is illustrated next. Consider a two-node prismatic plane beam element of span L with a hinge at its midsection as depicted in Figure 7.2(a). The beam bends in the xz plane. The Euler-Bernoulli (BE) beam model of Chapter 4 is used. The beam is fabricated of isotropic elastic material of elastic modulus E . The second moment of inertia with respect to the neutral axis y is I . The bending moment at the hinge section is taken to be zero.

The beam is referred to a Cartesian coordinate system (x, y, z) with axis x placed along the longitudinal axis of the beam and z along the plane beam transverse direction. Note that for the beam to be plane, the cross section must be symmetric with respect to the z axis, while all applied forces must act on the xz plane.

As discussed in Chapter 4, the field variables that appear in BE plane beam theory are: the internal bending moment $M = M(x)$, the cross-section transverse displacement $w = w(x)$, the cross-section rotation $\theta = \theta(x) = dw/dx = w'$ and the curvature $\kappa = \kappa(x) = d^2w/dx^2 = w''$.

§7.3.2. Element Formulation

A two-node BE beam element with a midsection hinge is depicted in Figure 7.2(b). The four degrees of freedom are the transverse node displacements w_1 and w_2 , and the about- y end rotations θ_1 and θ_2 (positive counterclockwise when viewed from the $-y$ direction) at the two end nodes.

The associated node forces are f_1, f_2, m_1, m_2 . The natural coordinate $\xi = (2x - 1)/L$ takes the values $-1, 1$ and 0 at the end nodes and at the hinge location, respectively.

Assuming zero body forces, the VHW functional for this beam model reduces to

$$\Pi_{\text{VHW}}[w, M, \kappa] = \int_L \left[M(\kappa^w - \kappa) + \frac{1}{2} E I \kappa^2 \right] dx - f_1 w_1 - f_2 w_2 - m_1 \theta_1 - m_2 \theta_2, \quad (7.10)$$

where M and κ are the assumed bending-moment and curvature functions, respectively, $\kappa^w = w''$ is the curvature derived from the assumed transverse displacement, and other quantities are defined in Figure 7.5.

Inspection of (7.10) shows that the variational indices for M, κ and w are $0, 0$, and 2 , respectively.⁵ It follows that the continuity requirements for these functions are C^{-1}, C^{-1} and C^1 , respectively. Consequently M and κ may be discontinuous between elements.

The assumed variation of the bending moment and curvature is linear:

$$M = \bar{M}^{(e)} \xi, \quad \kappa = \bar{\kappa}^{(e)} \xi, \quad (7.11)$$

both of which satisfy the hinge condition $M = \kappa = 0$ at $\xi = 0$. As for the transverse displacement we shall take the usual cubic Hermite interpolation for BE beam elements:

$$w = \frac{1}{4}(1 - \xi)^2(2 + \xi) w_1^{(e)} + \frac{1}{8}L(1 - \xi)^2(1 + \xi) \theta_1^{(e)} + \frac{1}{4}(1 + \xi)^2(2 - \xi) w_2^{(e)} - \frac{1}{8}L(1 + \xi)^2(1 - \xi) \theta_2^{(e)}. \quad (7.12)$$

The displacement-derived curvature is obtained by differentiating (7.12) twice with respect to x :

$$\kappa^w = w'' = \left[(6\xi/L) w_1^{(e)} + (3\xi - 1) \theta_1^{(e)} - (6\xi/L) w_2^{(e)} + (3\xi + 1) \theta_2^{(e)} \right] / L \quad (7.13)$$

Inserting (7.12)-(7.13) into (7.10), integrating over the length to evaluate the internal energy, and equating to zero the partials of the resulting expression of Π_{VHW} with respect to the element degrees of freedom $\bar{M}^{(e)}, \bar{\kappa}^{(e)}, w_1^{(e)}, \theta_1^{(e)}, w_2^{(e)}$ and $\theta_2^{(e)}$, we obtain the following finite element equations:

$$\begin{bmatrix} \frac{1}{3} E I L & -\frac{1}{3} L & 0 & 0 & 0 & 0 \\ -\frac{1}{3} L & 0 & \frac{2}{L} & 1 & -\frac{2}{L} & \frac{1}{2} \\ 0 & \frac{2}{L} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{L} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\kappa}^{(e)} \\ \bar{M}^{(e)} \\ w_1^{(e)} \\ \theta_1^{(e)} \\ w_2^{(e)} \\ \theta_2^{(e)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f_1^{(e)} \\ m_1^{(e)} \\ f_2^{(e)} \\ m_2^{(e)} \end{bmatrix}. \quad (7.14)$$

⁵ The displacement variational index has increased from 1 in (7.28) to 2 because of the introduction of the BE beam theory assumptions.


```

ClearAll[EI,L,n,m]; n=1; m=1;
κW= 6*ξ/L^2*w1+(3*ξ-1)/L*θ1+(-6*ξ/L^2*w2)+(3*ξ+1)/L*θ2;
κ= κ0*ξ^m; M=M0*ξ^n; W=f1*w1+m1*θ1+f2*w2+m2*θ2;
Π=(L/2)*Integrate[M*(κW-κ)+EI*κ^2/2,{ξ,-1,1}]-W;
Π=Simplify[Π]; Print["VHW functional Π=", Π];
r={D[Π,κ0],D[Π,M0],D[Π,w1],D[Π,θ1],D[Π,w2],D[Π,θ2]};
K={D[r,κ0],D[r,M0],D[r,w1],D[r,θ1],D[r,w2],D[r,θ2]};
Print["Full Ke=",K//MatrixForm];
K11=Table[K[[i,j]],{i,1,2},{j,1,2}];
K12=Table[K[[i,j]],{i,1,2},{j,3,6}];
K22=Table[K[[i,j]],{i,3,6},{j,3,6}];
Ke = Simplify[K22-Transpose[K12].Inverse[K11].K12];
Print["Condensed Ke=",Ke//MatrixForm];
Print["Eigenvalues of Cond Ke=",Eigenvalues[Ke]];

```

Figure 7.3. *Mathematica* script to derive the stiffness equations of the hinged plane beam element. Exponents m and n in expressions of κ and M are for Exercises.

§7.3.3. The Stiffness Equations

As previously explained both $\bar{\kappa}$ and $\bar{M}$ need not be continuous between elements. Static condensation of these two freedoms yields the following element stiffness equations in terms of the node displacements:

$$\frac{3EI}{L^3} \begin{bmatrix} 4 & 2L & -4 & 2L \\ 2L & L^2 & -2L & L^2 \\ -4 & -2L & 4 & -2L \\ 2L & L^2 & -2L & L^2 \end{bmatrix} \begin{bmatrix} w_1^{(e)} \\ \theta_1^{(e)} \\ w_2^{(e)} \\ \theta_2^{(e)} \end{bmatrix} = \begin{bmatrix} f_1^{(e)} \\ m_1^{(e)} \\ f_2^{(e)} \\ m_2^{(e)} \end{bmatrix}. \quad (7.15)$$

in which the generic element identifier has been inserted; or in compact form

$$\mathbf{K}^{(e)} \mathbf{u}^{(e)} = \mathbf{f}^{(e)}. \quad (7.16)$$

It can be verified that this 4×4 element stiffness matrix has only rank 1. The element has the two usual rigid-body modes of a plane beam element (rigid translation along z and rigid rotation about y), plus one zero-energy mode caused by the presence of the hinge.

The foregoing computations were done with the *Mathematica* script shown in Figure 7.3.

Homework Exercises for Chapter 7
The Three-Field Mixed Principle of Elastostatics

EXERCISE 7.1 [A:20] Explain how to reduce the VWH principle (7.7) to HR.

EXERCISE 7.2 [A:15] Explain how to reduce the VWH principle (7.7) to TPE. (This is easier than the previous one)

EXERCISE 7.3 [A:20] Use (7.8) to transform (7.7) keeping PBC strong. Show the form of the variational principle. (This transformation was the subject of a recent Ph. D. thesis in CVEN; 3 years of work for an exercise in variational calculus).

EXERCISE 7.4 [A/C:15] Can the hinged beam element be derived directly from HR? Does it give the same answers for the stiffness matrix?

EXERCISE 7.5 [A/C:15] Derive a “shearless” plane beam element by forcing the master moment M to be constant over the element. Compute its 4×4 condensed stiffness matrix $\mathbf{K}^{(e)}$. *Hint:* if you know how to use *Mathematica* use the script of Figure 7.3, setting $m=n=0$.

EXERCISE 7.6 [A:15] How would pickmoment and curvature distributions for deriving a *two-hinge* beam element using VHW? (Do not try to derive the element, it will blow up.)

EXERCISE 7.7 [A:15] The conventional plane beam element of IFEM can be obtained by setting $\kappa = \kappa''$ in VHW. Why?

EXERCISE 7.8 [A:25] Transform (7.12) by parts to reduce the variational index of the displacement w to one while raising that of the moment M to one. Which kind of beam element can one derive with this principle? Do you think it is worth a thesis?

8

Hybrid Variational Principles: Formulation

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§8.1. Nomenclature

In Chapter 6 the variational principles of linear elasticity were classified as single-field and multi-field. For the latter we expand now the classification as follows:

$$\text{Variational principles} \left\{ \begin{array}{l} \text{Single-field} \\ \text{Multifield} \end{array} \right. \left\{ \begin{array}{l} \text{Mixed (a.k.a. "pure mixed")} \\ \text{Hybrid} \end{array} \right. \left\{ \begin{array}{l} \text{Internally single field} \\ \text{Internally multifield} \end{array} \right. \quad (8.1)$$

Single-field and mixed principles have been covered in previous chapters. In this Chapter we begin the study of hybrid functionals with FEM applications in mind.

Hybrid functionals have one or more master fields that are defined only on interfaces. These principles represent an important extension to the classical principles of mechanics. As discussed below, these extensions were largely motivated by trying to improve and extend the power of finite elements models.

Finite elements based on hybrid functionals, called *hybrid elements*, were constructed in the early 1960s. The original elements were quite limited in their ability to treat nonlinear and dynamic problems, as well as treatment of complicated geometries. However, such limitations are gradually disappearing as the fundamental concepts are better understood. Presently hybrid principles represent an important area of research in the construction of high performance finite elements, especially for plates and shells.

§8.2. Motivation

Why hybrid functionals? The general objective is to *relax* continuity conditions of fields. This idea has taken root in a surprisingly large number of technical applications, not all of which involve finite elements.

§8.2.1. Early Work

The original development came almost simultaneously from two widely different contexts:

Solid Mechanics. Prager proposed¹ the variational treatment of discontinuity conditions in elastic bodies by adding an interface potential. This extension was intended to handle physical discontinuities such as cracks, dislocations or material interfaces, at which internal field components, notably stresses, may jump.

Finite Elements. Pian² constructed continuum finite elements with stress assumptions but with displacement degrees of freedom. These are now known as *stress hybrids*, representing a tiny subclass of a vast population. Originally hybrids were constructed following a virtual-work recipe. A variational framework was not developed until the late sixties by Pian and Tong.³

¹ W. Prager, Variational principles for linear elastostatics for discontinuous displacements, strains and stresses, in *Recent Progress in Applied Mechanics*, The Folke-Odgvist Volume, ed. by B. Broger, J. Hult and F. Niordson, Almqvist and Wiksell, Stockholm, 463–474, 1967

² T. H. H. Pian, Derivation of element stiffness matrices by assumed stress distributions, *AIAA J.*, **2**, 1964, pp. 1333–1336.

³ T. H. H. Pian and P. Tong, Basis of finite element methods for solid continua, *Int. J. Numer. Meth. Engrg.*, **1**, 1969, pp.

§8.2.2. Recent Developments

For the next two decades (1970-1990) hybrid variational forms made slow progress in finite element applications. The mathematical basis is not easily accessible to students.⁴ The topic is plagued with “variational glitches” that have often led FEM researchers astray. There have been questions on the applicability to nonlinear and dynamic analysis.

The topic has revived over the past decade because of increasing interest in *model decomposition methods* for a myriad of applications: massively parallel computation, system identification, damage detection, optimization, coupling of nonmatched meshes, and multiscale analysis. All of these applications have in common the breakdown of a FEM model into pieces (substructures or subdomains) separated by interfaces. Hybrid functionals provide a general and elegant way of “gluing” those interfaces together.

§8.2.3. Improving FEM Models

The original motivation of hybrids for FEM was to alleviate the following difficulties noted in displacement-assumed elements.

Relaxed continuity requirements. Meeting the variationally-dictated continuity requirements in the construction of fully conforming displacement shape functions of some structural models, notably plate and shell elements, is difficult.⁵ Furthermore, continuity across elements of different type (for example, a beam element linked to a solid or shell element) is not easy to achieve.

Better displacement solution. Even after conforming plate and shell elements were developed, it was noted that performance for coarse meshes or irregular meshes was disappointing. The elements were generally over stiff, requiring computationally expensive fine meshes to deliver engineering accuracy.

Better stress solution. Not only the conforming elements tended to be over stiff in displacements, but stresses derived from them were often of poor accuracy and — worse of all from an engineering standpoint — unconservative in the sense that they underestimated the analytical values.

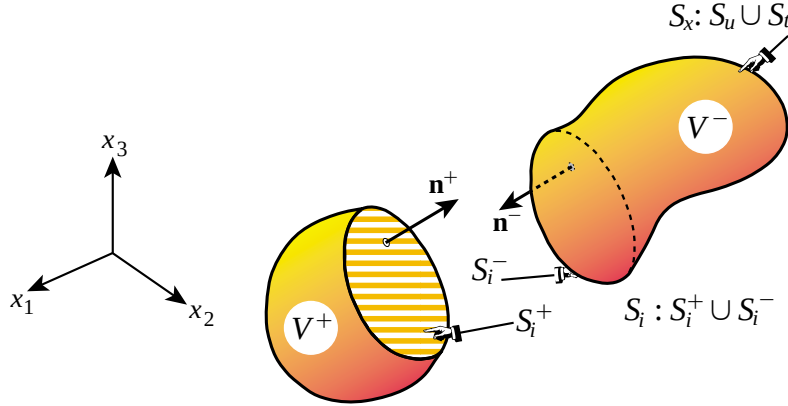
These three goals were accomplished for linear static analysis using hybrid elements. The extension to dynamic and nonlinear analysis was hampered initially by the lack of knowledge of interior displacements, which are needed to get mass and geometric stiffness matrices, respectively. This is gradually being solved with more powerful techniques.

At this point one may ask: why not use *mixed* variational principles instead of hybrids? Mixed principles are simpler to understand, and appear to address the goal of balanced accuracy directly.

3–29. See also T. H. H. Pian, Finite element methods by variational principles with relaxed continuity requirements, in *Variational Methods in Engineering*, Vol. 1, ed. by C. A. Brebbia and H. Tottenham, Southampton University Press, Southampton, U.K., 1973. A systematic classification of hybrid elements was undertaken by S. N. Atluri, On “hybrid” finite-element models in solid mechanics, In: *Advances in Computer Methods for Partial Differential Equations*. Ed. by R. Vichnevetsky, AICA, Rutgers University, 346–356, 1975.

⁴ Few textbooks deal with the subject more than a superficial “recipe” level. The terminology is not standardized. A typical example is Cook, Malkus and Plesha, who present stress hybrids following Pian’s original 1964 treatment and stop there. Even a more advanced monograph such as Oden and Reddy, cited in §3.6.2, covers hybrid functionals as an afterthought.

⁵ And sometimes, in the case of curved shell elements based on shell theory, impossible.

FIGURE 8.1. Slicing a body of volume V by an internal boundary S_i .

Indeed mixed methods do a good job for one-dimensional elements, as exemplified in Chapters 6 and 7. These improvements can be extended to 2D and 3D elements of particularly simple shapes, such as rectangles and cubes.

For 2D and 3D continuum elements of general shape, however, pure mixed methods run into implementation and numerical difficulties, which are too complex to describe here. To date they have not achieved the success of hybrid methods and there are reasons to argue that they never will.⁶

Note that the foregoing statement is qualified: it says “pure mixed functionals” — see the classification in (8.1). The classical HR and VHW functionals covered in the previous Chapters belong to this category. But if mixed and hybrid functionals are combined, in the sense that the former are used for the interior region, very powerful element formulation methods emerge. Therefore, learning mixed functionals is not a loss of time.

§8.3. Slicing a Potato

To understand the idea behind hybrid functionals, consider again a potato-shaped elastic body of volume V and surface S . Slice it by a smooth internal interface S_i , as depicted in Figure 8.1. This allows the consideration of certain field discontinuities. Those discontinuities may be of physical or computational nature, as discussed later.

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To visualize the following property of internal boundaries it is convenient to consider a two-dimensional domain as in Figure 8.2. This is broken up into four pieces or subdomains as illustrated

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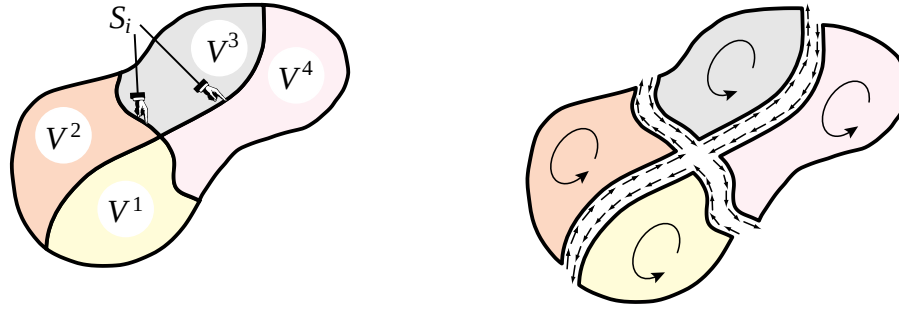


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The surface integral of a function g is split into contributions from the three boundaries

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As noted, the integral over S_i traverses twice over each face: $+$ and $-$, of the interface. Frequently the integrand g is of the flux form $g = \mathbf{f} \cdot \mathbf{n}$. Then if the components of $\mathbf{f}$ are *continuous* on S_i , that integral cancels out because $\mathbf{n}^+ = -\mathbf{n}^-$ and consequently $\mathbf{f} \cdot (\mathbf{n}^+ + \mathbf{n}^-) dS \equiv 0$.

But if some components of the integrand are discontinuous, the interface integral will not necessarily cancel. This is the origin of hybrid principles.

§8.4. A Stress Hybrid Principle

A hybrid principle is obtained by adding two functionals:

Hybrid Principle = Interior Functional + Interface Potential	(8.4)

The *interior functional* is of the classical type studied in previous Chapters. The new ingredient is the *interface potential*, which comes from the contribution of the interface.

Rather than going for the most general form possible, in the following we construct the particular hybrid variational principle that gives rise to *equilibrium-stress hybrid elements*. Historically this was the first one derived by Pian (see footnotes in §8.2 and Exercise 8.3). It is still good for instructional purposes because it has a minimum number of ingredients. This principle is used to formulate a four-node plane stress quadrilateral element in the next chapter.

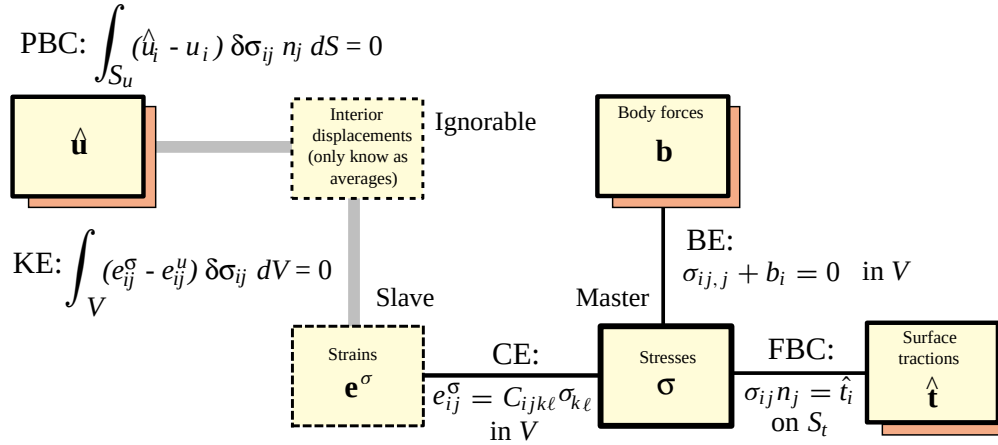


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The *interior functional* for this example is that of the total complementary potential energy (TCPE) principle of linear elastostatics:

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Here U_C is the internal complementary energy in terms of stresses

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which is stored in the body as elastic internal energy, and W_C is the work potential term of (8.5). This is a single-field functional with stresses as the only master field. The Weak Form for this principle is shown in Figure 8.3.

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To hybridize this principle, split V into M subvolumes V^m , $m = 1, 2, \dots, M$ by internal interfaces collected in S_i . (In the finite element applications, these subvolumes become the individual elements, to be relabeled with superscript e). Take a boundary displacement field d_i over S_i as additional master. This s-called *connector displacement field* must be unique on S_i . Its function is to link or connect subvolumes, functioning as a *frame*. The master stress field σ_{ij} is “glued” to the frame by

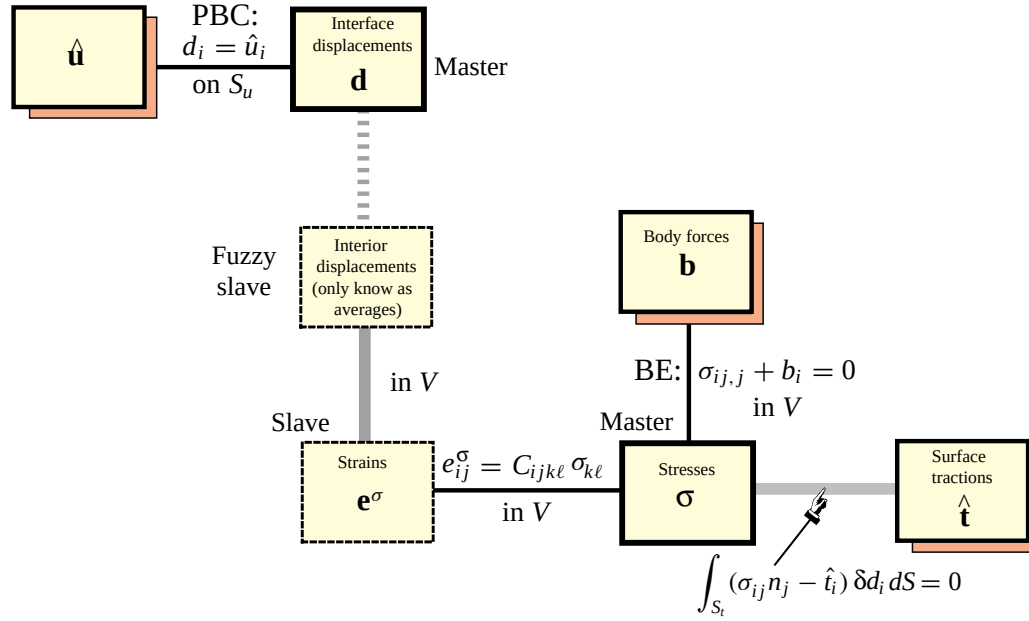


FIGURE 8.4. Schematics of Weak Form of the equilibrium-stress-hybrid principle. As a Tonti diagram this is still unsatisfactory; needs to be improved: suggestions welcome.

adding an integral π_d over S_i , called the *interface potential*, which measures the work lost or stored on S_i :

$$\Pi_C^d[\sigma_{ij}, d_i] = \Pi_C[\sigma_{ij}] + \pi_d[\sigma_{ij}, d_i] = \Pi_C[\sigma_{ij}] + \int_{S_i} d_i \sigma_{ij} n_j dS. \quad (8.7)$$

This is a *multifield* hybrid functional with two masters: the stresses σ_{ij} and the displacement field d_i . It is not a mixed functional because d_i is not an interior field, as it exists only over the interface S_i . The Weak Form for this principle is shown in the diagram of Figure 8.4.⁷ Comparing Figure 8.4 to Figure 8.3, it can be observed that link PBC has become strong whereas FBC is now weak. This is the result of the integral transformations worked out below.

Note that if the flux $t_j = \sigma_{ij} n_j$ is continuous across S_i , π_d vanishes, as explained after (8.3). This is characteristic of interface potentials: they vanish if there are no discontinuities.

§8.4.3. The Work Potential

The functional (8.7) can be decomposed into two functionally distinct parts

$$\Pi_C^d = -U_C + W_d \quad (8.8)$$

where U_C is the complementary energy (8.6), and W_d is the *work potential*

$$W_d = \int_{S_u} \hat{u}_i \sigma_{ij} n_j dS + \int_{S_i} d_i \sigma_{ij} n_j dS. \quad (8.9)$$

⁷ As noted in the legend, this diagram needs improvement. It does not show clearly the role of the interface potential. Suggestions welcome.

This term includes the work of the prescribed displacements on S_u as well as the energy stored or lost on the internal interface S_i . For finite element work it is necessary to transform the integral over S_i to one over $S = S_u \cup S_t \cup S_i$. Using the identity

$$\int_{S_i} d_i \sigma_{ij} n_j dS = \int_S d_i \sigma_{ij} n_j dS - \int_{S_u} d_i \sigma_{ij} n_j dS - \int_{S_t} d_i \sigma_{ij} n_j dS, \quad (8.10)$$

on the functional (8.9) we obtain

$$W_d = \int_S d_i \sigma_{ij} n_j dS - \int_{S_t} d_i \hat{t}_i dS \quad (8.11)$$

because the integral of $(\hat{u}_i - d_i) \sigma_{ij} n_j$ over S_u vanishes on account of the strong connection $d_i = \hat{u}_i$ on S_u . The last term comes from replacing $\sigma_{ij} n_j \rightarrow \hat{t}_i$ on S_t because of the original FBC strong connection (see Figure 8.3), which now becomes weak because d_i “interposes” between σ_{ij} and $\hat{t}_i$. Replacing into (8.8) we arrive at the final form

$$\Pi_C^d[\sigma_{ij}, d_i] = -U_C + W_d = -\frac{1}{2} \int_V \sigma_{ij} C_{ijkl} \sigma_{kl} dV + \int_S d_i \sigma_{ij} n_j dS - \int_{S_t} d_i \hat{t}_i dS. \quad (8.12)$$

The integral over S_u has disappeared while that over S_t , which has the same form as in the TPE functional except that u_i is replaced by d_i , comes into play. The specified displacement $\hat{u}_i$ disappears into the strong connection $d_i = \hat{u}_i$ on S_u . Most important of all: the interface potential is taken over the whole boundary S , not just S_i .

Remark 8.1. Several finite element papers and textbooks do this transformation incorrectly and end up with erroneous boundary terms. The error is often inconsequential, however, as most element derivations take the “save S_u and S_t for last” route described later. But in nonlinear analysis errors can have serious consequences.

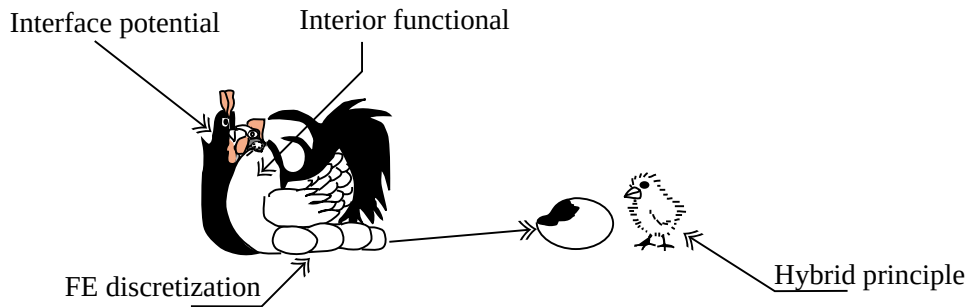


Figure 8.5. The chicken-and-egg story revisited: (a) An interior (non-hybrid) functional (the hen) and an interface potential (the rooster) beget a hybrid principle (the chick) in the sheltered framework of FEM.

§8.4.4. Hybrids or FEM: A Chicken and Egg Story

It has been said that finite elements are a byproduct of the advent of computers: no computers, no finite elements. A similar claim: “without finite elements there would be no hybrid variational principles” is too strong, because as noted in §8.2 these principles were also derived from a continuum mechanics standpoint. However, without finite elements they would have remained largely a mathematical curiosity. Figure 8.5 puts this observation into the context of the old chicken and egg story.

The conceptual steps in applying these principles to formulate individual finite elements are sketched in Figure 8.6. Note that the subdivision into elements comes *before* the principle is constructed; else there would be no S_i to integrate on. So the FE mesh is where the principle is realized and lives on.

Homework Exercises for Chapter 8

Hybrid Variational Principles: Formulation

EXERCISE 8.1 [A:15] Present the derivation steps of the stress hybrid principle for a prismatic bar of constant cross section and modulus. Is the hybrid functional different from Hellinger-Reissner's?

EXERCISE 8.2 [A:20] Present the derivation steps of the stress hybrid principle for a prismatic plane Bernoulli-Euler beam of constant inertia and modulus. Is the hybrid functional different from Hellinger-Reissner's?

EXERCISE 8.3 [A:20=10+10] In a recent article, Pian⁸ reminisces on some events that led, through serendipity, to the formulation of the first hybrid elements in 1963.⁹ He says that the initial investigation started from the Hellinger-Reissner principle for zero body forces ($b_i = 0$). Following is a variationally correct version of his arguments. In the notation of this course a “generalized HR” that extends HR with weak links to displacement BCs, reads

$$\Pi_{\text{HR}}^g[\sigma_{ij}, u_i] = -U^*[\sigma_{ij}] + \int_V \sigma_{ij} e_{ij}^u dV - \int_{S_t} \hat{t}_i u_i dS - \int_{S_u} \sigma_{ij} n_j (u_i - \hat{u}_i) dS, \quad (\text{E8.1})$$

where $U^*[\sigma_{ij}] = \frac{1}{2} \int_V \sigma_{ij} C_{ijkl} \sigma_{kl} dV$.

- (a) Assume that σ_{ij} satisfies strongly the zero-body-force equilibrium equations $\sigma_{ij,j} = 0$. Transform (E8.1) via integration by parts using

$$\int_V \sigma_{ij} e_{ij}^u dV = - \int_V \overset{0}{\sigma_{ij,j}} u_i dV + \int_S \sigma_{ij} n_j u_i dS = \int_{S_t} \sigma_{ij} n_j u_i dS + \int_{S_u} \sigma_{ij} n_j u_i dS + \int_{S_i} \sigma_{ij} n_j u_i dS, \quad (\text{E8.2})$$

to get

$$\Pi_{\text{HR}}^g[\sigma_{ij}, u_i] = -U^* + \int_{S_u} \sigma_{ij} n_j \hat{u}_i dS - \int_{S_t} (\hat{t}_i - \sigma_{ij} n_j) u_i dS + \int_{S_i} \sigma_{ij} n_j u_i dS. \quad (\text{E8.3})$$

(Check this out.)

- (b) Using the surface-integral identity (8.10) as appropriate, show that (E8.3) reduces to the stress hybrid functional (8.12) by identifying $d_i \equiv u_i$ on S and making $u_i = \hat{u}_i$ on S_u strong.

According to Pian that is roughly the way the stress hybrid principle eventually was linked to HR in the late 1960s.¹⁰

⁸ T. H. H. Pian, Some notes on the early history of hybrid stress finite element method, *Int. J. Numer. Meth. Engrg.*, **47**, 2000, 419–425.

⁹ During a Fall Semester 1963 graduate course entitled Variational and Matrix Methods in Structural Mechanics, offered at MIT's Aero & Astro Department. According to Pian, the method grew out of HW assignments for the last class of that course, which illustrated the use of the Hellinger-Reissner functional for the construction of element stiffness matrices. Eric Reissner was then a Professor at MIT and was of course influential in young Pian's research. Nobody else at the time had thought of using multifield functionals for FEM work, except for Len Herrmann at UC Davis.

¹⁰ In the article cited above there are several variational errors: omission of the S_u term of (E8.1), no S_i and no transformation of the interface term to the whole surface S . The errors seem to compensate (two wrongs made a right, at MIT). Pian's arguments are not easy to follow and are stated for a discrete functional, not a continuous one.

EXERCISE 8.4

[C:20] Form the 8×8 stiffness matrix $\mathbf{K}$ of the hybrid 4-node, plane stress element defined by 5-parameter linear stress assumptions. Restrict the geometry to a *rectangular* element of dimensions L along x and H along y . The material may be assumed isotropic, with elastic modules E and Poisson's ratio ν . The thickness h is uniform.

Although computations may be done by hand (with enough patience it would take a couple of days) the use of a symbolic algebra system is highly recommended. Note: a *Mathematica* script is posted in Chapter 8 index to help.

EXERCISE 8.5

[A:20] Extend the stress hybrid principle (8.12) to include linear isotropic thermoelasticity. Assume that if the temperature T changes by $\Delta T = T - T_0$ from a reference value T_0 , the body expands isotropically with coefficient α . Hint: the indicial-form strain-stress equations become

$$e_{ij} = C_{ijkl}\sigma_{kl} + \alpha\Delta T\delta_{ij}, \quad (\text{E8.4})$$

where δ_{ij} is the Kronecker delta. The total complementary energy to be used in Π_C^d is $U^*[\sigma_{ij}] = \frac{1}{2} \int_V (\sigma_{ij}C_{ijkl}\sigma_{kl} + \alpha\Delta T\delta_{ij}\sigma_{ij}) dV$. Here $\delta_{ij}\sigma_{ij} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \text{Trace}(\sigma_{ij}) = I_1$, the first invariant of the stress tensor, which is thrice the mean pressure.

EXERCISE 8.6

[A:20] Work out the inclusion of thermoelasticity effects, as outlined in the previous Exercise, into the formulation of the hybrid plane stress quadrilateral to be developed in Chapter 9. Show that an initial force vector $\mathbf{f}_i$ has to be added to $\mathbf{f}$, and find its expression in terms of $\mathbf{S}$, $\mathbf{G}$, ΔT and α .

9

Hybrid Variational Principles of Elastostatics I

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§9.1. Nomenclature

In Chapter 6 the variational principles of linear elasticity were classified as single-field and multi-field. For the latter we expand now the classification as follows:

$$\text{Variational principles} \left\{ \begin{array}{l} \text{Single-field} \\ \text{Multifield} \end{array} \right. \left\{ \begin{array}{l} \text{Mixed (a.k.a. "pure mixed")} \\ \text{Hybrid} \end{array} \right. \left\{ \begin{array}{l} \text{Internally single field} \\ \text{Internally multifield} \end{array} \right. \quad (9.1)$$

Single-field and mixed principles have been covered in previous chapters. In this Chapter we begin the study of hybrid functionals with FEM applications in mind.

Hybrid functionals have one or more master fields that are defined only on interfaces. These principles represent an important extension to the classical principles of mechanics. As discussed below, these extensions were largely motivated by trying to improve and extend the power of finite elements models.

Finite elements based on hybrid functionals, called *hybrid elements*, were constructed in the early 1960s. The original elements were quite limited in their ability to treat nonlinear and dynamic problems, as well as treatment of complicated geometries. However, such limitations are gradually disappearing as the fundamental concepts are better understood. Presently hybrid principles represent an important area of research in the construction of high performance finite elements, especially for plates and shells.

§9.2. Motivation

Why hybrid functionals? The general objective is to *relax* continuity conditions of fields. This idea has taken root in a surprisingly large number of technical applications, not all of which involve finite elements.

§9.2.1. Early Work

The original development came almost simultaneously from two widely different contexts:

Solid Mechanics. Prager proposed¹ the variational treatment of discontinuity conditions in elastic bodies by adding an interface potential. This extension was intended to handle physical discontinuities such as cracks, dislocations or material interfaces, at which internal field components, notably stresses, may jump.

Finite Elements. Pian² constructed continuum finite elements with stress assumptions but with displacement degrees of freedom. These are now known as *stress hybrids*, representing a tiny subclass of a vast population. Originally hybrids were constructed following a virtual-work recipe. A variational framework was not developed until the late sixties by Pian and Tong.³

¹ W. Prager, Variational principles for linear elastostatics for discontinuous displacements, strains and stresses, in *Recent Progress in Applied Mechanics*, The Folke-Odgvist Volume, ed. by B. Broger, J. Hult and F. Niordson, Almqvist and Wiksell, Stockholm, 463–474, 1967

² T. H. H. Pian, Derivation of element stiffness matrices by assumed stress distributions, *AIAA J.*, **2**, 1964, pp. 1333–1336.

³ T. H. H. Pian and P. Tong, Basis of finite element methods for solid continua, *Int. J. Numer. Meth. Engrg.*, **1**, 1969, pp.

§9.2.2. Recent Developments

For the next two decades (1970-1990) hybrid variational forms made slow progress in finite element applications. The mathematical basis is not easily accessible to students.⁴ The topic is plagued with “variational glitches” that have often led FEM researchers astray. There have been questions on the applicability to nonlinear and dynamic analysis.

The topic has revived over the past decade because of increasing interest in *model decomposition methods* for a myriad of applications: massively parallel computation, system identification, damage detection, optimization, coupling of nonmatched meshes, and multiscale analysis. All of these applications have in common the breakdown of a FEM model into pieces (substructures or subdomains) separated by interfaces. Hybrid functionals provide a general and elegant way of “gluing” those interfaces together.

§9.2.3. Improving FEM Models

The original motivation of hybrids for FEM was to alleviate the following difficulties noted in displacement-assumed elements.

Relaxed continuity requirements. Meeting the variationally-dictated continuity requirements in the construction of fully conforming displacement shape functions of some structural models, notably plate and shell elements, is difficult.⁵ Furthermore, continuity across elements of different type (for example, a beam element linked to a solid or shell element) is not easy to achieve.

Better displacement solution. Even after conforming plate and shell elements were developed, it was noted that performance for coarse meshes or irregular meshes was disappointing. The elements were generally over stiff, requiring computationally expensive fine meshes to deliver engineering accuracy.

Better stress solution. Not only the conforming elements tended to be over stiff in displacements, but stresses derived from them were often of poor accuracy and — worse of all from an engineering standpoint — unconservative in the sense that they underestimated the analytical values.

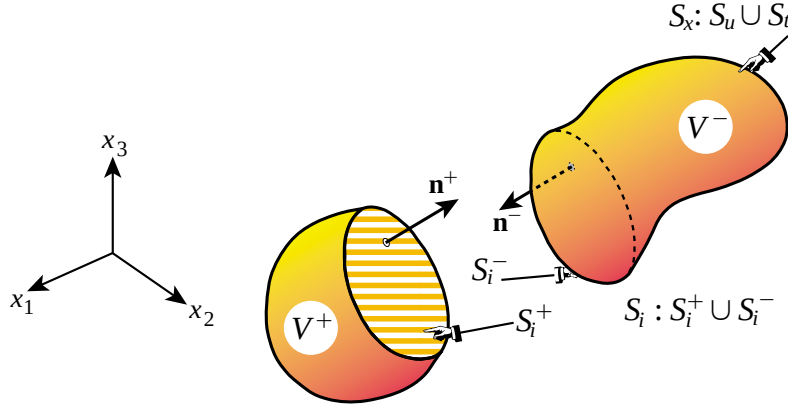
These three goals were accomplished for linear static analysis using hybrid elements. The extension to dynamic and nonlinear analysis was hampered initially by the lack of knowledge of interior displacements, which are needed to get mass and geometric stiffness matrices, respectively. This is gradually being solved with more powerful techniques.

At this point one may ask: why not use *mixed* variational principles instead of hybrids? Mixed principles are simpler to understand, and appear to address the goal of balanced accuracy directly.

3–29. See also T. H. H. Pian, Finite element methods by variational principles with relaxed continuity requirements, in *Variational Methods in Engineering*, Vol. 1, ed. by C. A. Brebbia and H. Tottenham, Southampton University Press, Southampton, U.K., 1973. A systematic classification of hybrid elements was undertaken by S. N. Atluri, On “hybrid” finite-element models in solid mechanics, In: *Advances in Computer Methods for Partial Differential Equations*. Ed. by R. Vichnevetsky, AICA, Rutgers University, 346–356, 1975.

⁴ Few textbooks deal with the subject more than a superficial “recipe” level. The terminology is not standardized. A typical example is Cook, Malkus and Plesha, who present stress hybrids following Pian’s original 1964 treatment and stop there. Even a more advanced monograph such as Oden and Reddy, cited in §3.6.2, covers hybrid functionals as an afterthought.

⁵ And sometimes, in the case of curved shell elements based on shell theory, impossible.

FIGURE 9.1. Slicing a body of volume V by an internal boundary S_i .

Indeed mixed methods do a good job for one-dimensional elements, as exemplified in Chapters 6 and 7. These improvements can be extended to 2D and 3D elements of particularly simple shapes, such as rectangles and cubes.

For 2D and 3D continuum elements of general shape, however, pure mixed methods run into implementation and numerical difficulties, which are too complex to describe here. To date they have not achieved the success of hybrid methods and there are reasons to argue that they never will.⁶

Note that the foregoing statement is qualified: it says “pure mixed functionals” — see the classification in (9.1). The classical HR and VHW functionals covered in the previous Chapters belong to this category. But if mixed and hybrid functionals are combined, in the sense that the former are used for the interior region, very powerful element formulation methods emerge. Therefore, learning mixed functionals is not a loss of time.

§9.3. Slicing a Potato

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This interface S_i , also called an interior boundary, divides V into two subdomains: V^+ and V^- so that $S_i : V^+ \cap V^-$. The outward normals to S_i that emanate from these subdomains are denoted by $\mathbf{n}^+$ and $\mathbf{n}^-$, respectively. Note that at corresponding locations they point in equal but opposite directions. The external boundary is relabeled S_x ; thus the complete boundary is $S : S_x \cup S_i$. For many derivations it is convenient to view V^+ and V^- as disconnected subvolumes with matching boundaries S_i^+ and S_i^- , as illustrated in Figure 9.1.

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To visualize the following property of internal boundaries it is convenient to consider a two-dimensional domain as in Figure 9.2. This is broken up into four pieces or subdomains as illustrated

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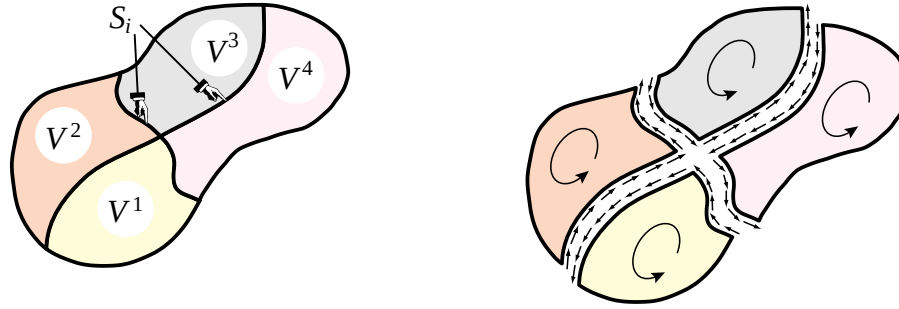


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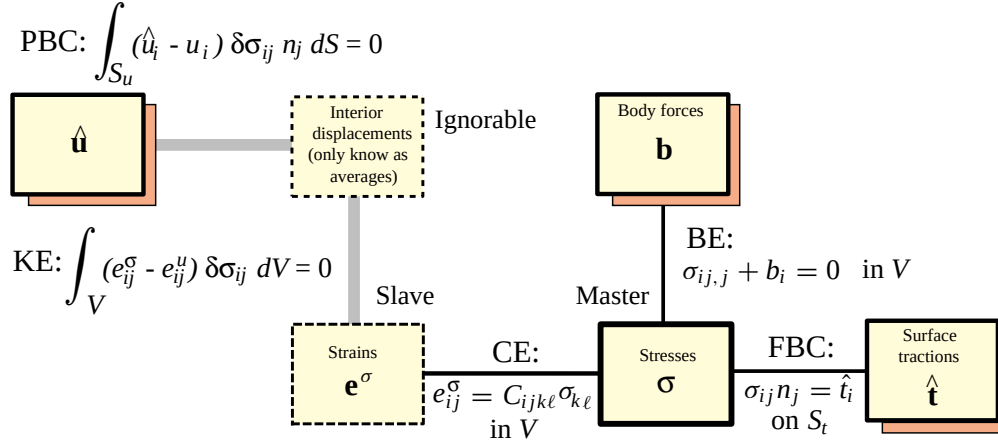


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§9.4.2. Hybridization

To hybridize this principle, split V into M subvolumes V^m , $m = 1, 2, \dots, M$ by internal interfaces collected in S_i . (In the finite element applications, these subvolumes become the individual elements, to be relabeled with superscript e). Take a boundary displacement field d_i over S_i as additional master. This s-called *connector displacement field* must be unique on S_i . Its function is to link or connect subvolumes, functioning as a *frame*. The master stress field σ_{ij} is “glued” to the frame by

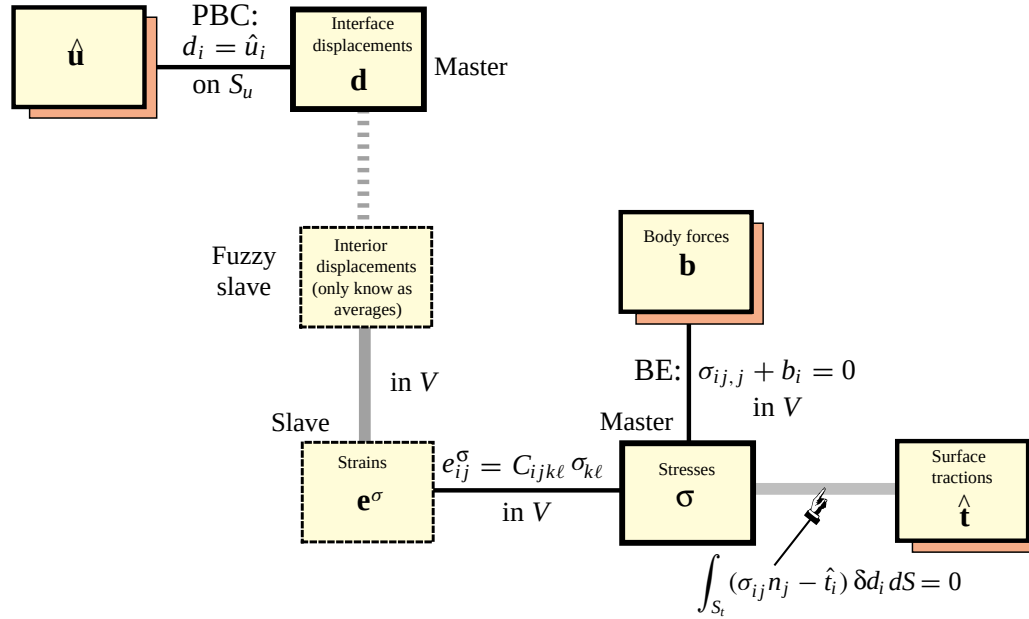


FIGURE 9.4. Schematics of Weak Form of the equilibrium-stress-hybrid principle. As a Tonti diagram this is still unsatisfactory; needs to be improved: suggestions welcome.

adding an integral π_d over S_i , called the *interface potential*, which measures the work lost or stored on S_i :

$$\Pi_C^d[\sigma_{ij}, d_i] = \Pi_C[\sigma_{ij}] + \pi_d[\sigma_{ij}, d_i] = \Pi_C[\sigma_{ij}] + \int_{S_i} d_i \sigma_{ij} n_j dS. \quad (9.7)$$

This is a *multifield* hybrid functional with two masters: the stresses σ_{ij} and the displacement field d_i . It is not a mixed functional because d_i is not an interior field, as it exists only over the interface S_i . The Weak Form for this principle is shown in the diagram of Figure 9.4.⁷ Comparing Figure 9.4 to Figure 9.3, it can be observed that link PBC has become strong whereas FBC is now weak. This is the result of the integral transformations worked out below.

Note that if the flux $t_j = \sigma_{ij} n_j$ is continuous across S_i , π_d vanishes, as explained after (9.3). This is characteristic of interface potentials: they vanish if there are no discontinuities.

§9.4.3. The Work Potential

The functional (9.7) can be decomposed into two functionally distinct parts

$$\Pi_C^d = -U_C + W_d \quad (9.8)$$

where U_C is the complementary energy (9.6), and W_d is the *work potential*

$$W_d = \int_{S_u} \hat{u}_i \sigma_{ij} n_j dS + \int_{S_i} d_i \sigma_{ij} n_j dS. \quad (9.9)$$

⁷ As noted in the legend, this diagram needs improvement. It does not show clearly the role of the interface potential. Suggestions welcome.

This term includes the work of the prescribed displacements on S_u as well as the energy stored or lost on the internal interface S_i . For finite element work it is necessary to transform the integral over S_i to one over $S = S_u \cup S_t \cup S_i$. Using the identity

$$\int_{S_i} d_i \sigma_{ij} n_j dS = \int_S d_i \sigma_{ij} n_j dS - \int_{S_u} d_i \sigma_{ij} n_j dS - \int_{S_t} d_i \sigma_{ij} n_j dS, \quad (9.10)$$

on the functional (9.9) we obtain

$$W_d = \int_S d_i \sigma_{ij} n_j dS - \int_{S_t} d_i \hat{t}_i dS \quad (9.11)$$

because the integral of $(\hat{u}_i - d_i) \sigma_{ij} n_j$ over S_u vanishes on account of the strong connection $d_i = \hat{u}_i$ on S_u . The last term comes from replacing $\sigma_{ij} n_j \rightarrow \hat{t}_i$ on S_t because of the original FBC strong connection (see Figure 9.3), which now becomes weak because d_i “interposes” between σ_{ij} and $\hat{t}_i$. Replacing into (9.8) we arrive at the final form

$$\Pi_C^d[\sigma_{ij}, d_i] = -U_C + W_d = -\frac{1}{2} \int_V \sigma_{ij} C_{ijkl} \sigma_{kl} dV + \int_S d_i \sigma_{ij} n_j dS - \int_{S_t} d_i \hat{t}_i dS.$$

(9.12)

The integral over S_u has disappeared while that over S_t , which has the same form as in the TPE functional except that u_i is replaced by d_i , comes into play. The specified displacement $\hat{u}_i$ disappears into the strong connection $d_i = \hat{u}_i$ on S_u . Most important of all: the interface potential is taken over the whole boundary S , not just S_i .

Remark 9.1. Several finite element papers and textbooks do this transformation incorrectly and end up with erroneous boundary terms. The error is often inconsequential, however, as most element derivations take the “save S_u and S_t for last” route described later. But in nonlinear analysis errors can have serious consequences.

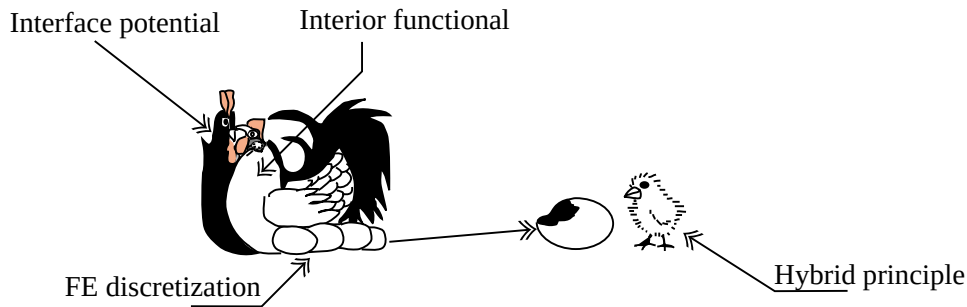


Figure 9.5. The chicken-and-egg story revisited: (a) An interior (non-hybrid) functional (the hen) and an interface potential (the rooster) beget a hybrid principle (the chick) in the sheltered framework of FEM.

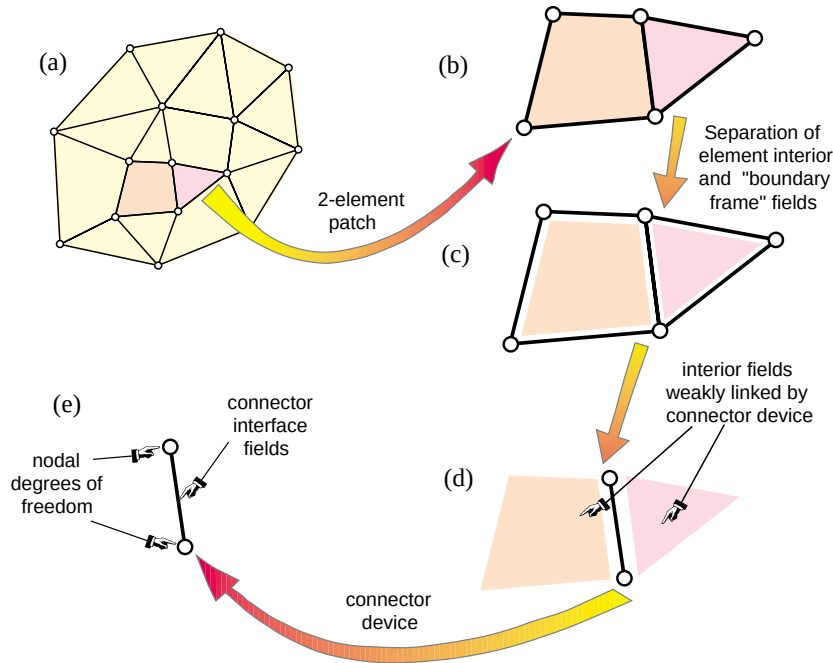


Figure 9.6. Conceptual steps in constructing hybrid finite elements. They are illustrated in 2D for visualization convenience.

§9.4.4. Hybrids or FEM: A Chicken and Egg Story

It has been said that finite elements are a byproduct of the advent of computers: no computers, no finite elements. A similar claim: “without finite elements there would be no hybrid variational principles” is too strong, because as noted in §9.2 these principles were also derived from a continuum mechanics standpoint. However, without finite elements they would have remained largely a mathematical curiosity. Figure 9.5 puts this observation into the context of the old chicken and egg story.

The conceptual steps in applying these principles to formulate individual finite elements are sketched in Figure 9.6. Note that the subdivision into elements comes *before* the principle is constructed; else there would be no S_i to integrate on. So the FE mesh is where the principle is realized and lives on.

§9.5. A 4-Node Plane Stress Hybrid Quadrilateral

We apply now Π_C^d to the construction of the 4-node plane-stress quadrilateral element shown in Figure 9.7. The element has constant thickness h and constant material properties characterized by the elastic compliance matrix $\mathbf{C} = \mathbf{E}^{-1}$ that relates strains to stresses: $\mathbf{e} = \mathbf{C}\boldsymbol{\sigma}$. For simplicity in the element construction we shall assume that the body force field $\mathbf{b}$ vanishes.

This element has historical importance as being the first one to be derived (by Pian in 1964, reference given in §9.2). Although as noted later the element does not have good performance, it serves to illustrate the derivation steps.

§9.5.1. The Stress Field

The first ingredient is the internal stress field, which is a master. We assume that each component of the stress field (σ_{xx} , σ_{yy} , σ_{xy}) varies linearly in x and y :

$$\begin{aligned} \sigma_{xx} &= a_1 + a_4 x + a_5 y, \\ \sigma_{yy} &= a_2 + a_6 x + a_7 y, \\ \sigma_{xy} &= a_3 + a_8 x + a_9 y. \end{aligned} \quad (9.13)$$

The a_i are called the *stress-amplitude parameters*, or simply *stress parameters*, which function as generalized coordinates. If this field is to satisfy the homogeneous equilibrium equations for zero body forces:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \quad (9.14)$$

then the stress coordinates cannot be independent but must verify the constraints

$$a_4 + a_9 = 0, \quad a_7 + a_8 = 0. \quad (9.15)$$

Substituting these into the expression for the shear stress in (9.13) gives $\sigma_{xy} = a_3 - a_7 x - a_4 y$. Consequently there are only *seven* independent stress parameters, which may be collected into a column vector $\mathbf{a}$. In matrix form:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & x & y \\ 0 & 0 & 1 & -y & 0 & 0 & -x \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix}, \quad (9.16)$$

or

$$\boxed{\boldsymbol{\sigma} = \mathbf{S} \mathbf{a}} \quad (9.17)$$

Remark 9.2. Why (9.13)? Short answer: invariance plus rank sufficiency. In the foregoing derivation x and y are assumed to be the global axes. If the orientation of these axes changes by a rotation about z , the equilibrium stress expansion (9.16) varies in the sense that the stress parameter values change, but the resulting element (and the finite element solution for the assembled model) is *independent* of the orientation of the axes. This is a consequence of the expansion (9.13) being a *complete polynomial* in x and y . Finite elements that comply with this condition (namely, that the solution be independent of the choice of global axes) are called *observer invariant* or simply *invariant*. Stress assumptions that are complete polynomials lead to invariant elements. The simplest such choice is a constant stress assumption (a complete polynomial of order 0) but as noted in the following Remark, that choice leads to rank deficiency.

Remark 9.3. Had only three stress parameters been retained in the assumption (9.16), namely a_1 , a_2 and a_3 , which obviously satisfy the homogeneous equilibrium equations (9.14), the element stiffness $\mathbf{K}^{(e)}$ derived later would have rank three at most (because the flexibility matrix $\mathbf{F}$ becomes 3×3). Since the target rank is $5 = 8 - 3$, the element stiffness matrix would be twice rank deficient and thus unacceptable.

Over each side the external normals have fixed direction; they will be identified by the notation of Figure 9.2. On side 1-2 the matrix form of (9.20) is

$$\begin{bmatrix} t_{x12} \\ t_{y12} \end{bmatrix} = \begin{bmatrix} n_{x12} & 0 & n_{y12} \\ 0 & n_{y12} & n_{x12} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \mathbf{N}_{12} \boldsymbol{\sigma}_{12} = \mathbf{N}_{12} \mathbf{S}_{12} \mathbf{a} \quad (9.21)$$

where $\mathbf{S}_{12}$ is $\mathbf{S}$ evaluated on side 1-2. Repeating this construction for the other three sides we build the relation

$$\mathbf{t} = \mathbf{T} \mathbf{a} \quad (9.22)$$

where $\mathbf{t}$ collects the 8 traction components, which are function of the coordinates through $\mathbf{S}$:

$$\mathbf{t} = [t_{x12} \quad t_{y12} \quad t_{x23} \quad t_{y23} \quad \cdots \quad t_{y41}]^T, \quad (9.23)$$

and $\mathbf{T}$ is an 8×7 matrix obtained by appropriately “row stacking” the four 2×7 matrices $\mathbf{N}_{12} \mathbf{S}_{12}$, $\mathbf{N}_{23} \mathbf{S}_{23}$, $\mathbf{N}_{34} \mathbf{S}_{34}$ and $\mathbf{N}_{41} \mathbf{S}_{41}$.

§9.5.4. Specified Boundary Traction

As forces acting on the element we consider boundary tractions $\hat{\mathbf{t}}$ acting on the four sides, and specified per unit of side length and thickness. These are collected to form the 8-vector

$$\hat{\mathbf{t}} = [\hat{t}_{x12} \quad \hat{t}_{y12} \quad \hat{t}_{x23} \quad \hat{t}_{y23} \quad \cdots \quad \hat{t}_{y41}]^T. \quad (9.24)$$

§9.5.5. The Discrete Equations

Inserting (9.17), (9.19), (9.22) and (9.24) into the functional Π_C^d for an individual element we get

$$\Pi_C^d = -\frac{1}{2} \mathbf{a}^T \mathbf{F} \mathbf{a} + \mathbf{a}^T \mathbf{G} \mathbf{u} - \mathbf{f}^T \mathbf{u}, \quad (9.25)$$

in which the element identification superscript has been omitted from matrices and vectors for brevity. The matrices in (9.25) are given by

$$\mathbf{F} = \int_{\Omega^{(e)}} h \mathbf{S}^T \mathbf{E}^{-1} \mathbf{S} d\Omega, \quad \mathbf{G} = \int_{\Gamma^{(e)}} h \mathbf{T}^T \mathbf{P} d\Gamma, \quad \mathbf{f} = \int_{S_i^{(e)}} h \hat{\mathbf{t}} \mathbf{P} d\Gamma. \quad (9.26)$$

Matrix $\mathbf{F}$ is often called a flexibility matrix, hence the identifying symbol.

Rendering Π_C^d (now an algebraic function) stationary with respect to the stress and displacement degrees of freedom we get

$$\frac{\partial \Pi_C^d}{\partial \mathbf{a}} = -\mathbf{F} \mathbf{a} + \mathbf{G} \mathbf{u} = \mathbf{0}, \quad \frac{\partial \Pi_C^d}{\partial \mathbf{u}} = \mathbf{G}^T \mathbf{a} - \mathbf{f} = \mathbf{0}. \quad (9.27)$$

The first equation is a discrete version of KE (the kinematic or compatibility relation), whereas the second one is a discrete form of BE, the balance or equilibrium equation.⁸

⁸ Recall that the KE and BE links are weak in this hybrid principle; cf. Figure 9.4.

Now if matrix $\mathbf{F}$ is invertible we may solve for the $\mathbf{a}$ vector at the element level from the first of (9.27) as $\mathbf{a} = \mathbf{F}^{-1}\mathbf{G}\mathbf{u}$, because the stress parameters are “disconnected” from element to element. Substituting into the second of (9.27) yields

$$\mathbf{G}^T \mathbf{F}^{-1} \mathbf{G} \mathbf{u} - \mathbf{f} = \mathbf{0}. \quad (9.28)$$

But these are formally the element stiffness equations, which on restoring the element superscript, become

$$\mathbf{K}^{(e)} \mathbf{u}^{(e)} = \mathbf{f}^{(e)}, \quad (9.29)$$

in which

$$\mathbf{K}^{(e)} = \mathbf{G}^T \mathbf{F}^{-1} \mathbf{G} \quad (9.30)$$

is the element stiffness matrix. The dimensions of $\mathbf{G}^T$, $\mathbf{F}$ and $\mathbf{G}$ are 8×7 , 7×7 and 7×8 , respectively. The expected rank of $\mathbf{K}^{(e)}$ is $5 = 8 - 3$, which is the dimension of $\mathbf{K}^{(e)}$ minus the number of independent rigid body modes. The rigid body modes are injected by the matrix $\mathbf{G}$.

Remark 9.4. $\mathbf{F}$ is called a *flexibility matrix* in terms of the stress parameters $\mathbf{a}$, whereas $\mathbf{G}$ is called the *connection matrix*, or *leverage matrix* in the literature. The transpose $\mathbf{G}^T$ is called the *equilibrium matrix*.

Remark 9.5. The supermatrix form of (9.27) is

$$\begin{bmatrix} -\mathbf{F} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}, \quad (9.31)$$

which displays the characteristic configuration for hybrid elements of this type. (Compare the discussion of “connector elements” in §6.4.) Static condensation of $\mathbf{a}$ by forward Gauss elimination yields the stiffness equation (9.29).

§9.5.6. Is This Element Any Good?

Historically the foregoing element was the first stress hybrid model for plane stress analysis. Numerical experiments show that the element is better for bending-like behavior than the 4-node isoparametric bilinear quadrilateral developed in IFEM. However, the improvement is marginal and would not justify the far more complex construction. Why? The key reason is that the proper rank of the stiffness matrix is five = $8 - 3$. That would be the ideal number of *independent* stress parameters a_i , no more and no less. But instead we have used seven in (9.16).

How can the number of stress parameters be cut to 5? One solution, discovered (and re-discovered) by many authors, is to set $a_4 = a_7 = 0$ so that after renumbering the parameters the equilibrium stress field assumption effectively reduces to

$$\begin{aligned} \sigma_{xx} &= a_1 + a_4 y, \\ \sigma_{yy} &= a_2 + a_5 x, \\ \sigma_{xy} &= a_3. \end{aligned} \quad (9.32)$$

which in matrix form is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}. \quad (9.33)$$

This assumption improves the element behavior while reducing formation cost. Unfortunately it has a significant drawback: the element is *no longer observer-invariant* with respect to the choice of axes x and y because (9.32) are not complete polynomials (cf. Remark 9.1). Aligning x and y with the global axes now would give *finite element solutions that depend on orientation*, which is obviously highly undesirable as well as scary to a naive user.

The usual solution to this dilemma: balanced stiffness versus invariance, is to choose (9.32) in a *local Cartesian system* $(\bar{x}, \bar{y})$ which is *attached* to the element in a “natural” way. For a rectangular geometry the obvious choice is to align $(\bar{x}, \bar{y})$ with the side directions. The resulting element is widely recognized to be the optimal one for this nodal DOF configuration.⁹ But for *arbitrary quadrilateral* geometries the choice of local system is far from obvious and has been the topic of substantial research.

Some authors have tried to recast the equilibrium equations (9.14) in isoparametric coordinates (ξ, η) , a process that automatically fulfills invariance but greatly complicates the stress assumption because such equations become quasilinear partial differential equations.

⁹ In fact the same quadrilateral element coalesces with those obtained from other high-performance element derivation methods, which is one of the characteristics of optimality.

Homework Exercises for Chapter 9

Hybrid Variational Principles of Elastostatics I

EXERCISE 9.1 [A:15] Present the derivation steps of the stress hybrid principle for a prismatic bar of constant cross section and modulus. Is the hybrid functional different from Hellinger-Reissner's?

EXERCISE 9.2 [A:20] Present the derivation steps of the stress hybrid principle for a prismatic plane Bernoulli-Euler beam of constant inertia and modulus. Is the hybrid functional different from Hellinger-Reissner's?

EXERCISE 9.3 [A:20=10+10] In a recent article, Pian¹⁰ reminisces on some events that led, through serendipity, to the formulation of the first hybrid elements in 1963.¹¹ He says that the initial investigation started from the Hellinger-Reissner principle for zero body forces ($b_i = 0$). Following is a variationally correct version of his arguments. In the notation of this course a “generalized HR” that extends HR with weak links to displacement BCs, reads

$$\Pi_{\text{HR}}^g[\sigma_{ij}, u_i] = -U^*[\sigma_{ij}] + \int_V \sigma_{ij} e_{ij}^u dV - \int_{S_t} \hat{t}_i u_i dS - \int_{S_u} \sigma_{ij} n_j (u_i - \hat{u}_i) dS, \quad (\text{E9.1})$$

where $U^*[\sigma_{ij}] = \frac{1}{2} \int_V \sigma_{ij} C_{ijkl} \sigma_{kl} dV$.

- (a) Assume that σ_{ij} satisfies strongly the zero-body-force equilibrium equations $\sigma_{ij,j} = 0$. Transform (E9.1) via integration by parts using

$$\int_V \sigma_{ij} e_{ij}^u dV = - \int_V \overset{0}{\sigma_{ij,j}} u_i dV + \int_S \sigma_{ij} n_j u_i dS = \int_{S_t} \sigma_{ij} n_j u_i dS + \int_{S_u} \sigma_{ij} n_j u_i dS + \int_{S_i} \sigma_{ij} n_j u_i dS, \quad (\text{E9.2})$$

to get

$$\Pi_{\text{HR}}^g[\sigma_{ij}, u_i] = -U^* + \int_{S_u} \sigma_{ij} n_j \hat{u}_i dS - \int_{S_t} (\hat{t}_i - \sigma_{ij} n_j) u_i dS + \int_{S_i} \sigma_{ij} n_j u_i dS. \quad (\text{E9.3})$$

(Check this out.)

- (b) Using the surface-integral identity (9.10) as appropriate, show that (E9.3) reduces to the stress hybrid functional (9.12) by identifying $d_i \equiv u_i$ on S and making $u_i = \hat{u}_i$ on S_u strong.

According to Pian that is roughly the way the stress hybrid principle eventually was linked to HR in the late 1960s.¹²

¹⁰ T. H. H. Pian, Some notes on the early history of hybrid stress finite element method, *Int. J. Numer. Meth. Engrg.*, **47**, 2000, 419–425.

¹¹ During a Fall Semester 1993 graduate course entitled Variational and Matrix Methods in Structural Mechanics, offered at MIT's Aero & Astro Department. According to Pian, the method grew out of assignments for the last class, which illustrated the use of the Hellinger-Reissner functional for the construction of element stiffness matrices. Eric Reissner was then a Professor at MIT and was of course influential in young Pian's research. Nobody else at the time had thought of using multifield functionals for FEM work, except for Len Herrmann at UC Davis.

¹² In the article cited above there are several variational errors: omission of the S_u term of (E8.1), no S_i and no transformation of the interface term to the whole surface S . The errors seem to compensate (two wrongs make a right). Pian's arguments are not easy to follow and are stated for a discrete functional, not a continuous one.

EXERCISE 9.4

[C:20] Form the 8×8 stiffness matrix $\mathbf{K}$ of the hybrid 4-node, plane stress element defined by the stress assumptions (9.27). Restrict the geometry to a *rectangular* element of dimensions L along x and H along y . The material may be assumed isotropic, with elastic modules E and Poisson's ratio ν . The thickness h is uniform.

Although computations may be done by hand (with enough patience it would take a couple of days) the use of a symbolic algebra system is highly recommended. Note: a *Mathematica* script is posted in Chapter 8 index to help.

EXERCISE 9.5

[A:20] Extend the stress hybrid principle (9.12) to include linear isotropic thermoelasticity. Assume that if the temperature T changes by $\Delta T = T - T_0$ from a reference value T_0 , the body expands isotropically with coefficient α . Hint: the indicial-form strain-stress equations become

$$e_{ij} = C_{ijkl}\sigma_{kl} + \alpha\Delta T\delta_{ij}, \quad (\text{E9.4})$$

where δ_{ij} is the Kronecker delta. The total complementary energy to be used in Π_C^d is $U^*[\sigma_{ij}] = \frac{1}{2} \int_V (\sigma_{ij}C_{ijkl}\sigma_{kl} + \alpha\Delta T\delta_{ij}\sigma_{ij}) dV$. Here $\delta_{ij}\sigma_{ij} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \text{Trace}(\sigma_{ij}) = I_1$, the first invariant of the stress tensor, which is thrice the mean pressure.

EXERCISE 9.6

[A:20] Work out the inclusion of thermoelasticity effects, as outlined in the previous Exercise, into the formulation of the hybrid plane stress quadrilateral developed in §9.3. Show that an initial force vector $\mathbf{f}_i$ has to be added to $\mathbf{f}$, and find its expression in terms of $\mathbf{S}$, $\mathbf{G}$, ΔT and α .

10

Axisymmetric Solids (Structures of Revolution)

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§10.1. Introduction

In the Introduction to Finite Element Methods (IFEM) course *two-dimensional* problems were emphasized. The axisymmetric problem considered in this and following two Chapters of this course provides a “bridge” to the treatment of three-dimensional elasticity. Besides its instructional value, the treatment of axisymmetric structures has considerable practical interest in aerospace, civil, mechanical and nuclear engineering.

§10.1.1. The Axisymmetric Problem

The axisymmetric problem deals with the analysis of structures of revolution under axisymmetric loading. A *structure of revolution* or SOR is generated by a *generating cross section* that rotates 360° about an *axis of revolution*, as illustrated in Figure 10.1. Such structures are said to be *rotationally symmetric*.

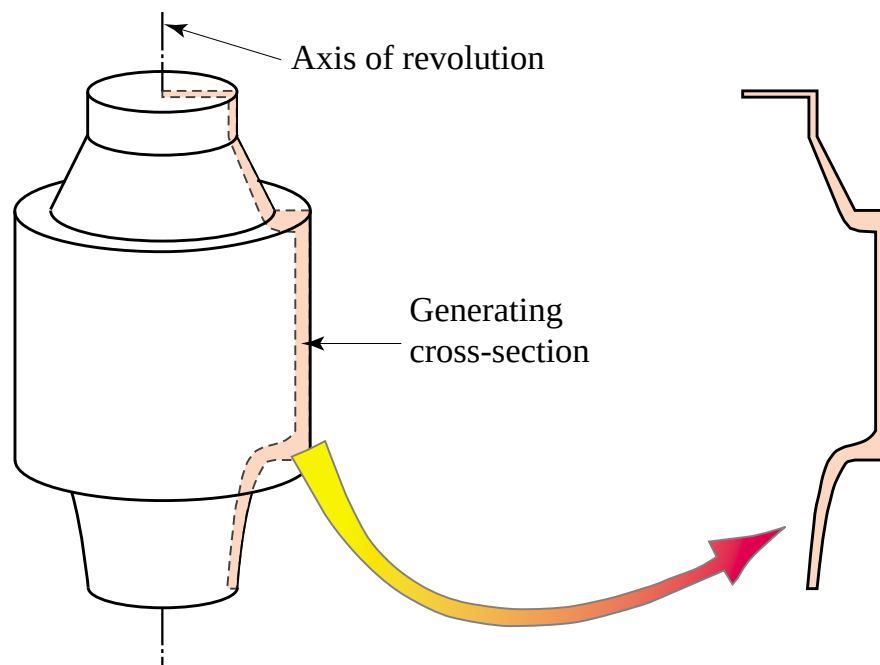


FIGURE 10.1. A structure of revolution is generated by rotating a generating cross section about an axis of revolution.

The technical importance of SOR's is considerable because of the following practical considerations:

1. **Fabrication:** axisymmetric bodies are usually easier to manufacture than bodies with more complex geometries. Think for example of pipes, piles, axles, wheels, bottles, cans, cups, nails.
2. **Strength:** axisymmetric configurations are often optimal in terms of strength to weight ratio because of the favorable distribution of the structural material. (Recall that the strongest columns and shafts, if wall buckling is ignored, have annular cross sections.)

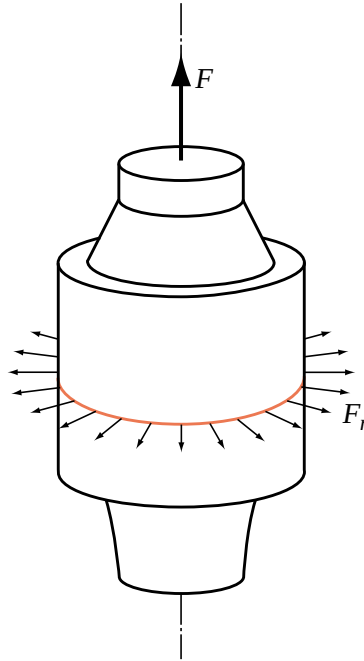


FIGURE 10.2. Axisymmetric loading on a SOR: F = concentrated load, F_r = radial component of “ring” line load.

3. Multipurpose: hollow axisymmetric bodies can assume a dual purpose as both *structure* and *shelter*, as in containers, vessels, tanks, rockets, etc.

Perhaps the most important application of SORs is containment and transport of liquid and gasses. Specific examples of such structures are pressure vessels, containment vessels, pipes, cooling towers, and rotating machinery (turbines, generators, shafts, etc.).

But a SOR by itself does not necessarily define an axisymmetric problem. It is also necessary that the loading, as well as the support boundary conditions, be rotationally symmetric. This is illustrated in Figure 10.2 for loads.

If these two conditions are met:

axisymmetric geometry and *axisymmetric loading*

the response of the structure is axisymmetric (also called radially symmetric). By this is meant that all quantities of interest in structural analysis: displacement, strains, and stresses, are *independent of the circumferential coordinate* defined below.

Remark 10.1. A linear SOR under non-axisymmetric loading can be treated by a Fourier decomposition method. This involves decomposing the load into a Fourier series in the circumferential direction, calculating the response of the structure to each harmonic term retained in the series, and superposing the results. The axisymmetric problem considered here may be viewed as computing the response to the zero-th harmonic. This superposition technique, however, is limited to linear problems.

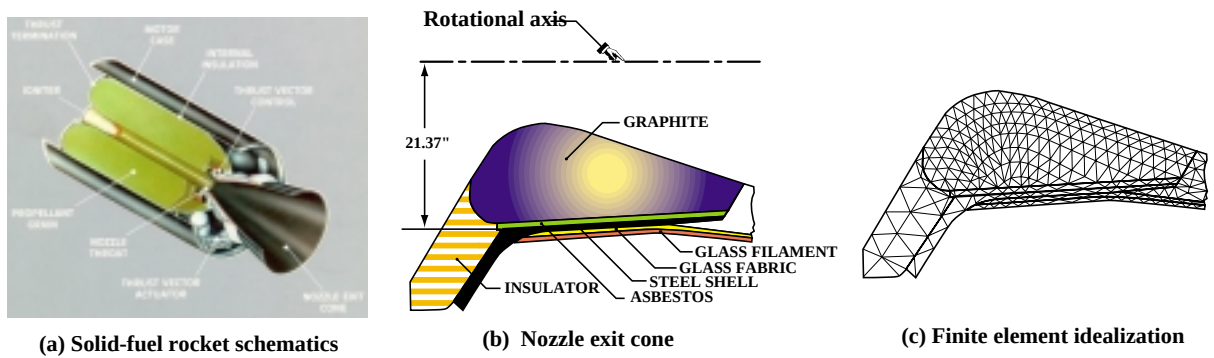


FIGURE 10.3. Axisymmetric FE analysis of a typical rocket nozzle (carried out by E. L. Wilson at Aerojet Corporation, circa 1963). Figure from paper cited in footnote 1.

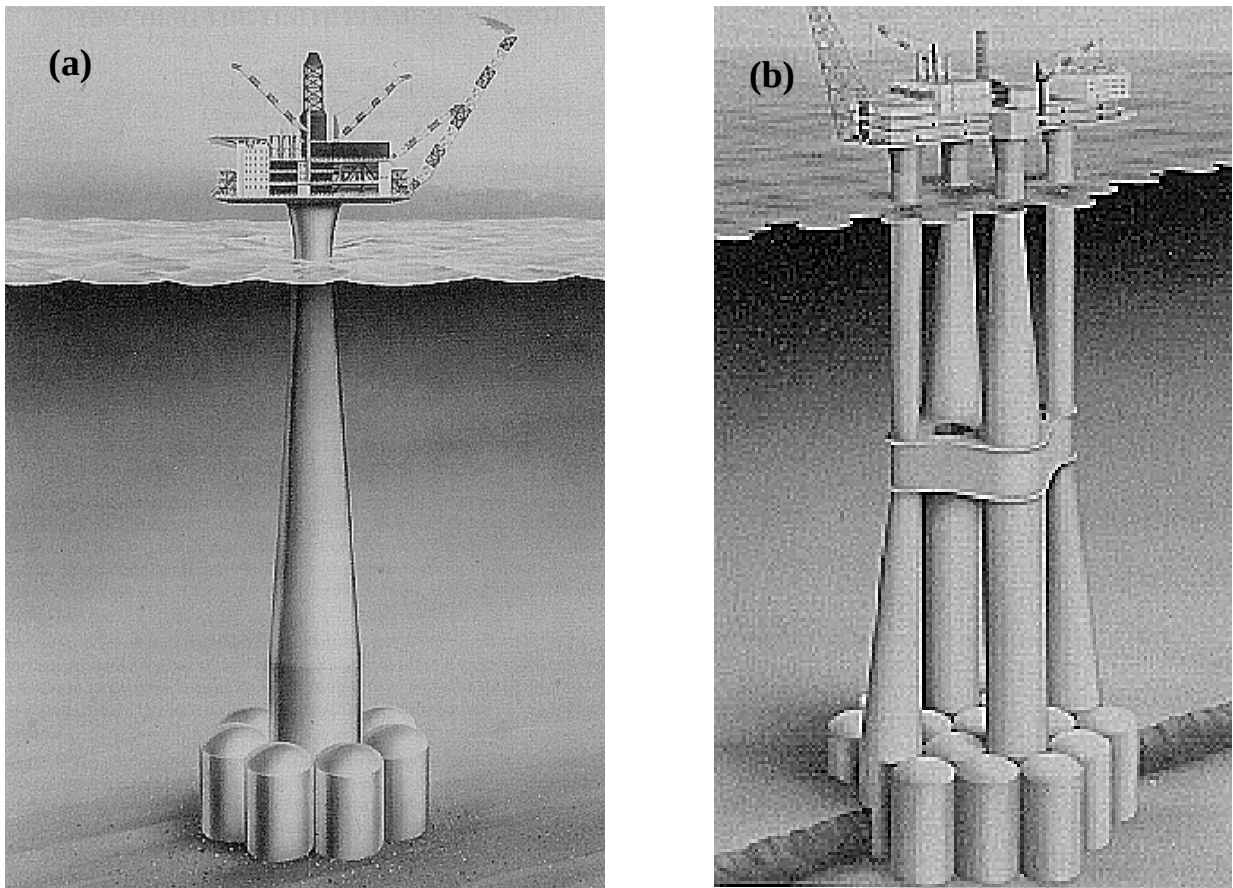


FIGURE 10.4. Two quasi-axisymmetric marine structures. (a) The Draugen oil-drilling platform (artist's sketch). The first monotower concrete platform built by Norwegian Contractors. The concrete structure is 295 m high. First deployed in 1993. The seven cells at the bottom of the sea form a reservoir system that can store up to 1.4 M barrels of oil. (b) The Troll oil-drilling platform (artist sketch). The tallest concrete platform built to date. It is 386 m tall and has 220,000 m³ of concrete. The foundation consists of 36 m tall concrete skirts that penetrate into the soft seabed.

§10.1.2. Some SOR Examples

Rocket Analysis. The analysis of axisymmetric structures by the Finite Element Method (FEM) has a long history that may be traced back to the early 1960s. Recall that the FEM originated in the aircraft industry in the mid 1950s. Aircraft are not SORs, but several structures of interest in aerospace are, notably rockets. As the FEM began to disseminate throughout the aerospace industry, interest in application to rocket analysis prompted the development of the first *axisymmetric finite elements* during the period 1960-1965. These elements were of shell and solid type. The first archival-journal paper on axisymmetric solid elements, by E. L. Wilson, appeared in 1965.¹ Figure 10.3 shows a realistic application to a rocket nozzle presented in that first paper.

SOR Members as Major Structural Components. Often important structural *components* are have axisymmetric geometry such as pipes, but the entire structure is not SOR. Two examples taken from the field of petroleum engineering are shown in Figure 10.4.² These are two recent designs of oil-drilling platforms intended for water depths of 300 to 400m. As can be observed the main structural members are axisymmetric (reinforced concrete cylindrical shells). This kind of structure is often analyzed by *global-local techniques*. In the global analysis such members are treated with beam or simplified shell models. Forces computed from the global analysis are then applied to individual members for a more detailed 3D analysis that may take advantage of axisymmetry.

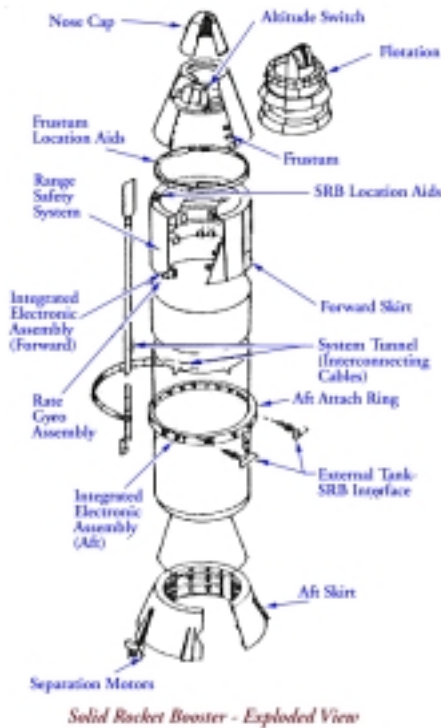


FIGURE 10.5. Solid Rocket Booster (SRB) of Space Shuttle orbiter: a quasi-axisymmetric structure.

¹ E. L. Wilson, Structural Analysis of Axisymmetric solids, *AIAA Journal*, Vol. 3, No. 12, 1965.

² From the article by B. Jacobsen, 'The evolution of the offshore concrete platform,' in *From Finite Elements to the Troll Platform — a book in honor of Ivar Holand's 70th Anniversary*, ed. by K. Bell, Dept. of Structural Engineering, The Norwegian Institute of Technology, Trondheim, Norway, 1994.

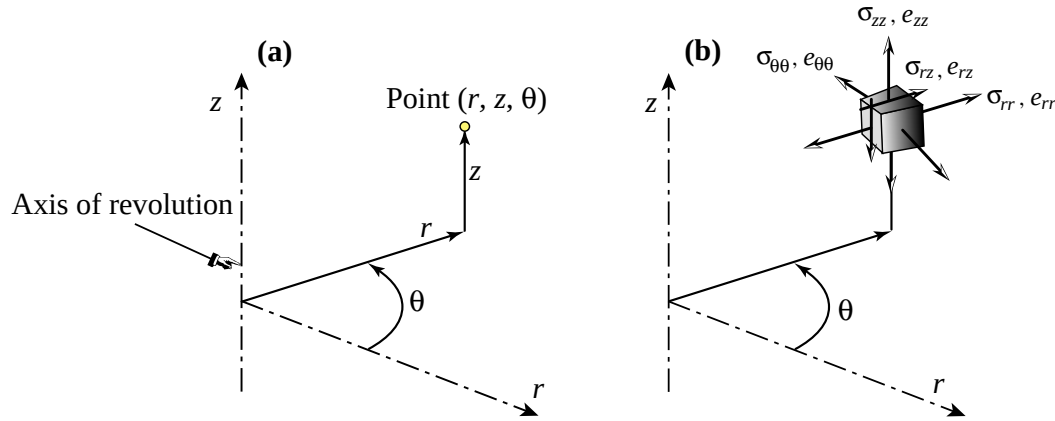


FIGURE 10.6. (a) Global cylindrical coordinate system (r, z, θ) for axisymmetric structural analysis; (b) strains and stresses with respect to cylindrical coordinate system.

Quasi-axisymmetric structures. There is an important class of structures that may be termed “quasi-axisymmetric,” in which the axisymmetric geometry is locally perturbed by non-axisymmetric features such as access openings, foundations and nonstructural attachments. Important examples are cooling towers, container vehicles, jet engines and rockets. See for example the SRB of Figure 10.5. Such structures may benefit from a global-local analysis if the axisymmetric characteristics dominate. In this case the global analysis is axisymmetric but the local analyses are not.

§10.2. The Governing Equations

§10.2.1. Global Coordinate System

To simplify the governing equations of the axisymmetric problem it is natural to use a global *cylindrical* coordinate system (r, z, θ) where

- r the *radial* coordinate: distance from the axis of revolution; always $r \geq 0$.
- z the *axial* coordinate: directed along the axis of revolution.
- θ the *circumferential* coordinate, also called the longitude.

The global coordinate system is sketched in Figure 10.6(a).

Remark 10.2. Note that $\{r, z\}$ form a right-handed Cartesian coordinate system on the $\theta = \text{const}$ planes, whereas $\{r, \theta\}$ form a polar coordinate system on the $z = \text{const}$ planes.

§10.2.2. Displacement, Strains, Stresses

The displacement field is a function of r and z only, defined by two components:

$$\mathbf{u}(r, z) = \begin{bmatrix} u_r(r, z) \\ u_z(r, z) \end{bmatrix} \quad (10.1)$$

u_r is called the *radial displacement* and u_z is the *axial displacement*. The *circumferential displacement* component, u_θ , is zero on account of rotational symmetry.

The infinitesimal strain tensor in cylindrical coordinates is represented by the symmetric matrix:

$$[\mathbf{e}] = \begin{bmatrix} e_{rr} & e_{rz} & e_{r\theta} \\ e_{rz} & e_{zz} & e_{z\theta} \\ e_{z\theta} & e_{z\theta} & e_{\theta\theta} \end{bmatrix} \quad (10.2)$$

Because of the assumed axisymmetric state, $e_{r\theta}$ and $e_{z\theta}$ vanish, leaving only four distinct components:

$$[\mathbf{e}] = \begin{bmatrix} e_{rr} & e_{rz} & 0 \\ e_{rz} & e_{zz} & 0 \\ 0 & 0 & e_{\theta\theta} \end{bmatrix} \quad (10.3)$$

Each of these vanishing components is a function of r and z only. As usual in preparation for finite element work, the nonvanishing components are arranged as a 4×1 strain vector:

$$\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} \quad (10.4)$$

in which $\gamma_{rz} = e_{rz} + e_{zr} = 2e_{rz}$. This differs from the plane stress case considered in the introductory course in the appearance of $e_{\theta\theta}$, the “hoop” or circumferential strain.

The stress tensor in cylindrical coordinates is represented by the symmetric matrix

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} & \sigma_{r\theta} \\ \sigma_{rz} & \sigma_{zz} & \sigma_{z\theta} \\ \sigma_{r\theta} & \sigma_{z\theta} & \sigma_{\theta\theta} \end{bmatrix} \quad (10.5)$$

Again because of axisymmetry the components $\sigma_{r\theta}$ and $\sigma_{z\theta}$ vanish, leaving four nontrivial components:

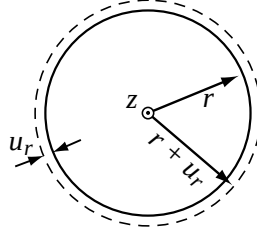
$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} & 0 \\ \sigma_{rz} & \sigma_{zz} & 0 \\ 0 & 0 & \sigma_{\theta\theta} \end{bmatrix} \quad (10.6)$$

Each of the nonvanishing components is a function of r and z . Collecting these four components into a stress vector:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} \quad (10.7)$$

where $\sigma_{rz} \equiv \sigma_{zr}$. The difference with respect to the plane stress problem is again the appearance of the “hoop” or circumferential stress $\sigma_{\theta\theta}$.

The stresses and strains over an infinitesimal volume are depicted in Figure 10.7.

FIGURE 10.7. A uniform radial displacement u_r induces a circumferential strain u_r/r .

§10.3. Governing Equations

The elasticity equations for the axisymmetric problem are the field equations: strain-displacement, stress-strain, and stress equilibrium equations, complemented by displacement and stress boundary conditions.

§10.3.1. Kinematic Equations

The strain-displacement equations for the axisymmetric problem are:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{\theta\theta} = \frac{u_r}{r}, \quad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = e_{rz} + e_{zr} = 2e_{rz}. \quad (10.8)$$

In matrix form:

$$\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} = \mathbf{D} \mathbf{u}. \quad (10.9)$$

where $\mathbf{D}$ is the 4×2 strain-displacement (symmetric-gradient) operator. A noteworthy difference with respect to the plane stress case is the appearance of the hoop strain $e_{\theta\theta} = u_r/r$. Thus a uniform radial displacement is *no longer a rigid body motion, but produces a circumferential strain*. The physical reason behind this phenomenon is illustrated in Figure 10.7. The length of the original circumference is $2\pi r$, which grows to $2\pi(r + u_r)$, inducing a strain $2\pi u_r / 2\pi r = u_r/r$.

§10.3.2. Constitutive Equations

For a linear hyperelastic material, and ignoring thermal and prestress effects, the most general constitutive equation consistent with axisymmetry takes the form:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \mathbf{E} \mathbf{e} \quad (10.10)$$

To retain axisymmetry, the cross-coupling between the shear strain and hoop stress must vanish. Consequently $E_{34} = E_{43} = 0$.

For an isotropic material of elastic modulus E and Poisson's ratio ν ,

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \quad (10.11)$$

Remark 10.3. The coefficients of $\mathbf{E}$ go to infinity if $\nu \rightarrow 1/2$, which characterizes an incompressible material. This behavior is a consequence of the “confinement” effect in solids and appears also in general 3D analysis. On the other hand, the plane stress constitutive matrix remains finite for $\nu = \frac{1}{2}$, a behavior that is characteristic of thin bodies such as plates and shells. Physically, the small transverse dimension of bodies in plane stress (plates) allows the material to freely expand or contract in the z direction.

§10.3.3. Equilibrium Equations

The general (three dimensional) differential equations of equilibrium in cylindrical coordinates are

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta}(\sigma_{r\theta}) + \frac{\partial}{\partial z}\sigma_{rz} - \frac{\sigma_{\theta\theta}}{r} + b_r &= 0 \\ \frac{1}{r} \frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{1}{r}\sigma_{z\theta} + \frac{\partial}{\partial z}\sigma_{zz} + b_z &= 0 \\ \frac{1}{r^2} \frac{\partial}{\partial r}(r^2\sigma_{\theta r}) + \frac{1}{r} \frac{\partial}{\partial \theta}\sigma_{\theta\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + b_\theta &= 0 \end{aligned} \quad (10.12)$$

where b_r , b_z , b_θ are the components of the body force field in the r , z and θ directions, respectively. For the axisymmetric problem these equations reduce to

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial}{\partial z}\sigma_{rz} - \frac{\sigma_{\theta\theta}}{r} + b_r &= 0 \\ \frac{1}{r} \frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{\partial}{\partial z}\sigma_{zz} + b_z &= 0 \end{aligned} \quad (10.13)$$

The third equation in (10.12) is identically satisfied if $b_\theta = 0$, because $\sigma_{\theta r} = \sigma_{\theta z} = 0$ and $\sigma_{\theta\theta}$ is independent of θ . If $b_\theta \neq 0$ the problem cannot be treated as axisymmetric.

§10.3.4. Boundary Conditions

As usual boundary conditions can be of displacement (PBC) or of stress or traction (FBC) type. They are specified on portions S_u and S_t of the boundary, respectively. The reduction of the stress BCs to two dimension is further discussed in §10.4.2 and §10.4.3.

§10.4. Variational Formulation

The variational form of the axisymmetric problem is illustrated with the widely used Total Potential Energy (TPE) form. The delicate part of the formulation is the dimensionality reduction step.

§10.4.1. The TPE Functional

The Total Potential Energy (TPE) functional contains only displacements as master field:

$$\Pi[\mathbf{u}] = U[\mathbf{u}] - W[\mathbf{u}]. \quad (10.14)$$

Here the strain energy functional is

$$U[\mathbf{u}] = \frac{1}{2} \int_V \boldsymbol{\sigma}^T \mathbf{e} dV = \frac{1}{2} \int_V \mathbf{e}^T \mathbf{E} \mathbf{e} dV = \frac{1}{2} \int_V \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} dV. \quad (10.15)$$

In (10.15) the strains are a slave field are derived from displacements. Superscript u used in Chapter 3–8 to identify the master field is omitted to reduce clutter.

The external work potential is the sum of contributions due to body force and prescribed surface tractions:

$$\begin{aligned} W[\mathbf{u}] &= W_b[\mathbf{u}] + W_t[\mathbf{u}] \\ W_b[\mathbf{u}] &= \int_V \mathbf{b}^T \mathbf{u} dV = \int_V [b_r \quad b_z] \begin{bmatrix} u_r \\ u_z \end{bmatrix} dV \\ W_t[\mathbf{u}] &= \int_{S_t} \hat{\mathbf{t}}^T \mathbf{u} dS = \int_{S_t} [\hat{t}_r \quad \hat{t}_z] \begin{bmatrix} u_r \\ u_z \end{bmatrix} dS \end{aligned} \quad (10.16)$$

Here $\mathbf{b}$ is the body force vector and $\hat{\mathbf{t}}$ the vector of surface tractions.

§10.4.2. Dimensionality Reduction

The element of volume dV that appears in U and W_b can be expressed as the “ring element”

$$dV = 2\pi r dA \quad (10.17)$$

where dA is the element of area in the generating cross section. Insertion in (10.15) and the second of (10.16) reduces U and W_b to area integrals:

$$U = \frac{1}{2} 2\pi \int_A r \mathbf{e}^T \mathbf{E} \mathbf{e} dA \quad (10.18)$$

$$W_b = 2\pi \int_A r \mathbf{b}^T \mathbf{u} dA \quad (10.19)$$

Notice the appearance of r in the integrand.

Similarly, the element of surface dS in W_t can be expressed as

$$dS = 2\pi r ds \quad (10.20)$$

where ds is an arclength element. Inserting in the last of (10.16) reduces W_t to a one-dimensional (line) integral

$$W_t = 2\pi \int_{s_t} r \mathbf{t}^T \mathbf{u} ds \quad (10.21)$$

The common factor 2π in these integrals is (usually) suppressed in the finite element implementation. This should not cause difficulties except for the case of a concentrated load, as discussed in the following subsection.

We summarize the outcome of this dimensionality reduction by saying that the original three-dimensional problem has been reduced to a two-dimensional one.

§10.4.3. Line and Point Forces

Body forces (e.g. gravity or centrifugal forces) and distributed surface forces (e.g. pressure) are handled like in plane elasticity case explained in LFEM, but concentrated loads require more careful treatment. There are two possibilities: a line load and an actual concentrated load.

A line load is actually a “ring” load (see Figure 10.2) acting on a circle described by a point of the generating cross section. If the global components of this load are F_r and F_z , the appropriate energy contribution to the loads potential W is

$$W_F = 2\pi r (F_r u_r + F_z u_z) \quad (10.22)$$

where (u_r, u_z) are the displacements of the “ring” point. Thus the ubiquitous 2π term can be suppressed

A concentrated or point load F , however, can only act along the z direction at points on the axis of revolution as illustrated in Figure 10.2. The corresponding work term is

$$W_F = F u_z \quad (10.23)$$

so the factor 2π is missing. To render this compatible with the other energy terms the load is divided by 2π , so the contribution to the external loads potential is

$$W_F = 2\pi \left(\frac{F}{2\pi} \right) u_z \quad (10.24)$$

This device can be visualized by regarding F as the limit of a z -directed ring load F_z as $r \rightarrow 0$.

Remark 10.4. What the last equation means in practice is that if a concentrated force of, say, 1000 lb acts on the z axis, it has to be divided by 2π (that is, $1000/2\pi$) before giving it to a SOR finite element program if the factor of 2π has been suppressed. (It is important to read the users manual to see if that is the case.)

§10.4.4. Other Variational Forms

The Hellinger-Reissner (HR) functional and the equilibrium-stress hybrid functionals are derived in the Exercises.

§10.5. Treating Plane Strain as a Limit Case

The problem of *plane strain* may be viewed as the limit of the axisymmetric case in which the axis of revolution is moved to infinity so that $r \rightarrow \infty$, and a “slice” of unit thickness is taken.

Thus a finite element program that handles the axisymmetric problem may be used to solve problems of plane strain with acceptable approximation.

Homework Exercises for Chapter 10
Axisymmetric Solids (Structures of Revolution)

EXERCISE 10.1 [A:20] Derive the HR functional for the axisymmetric solid problem. Use compact matrix notation, as done in §10.4.1 for the TPE form, because indicial notation does not fit this particular problem well. In matrix notation, the complementary energy density is $\mathcal{U}^* = \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{C} \boldsymbol{\sigma}$, in which $\boldsymbol{\sigma}$ is the stress vector (10.7) and $\mathbf{C} = \mathbf{E}^{-1}$ the 4×4 elastic compliance matrix, with $\mathbf{E}$ given by (10.10).

Is there any difference in the treatment of body forces and surface tractions with respect to the TPE form?

EXERCISE 10.2 [A:20] Derive the equilibrium-stress hybrid functional for the axisymmetric solid problem. Use compact matrix notation, as done in §10.4.1 for the TPE form, because indicial notation does not fit this particular problem well. In matrix notation, the complementary energy density is $\mathcal{U}^* = \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{C} \boldsymbol{\sigma}$, in which $\boldsymbol{\sigma}$ is the stress vector (10.7) and $\mathbf{C} = \mathbf{E}^{-1}$ the 4×4 elastic compliance matrix, with $\mathbf{E}$ given by (10.10).

Is there any difference in the treatment of body forces and surface tractions with respect to the TPE form?

11

Axisymmetric Solid Iso-P Elements

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§11.1. Introduction

In this Chapter we consider the finite element discretization of axisymmetric solids that form structures of revolution (SOR). The focus is on isoparametric (iso-P) elements. This focus links up with matter covered in the Introduction to Finite Elements (IFEM) book.

The dimensionality reduction process described in the previous Chapter “folds” the three-dimensional problem into integrals taken over the generating cross section and its boundaries. The finite element discretization can be therefore confined to the $\{r, z\}$ plane. The circumferential (θ) dimension conceptually disappears from the FEM discretization. The resulting finite elements are called *axisymmetric solid elements*, SOR elements, or “ring” elements in the literature. In this and following chapters, the term *ring element* will be often used for brevity.

A ring element is completely defined geometry of its generating cross section in the (r, z) plane, as illustrated in Figure 11.1.

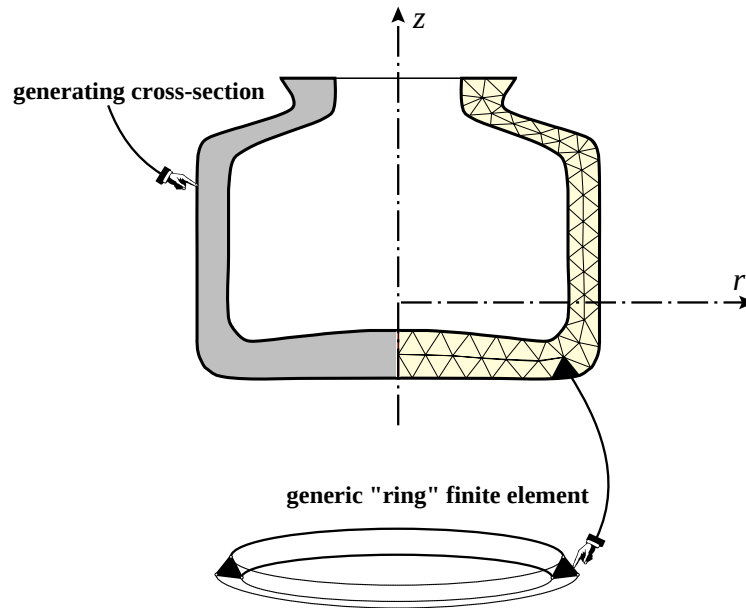


FIGURE 11.1. Axisymmetric solid “ring” element.

Since this cross section is plane, the element geometry definition is two-dimensional. It follows that the two-dimensional geometrical configurations previously studied in IFEM (for example, 3 and 6-node triangles, 4-, 8-, and 9-node quadrilaterals, etc) for the plane stress case can be reused.

The major modeling difference with respect to the plane stress case is the appearance of the circumferential or “hoop” strain and stress, which together contribute a term

$$\frac{1}{2} \sigma_{\theta\theta} \epsilon_{\theta\theta} \quad (11.1)$$

to the strain energy density. This energy term injects an extra row in the strain-displacement matrix **B**. Furthermore the constitutive matrix must be adjusted to reflect the presence of hoop effects as well as absence of the plane stress assumption. Taken together, these changes must be accounted for in the formation of the element stiffness matrix as well as the displacement-to-stress recovery.

§11.2. Isoparametric Definition

Ring finite elements for axisymmetric solids are developed in this and next two Chapters using the displacement-based isoparametric (iso-P) formulation, which was introduced in Chapter 16 of IFEM. The key concept is that both the element cross section geometry and the displacement field are interpolated by the same shape functions. Two degrees of freedom: the radial and axial displacements $\{u_r, u_z\}$ are specified at each node. The iso-P definition of an element with n nodes is

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ z_1 & z_2 & \cdots & z_n \\ u_{r1} & u_{r2} & \cdots & u_{rn} \\ u_{z1} & u_{z2} & \cdots & u_{zn} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}. \quad (11.2)$$

In the left-hand side, $u_r = u_r(r, z)$ and $u_z = u_z(r, z)$ are element displacements in terms of their node values u_{ri} and u_{zi} , respectively and of the shape functions N_i^e collected on the rightmost column vector. The element geometry, represented by the position coordinates r and z in the left hand side, are defined by the same interpolation. The first row of (11.2) expresses that the sum of shape functions $N_1^e + N_2^e + \dots + N_n^e$ is identically one.

The element shape functions N_i are written in terms of the element natural coordinates introduced in IFEM: $\{\zeta_1, \zeta_2, \zeta_3\}$ for triangles and $\{\xi, \eta, \zeta\}$ for quadrilaterals. The same continuity and completeness requirements discussed in that course apply with minor differences (see remarks below).

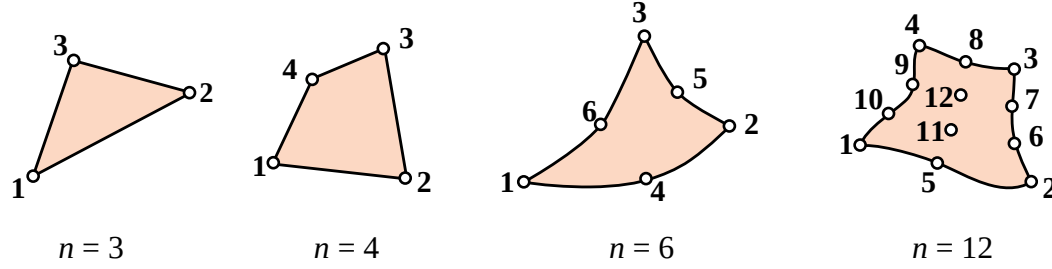
Remark 11.1. Displacement based elements are based on the TPE functional. The variational index for the only master field: displacements, is one. Consequently the interelement continuity required is C^0 . That is, the displacement components u_r and u_z must be continuous between adjacent elements. Element boundaries lying on the axis of revolution are special: at such points the radial displacement u_r must vanish if the structure is continuous there (that is, a tiny hole at $r = 0$ is precluded) although there is no need to make that condition explicit at the element level. That constraint is applied through displacement boundary conditions at the assembly level.

Remark 11.2. The completeness criterion demands that all rigid body modes and strain states be exactly represented. This is met if the element shape functions can represent any displacement field that is linear in r and z . (In fact, a more detailed analysis shows that this is a slight overkill because the hoop strain is not a displacement differential, but will do for now.)

§11.3. The Element Stiffness Matrix

We continue to assume a generic iso-P ring element with n nodes. Some specific geometric configurations are shown in Figure 11.2. Two freedoms are chosen at each node: the radial and axial displacement components. Thus the element has $2n$ degrees of freedom. As usual, the node displacement vector $\mathbf{u}^e$ is a column $2n$ -vector constructed by arranging those freedoms node by node:

$$\mathbf{u}^e = [u_{r1} \quad u_{z1} \quad u_{r2} \quad u_{z2} \quad \cdots \quad u_{rn} \quad u_{zn}]^T \quad (11.3)$$

FIGURE 11.2. Ring element cross sections characterized by their number of nodes n .

§11.3.1. Displacement Interpolation

As displayed in the iso-P definition (11.2), both displacement components $u_r(r, z)$ and $u_z(r, z)$ are interpolated over the element by the same shape functions. It is convenient to express that interpolation in terms of the node displacement vector (11.3). Extracting the corresponding rows of (11.2) and transposing we obtain

$$\begin{bmatrix} u_r \\ u_z \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \dots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \dots & 0 & N_n^e \end{bmatrix} \mathbf{u}^e = \mathbf{N} \mathbf{u}^e. \quad (11.4)$$

This $\mathbf{N}$ (with superscript e omitted to reduce clutter) is called the *shape function matrix*. It has dimensions $2 \times 2n$. For example, if the element has 4 nodes (as in the second one from the left in Figure 11.2), $\mathbf{N}$ is 2×8 .

§11.3.2. The Strain-Displacement Matrix

Differentiating the finite element displacement interpolation (11.4) to get the strains $e_{rr} = \partial u_r / \partial r$, $e_{zz} = \partial u_z / \partial z$, $e_{\theta\theta} = u_r / r$ and $2e_{rz} = \partial u_r / \partial z + \partial u_z / \partial r$ yields the strain-displacement relations:

$$\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial r} & 0 & \frac{\partial N_2^e}{\partial r} & 0 & \dots & \frac{\partial N_n^e}{\partial r} & 0 \\ 0 & \frac{\partial N_1^e}{\partial z} & 0 & \frac{\partial N_2^e}{\partial z} & \dots & 0 & \frac{\partial N_n^e}{\partial z} \\ \frac{N_1^e}{r} & 0 & \frac{N_2^e}{r} & 0 & \dots & \frac{N_n^e}{r} & 0 \\ \frac{\partial N_1^e}{\partial z} & \frac{\partial N_1^e}{\partial r} & \frac{\partial N_2^e}{\partial z} & \frac{\partial N_2^e}{\partial r} & \dots & \frac{\partial N_n^e}{\partial z} & \frac{\partial N_n^e}{\partial r} \end{bmatrix} \mathbf{u}^e = \mathbf{B} \mathbf{u}^e. \quad (11.5)$$

This $\mathbf{B} = \mathbf{D} \mathbf{N}$ is called the *strain-displacement matrix*. It is dimensioned $3 \times 2n$. For example, if the element has 6 nodes, $\mathbf{B}$ is 3×12 . Note that terms in the third row of $\mathbf{B}$ incur division by zero if $r = 0$, which happens at points on the axis of revolution. This can be the source of numerical difficulties that are discussed later.

The stresses are given in terms of strains and displacements by

$$\boldsymbol{\sigma} = \mathbf{E} \mathbf{e} = \mathbf{E} \mathbf{B} \mathbf{u}^e. \quad (11.6)$$

The expression of the 4×4 elasticity matrix $\mathbf{E}$ is given in the previous Chapter for a general anisotropic material and its isotropic specialization.

Both (11.5) and (11.6) are assumed to hold at all points of the element domain, since they correspond to the strong links in the TPE diagram.

Comparing these expressions with those in Chapter 14 of IFEM, we see that $\mathbf{N}$ is the same but $\mathbf{B}$ has an extra row, which is associated with the hoop strain.

§11.3.3. The Element Stiffness Equations

Inserting the foregoing expressions into the TPE functional and integrating gives the quadratic matrix form

$$\Pi^e = \frac{1}{2} \mathbf{u}^{eT} \mathbf{K}^e \mathbf{u}^e - \mathbf{u}^{eT} \mathbf{f}^e. \quad (11.7)$$

Here the stiffness matrix is

$$\mathbf{K}^e = \int_{\Omega^e} r \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega, \quad (11.8)$$

while the consistent element nodal force vector combines body force and surface traction effects:

$$\mathbf{f}^e = \mathbf{f}_b^e + \mathbf{f}_t^e = \int_{\Omega^e} r \mathbf{N}^T \mathbf{b} d\Omega + \int_{\Gamma^e} r \mathbf{N}^T \hat{\mathbf{t}} d\Gamma. \quad (11.9)$$

In these expressions Ω^e and Γ^e denote the element domain and boundary, respectively. To justify the names given to $\mathbf{K}^e$ and $\mathbf{f}^e$, take the variation of (11.7) with respect to the node displacements:

$$\delta \Pi^e = (\delta \mathbf{u}^e)^T [\mathbf{K}^e \mathbf{u}^e - \mathbf{f}^e] = \mathbf{0}. \quad (11.10)$$

Since $\delta \mathbf{u}^e$ is arbitrary, the expression in brackets must vanish, which yields the element stiffness equations

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e. \quad (11.11)$$

Remark 11.3. Comparing (11.8) and (11.9) with the expressions given for the plane stress problem in Chapter 14 of IFEM, we note that the plate thickness h has been replaced by the radial coordinate r . Unlike h , this coordinate cannot generally be taken out of the integral. Note also that the circumferential factor 2π has been dropped from the integrals that appear (11.8) and (11.9); the main consequence being that axial point along the axis of revolution forces must be divided by 2π as discussed in the previous Chapter.

§11.3.4. *Thermal Effects

Suppose the temperature of the structure changes axisymmetrically by $\Delta T(r, z)$ from a reference temperature. The constitutive equations become

$$\boldsymbol{\sigma} = \mathbf{E}(\mathbf{e} - \boldsymbol{\alpha} \Delta T) \quad (11.12)$$

where $\boldsymbol{\alpha}$ is an array of dilatation coefficients that provide thermal strains in a mechanically unconstrained body. (A linear relation between strains and temperature changes is assumed.) For isotropic materials

$$\boldsymbol{\alpha}^T = [\alpha \quad \alpha \quad \alpha \quad 0] \quad (11.13)$$

where α is the usual coefficient of dilatation of the material. On inserting (11.13) into the potential energy formulation for a generic element yields element stiffness equations that account for thermal effects:

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e = \mathbf{f}_M^e + \mathbf{f}_T^e \quad (11.14)$$

where $\mathbf{f}_M$ are the mechanical forces considered so far, and $\mathbf{f}_T$, called the *thermal forces* or *equivalent thermal loads*, account for contribution from the temperature changes:

$$\mathbf{f}_T^e = \int_A \mathbf{B}^T \alpha \Delta T r dA \quad (11.15)$$

This term can be evaluated by numerical integration.

The sum of mechanical and thermal node forces is sometimes called the *effective node force vector* or simply *effective forces* in the FEM literature.

§11.4. Element Computations

The entries of the element stiffness matrix $\mathbf{K}^e$ and consistent node force vector $\mathbf{f}^e$ are usually computed by numerical integration techniques that rely on the Gauss quadrature formulas covered in Chapters 17, 23 and 24 of IFEM. (Some of that material is reproduced in §11.5 for convenience.)

§11.4.1. Stiffness Matrix Integration

For specificity suppose that the stiffness matrix of a *quadrilateral* isoparametric element is to be numerically integrated by a p -point Gauss quadrature rule along each isoparametric coordinate. Denote by ξ_k and η_ℓ the Gauss points abscissae whereas w_k and w_ℓ denote the corresponding integration weights, with indices k and ℓ running from 1 through p . Then

$$\mathbf{K}^e = \sum_{k=1}^p \sum_{\ell=1}^p w_k w_\ell \mathbf{B}^T(\xi_k, \eta_\ell) \mathbf{E} \mathbf{B}(\xi_k, \eta_\ell) r(\xi_k, \eta_\ell) J_\Omega(\xi_k, \eta_\ell). \quad (11.16)$$

Here $\mathbf{B}(\xi_k, \eta_\ell)$ means $\mathbf{B} = \mathbf{B}(\xi, \eta)$ evaluated at the Gauss point; likewise for $r(\xi_k, \eta_\ell)$ and $J_\Omega(\xi_k, \eta_\ell)$. The latter is the Jacobian determinant that maps the $\{r, z\}$ area element to that in the natural coordinates $\{\xi, \eta\}$ as discussed in Chapter 17 of IFEM. Usually the elasticity matrix $\mathbf{E}$ is assumed constant over the element, which explains its lack of arguments in (11.16).

§11.4.2. Consistent Body Force Vector Integration

Body forces (also called volume forces) arise frequently in analysis of SOR. The most important loads of this type are

1. Gravity (own weight). This effect is important in massive SOR, as encountered in civil, geophysical and nuclear applications.
2. Centrifugal forces in rotating structures. These are important in aerospace and mechanical structures (for example, high-speed rotating machinery such as turbines).
3. Thermal, shrinkage and prestress effects. These may be important depending on fabrication techniques, material, and the environment to which the structure will be exposed.

The consistent force vector for body forces may be also evaluated by numerical Gauss quadrature resulting in an expression similar to (11.16):

$$\mathbf{f}_b^e = \sum_{k=1}^p \sum_{\ell=1}^p w_k w_\ell \mathbf{N}^T(\xi_k, \eta_\ell) \mathbf{b}(\xi_k, \eta_\ell) r(\xi_k, \eta_\ell) J_\Omega(\xi_k, \eta_\ell). \quad (11.17)$$

Note that here we evaluate the shape function matrix $\mathbf{N}$ and the body force vector $\mathbf{b}$ at the Gauss point. While the former is supplied as part of the iso-P definition (11.9), the question arises on how to interpolate $\mathbf{b}$ to get $\mathbf{b}(\xi_k, \eta_\ell)$. Two possibilities:

Element Level Specification. $\mathbf{b}$ is defined element by element; for example constant b_r and b_z that may jump from element to element.

Node Level Specification. $\mathbf{b}$ is defined by nodal values, and interpolated by the same shape functions as displacements.

Both choices are common in practice, but the first is more flexible in accomodating interelement force discontinuities. For example the body force due to gravity or centrifugal effects jumps between materials of different density. The second choice is appropriate for a continuous body force field, and especially when $\mathbf{b}$ can be defined as a function of the position coordinates.

§11.4.3. Consistent Surface Traction Integration

The consistent force vector associated with surface tractions is often simple enough that it can be precomputed by hand using NbN or EbE lumping, as explained in Chapter 7 of IFEM, and fed to the FEM program as node forces.

For more difficult cases involving for example space-varying pressures or oblique surfaces, unidimensional numerical integration may also be applied:

$$\mathbf{f}_t^e = \sum_{k=1}^p w_k \mathbf{N}^T(\xi_k) \hat{\mathbf{t}}(\xi_k) r(\xi_k) J_\Gamma(\xi_k). \quad (11.18)$$

Here ξ is an isoparametric arclength coordinate, customized to traverse element sides on which a nonzero traction vector $\hat{\mathbf{t}}$ is given, and J_Γ the associated arclength Jacobian. If this procedure is implemented, $\hat{\mathbf{t}}$ is generally defined at designated elements traversed side by side.

§11.4.4. Preventing Division by Zero

One glance at the strain displacement matrix $\mathbf{B}$ in (11.5) shows that the third row terms involve a division by zero if $r = 0$, that is, at points on the axis of revolution. It follows that one should avoid evaluating that matrix there, as that could abort the calculations. This may happen if one or more elements extends to the axis, as is the case in the lower portion of the structure shown in Figure 11.1. Fortunately it is not difficult to avoid trouble by sticking to two common sense rules:

1. When numerically integrating over element areas, as in the computation of $\mathbf{K}^e$ in (11.16), use only quadrature rules with *internal* sample points. For the quadrilateral geometry, all Gauss-product rules comply with this restriction.
2. In the postprocessing stage, avoid evaluating $\mathbf{B}$ directly at element nodes for strain recovery from displacements. Instead, evaluate at interior points (usually Gauss points) and extrapolate to nodes as discussed in Chapter 30 of IFEM. The stress recovery routines described in the next two Chapters make use of this avoidance technique.

§11.5. Numerical Integration by Gauss Rules

The following material on numerical integration is transcribed here from Chapter 17 of the IFEM Notes for convenience. The Gauss quadrature information modules listed below are used in the element implementations covered in the next two Chapters.

The use of numerical integration is essential for practical evaluation of integrals over isoparametric element domains. The standard practice has been to use *Gauss integration* because such rules use a *minimal number of sample points to achieve a desired level of accuracy*. This economy is important for efficient element calculations, since a *matrix product* is evaluated at each sample point. The fact that the location of the sample points in Gauss rules is usually given by non-rational numbers is of no concern in digital computation.

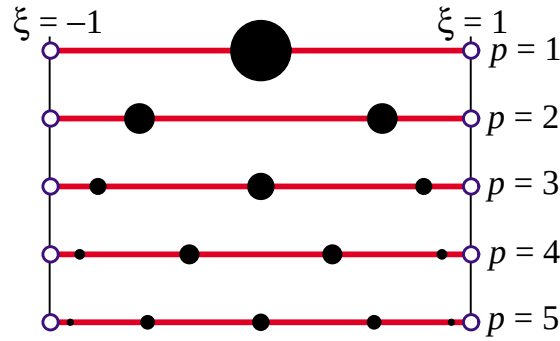


FIGURE 11.3. The first five unidimensional Gauss rules $p = 1, 2, 3, 4, 5$ depicted over the line segment $\xi \in [-1, +1]$. Sample point locations are marked with black circles. The radii of those circles are proportional to the integration weights.

§11.5.1. One Dimensional Rules

The classical Gauss integration rules are defined by

$$\int_{-1}^1 F(\xi) d\xi \approx \sum_{i=1}^p w_i F(\xi_i). \quad (11.19)$$

Here $p \geq 1$ is the number of Gauss integration points (also known as sample points), w_i are the integration weights, and ξ_i are sample-point abscissae in the interval $[-1, 1]$. The use of the canonical interval $[-1, 1]$ is no restriction, because an integral over another range, say from a to b , can be transformed to $[-1, +1]$ via a simple linear transformation of the independent variable, as shown in the Remark below.

The first five unidimensional Gauss rules, illustrated in Figure 11.3, are listed in Table 11.1. These integrate exactly polynomials in ξ of orders up to 1, 3, 5, 7 and 9, respectively. In general a unidimensional Gauss rule with p points integrates exactly polynomials of order up to $2p - 1$. This is called the *degree* of the formula.

Table 11.1 - One Dimensional Gauss Rules with 1 through 5 Sample Points

Points	Rule
1	$\int_{-1}^1 F(\xi) d\xi \approx 2F(0)$
2	$\int_{-1}^1 F(\xi) d\xi \approx F(-1/\sqrt{3}) + F(1/\sqrt{3})$
3	$\int_{-1}^1 F(\xi) d\xi \approx \frac{5}{9}F(-\sqrt{3/5}) + \frac{8}{9}F(0) + \frac{5}{9}F(\sqrt{3/5})$
4	$\int_{-1}^1 F(\xi) d\xi \approx w_{14}F(\xi_{14}) + w_{24}F(\xi_{24}) + w_{34}F(\xi_{34}) + w_{44}F(\xi_{44})$
5	$\int_{-1}^1 F(\xi) d\xi \approx w_{15}F(\xi_{15}) + w_{25}F(\xi_{25}) + w_{35}F(\xi_{35}) + w_{45}F(\xi_{45}) + w_{55}F(\xi_{55})$

For the 4-point rule, $\xi_{34} = -\xi_{24} = \sqrt{(3 - 2\sqrt{6/5})/7}$, $\xi_{44} = -\xi_{14} = \sqrt{(3 + 2\sqrt{6/5})/7}$,
 $w_{14} = w_{44} = \frac{1}{2} - \frac{1}{6}\sqrt{5/6}$, and $w_{24} = w_{34} = \frac{1}{2} + \frac{1}{6}\sqrt{5/6}$.

For the 5-point rule, $\xi_{55} = -\xi_{15} = \frac{1}{3}\sqrt{5 + 2\sqrt{10/7}}$, $\xi_{45} = -\xi_{35} = \frac{1}{3}\sqrt{5 - 2\sqrt{10/7}}$, $\xi_{35} = 0$,
 $w_{15} = w_{55} = (322 - 13\sqrt{70})/900$, $w_{25} = w_{45} = (322 + 13\sqrt{70})/900$ and $w_{35} = 512/900$.

```

LineGaussRuleInfo[{rule_, number_}, point_] := Module[
{g2={-1,1}/Sqrt[3], w3={5/9, 8/9, 5/9},
g3={-Sqrt[3/5], 0, Sqrt[3/5]},
w4={(1/2)-Sqrt[5/6]/6, (1/2)+Sqrt[5/6]/6,
(1/2)+Sqrt[5/6]/6, (1/2)-Sqrt[5/6]/6},
g4={-Sqrt[(3+2*Sqrt[6/5])/7], -Sqrt[(3-2*Sqrt[6/5])/7],
Sqrt[(3-2*Sqrt[6/5])/7], Sqrt[(3+2*Sqrt[6/5])/7]},
g5={-Sqrt[5+2*Sqrt[10/7]], -Sqrt[5-2*Sqrt[10/7]], 0,
Sqrt[5+2*Sqrt[10/7]], Sqrt[5-2*Sqrt[10/7]]}/3,
w5={322-13*Sqrt[70], 322+13*Sqrt[70], 512,
322+13*Sqrt[70], 322-13*Sqrt[70]}/900,
i=point, p=rule, info={{Null, Null}, 0}},
If [p==1, info={0, 2}];
If [p==2, info={g2[[i]], 1}];
If [p==3, info={g3[[i]], w3[[i]]}];
If [p==4, info={g4[[i]], w4[[i]]}];
If [p==5, info={g5[[i]], w5[[i]]}];
If [number, Return[N[info]], Return[Simplify[info]]];
];

```

FIGURE 11.4. A *Mathematica* module that returns the first five unidimensional Gauss rules.

Remark 11.4. A more general integral, such as $F(x)$ over $[a, b]$ in which $\ell = b - a > 0$, is transformed to the canonical interval $[-1, 1]$ through the mapping $x = \frac{1}{2}a(1 - \xi) + \frac{1}{2}b(1 + \xi) = \frac{1}{2}(a + b) + \frac{1}{2}\ell\xi$, or $\xi = (2/\ell)(x - \frac{1}{2}(a + b))$. The Jacobian of this mapping is $J = dx/d\xi = 1/2\ell$. Thus

$$\int_a^b F(x) dx = \int_{-1}^1 F(\xi) J d\xi = \int_{-1}^1 F(\xi) \frac{1}{2}\ell d\xi. \quad (11.20)$$

Remark 11.5. Higher order Gauss rules are tabulated in standard manuals for numerical computation. For example, the widely used Handbook of Mathematical Functions [1] lists (in Table 25.4) rules with up to 96 points. For $p > 6$ the abscissas and weights of sample points are not expressible as rational numbers or radicals, and can only be given as floating-point numbers.

§11.5.2. Implementation of 1D Rules

The *Mathematica* module shown in Figure 11.4 returns either exact or floating-point information for the first five unidimensional Gauss rules. To get information for the i^{th} point of the p^{th} rule, in which $1 \leq i \leq p$ and $p = 1, 2, 3, 4, 5$, call the module as

$$\{x_{ii}, w_i\} = \text{LineGaussRuleInfo}[\{p, \text{numer}\}, i] \quad (11.21)$$

Logical flag *numer* is *True* to get numerical (floating-point) information, or *False* to get exact information. The module returns the sample point abscissa ξ_i in x_{ii} and the weight w_i in w_i . If p is not in the implemented range 1 through 5, the module returns $\{\text{Null}, 0\}$.

Example 11.1. $\{x_i, w\} = \text{LineGaussRuleInfo}[\{3, \text{False}\}, 2]$ returns $x_i=0$ and $w=8/9$, whereas $\{x_i, w\} = \text{LineGaussRuleInfo}[\{3, \text{True}\}, 2]$ returns (to 16 places) $x_i=0.$ and $w=0.888888888888889$.

§11.5.3. Two Dimensional Rules

The simplest two-dimensional Gauss rules are called *product rules*. They are obtained by applying the unidimensional rules to each independent variable in turn. To apply these rules we must first reduce the integrand to the canonical form:

$$\int_{-1}^1 \int_{-1}^1 F(\xi, \eta) d\xi d\eta = \int_{-1}^1 d\eta \int_{-1}^1 F(\xi, \eta) d\xi. \quad (11.22)$$

Once this is done we can process numerically each integral in turn:

$$\int_{-1}^1 \int_{-1}^1 F(\xi, \eta) d\xi d\eta = \int_{-1}^1 d\eta \int_{-1}^1 F(\xi, \eta) d\xi \approx \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} w_i w_j F(\xi_i, \eta_j). \quad (11.23)$$

where p_1 and p_2 are the number of Gauss points in the ξ and η directions, respectively. Usually the same number $p = p_1 = p_2$ is chosen if the shape functions are taken to be the same in the ξ and η directions. This is in fact the case for all quadrilateral elements presented here. The first four two-dimensional Gauss product rules with $p = p_1 = p_2$ are illustrated in Figure 11.5.

§11.5.4. Implementation of 2D Gauss Rules

The *Mathematica* module listed in Figure 11.6 implements two-dimensional product Gauss rules having 1 through 5 points in each direction. The number of points in each direction may be the same or different. If the rule has the same number of points p in both directions the module is called in either of two ways:

$$\begin{aligned} \{\{x_{ii}, \eta_{aj}\}, w_{ij}\} &= \text{QuadGaussRuleInfo}[\{p, \text{numer}\}, \{i, j\}] \\ \{\{x_{ii}, \eta_{aj}\}, w_{ij}\} &= \text{QuadGaussRuleInfo}[\{p, \text{numer}\}, k] \end{aligned} \quad (11.24)$$

The first form is used to get information for point $\{i, j\}$ of the $p \times p$ rule, in which $1 \leq i \leq p$ and $1 \leq j \leq p$. The second form specifies that point by a “visiting counter” k that runs from 1 through p^2 ; if so $\{i, j\}$ are internally extracted¹ as $j = \text{Floor}[(k-1)/p] + 1$; $i = k - p*(j-1)$.

¹ Indices i and j are denoted by $i1$ and $i2$, respectively, inside the module.

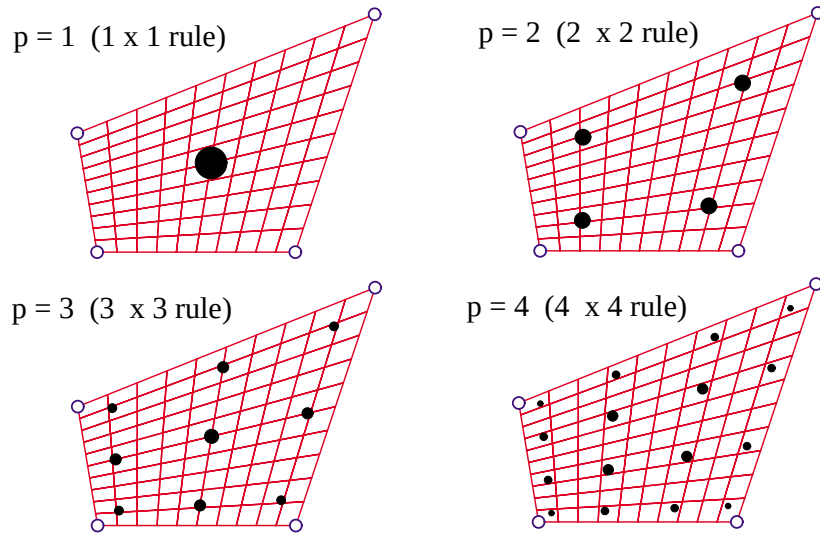


FIGURE 11.5. The first four two-dimensional Gauss product rules $p = 1, 2, 3, 4$ depicted over a straight-sided quadrilateral region. Sample points are marked with black circles. The areas of these circles are proportional to the integration weights.

If the integration rule has p_1 points in the ξ direction and p_2 points in the η direction, the module may be called also in two ways:

$$\begin{aligned} \{\{xii, etaj\}, wij\} &= \text{QuadGaussRuleInfo}[\{\{p1, p2\}, \text{numer}\}, \{i, j\}] \\ \{\{xii, etaj\}, wij\} &= \text{QuadGaussRuleInfo}[\{\{p1, p2\}, \text{numer}\}, k] \end{aligned} \quad (11.25)$$

The meaning of the second argument is as follows. In the first form i runs from 1 to p_1 and j from 1 to p_2 . In the second form k runs from 1 to $p_1 p_2$; if so i and j are extracted by $j = \text{Floor}[(k-1)/p1] + 1$; $i = k - p1*(j-1)$. In all four forms, logical flag *numer* is set to True if numerical information is desired and to False if exact information is desired.

The module returns ξ_i and η_j in *xii* and *etaj*, respectively, and the weight product $w_i w_j$ in *wij*. This code is used in the Exercises at the end of the chapter. If the inputs are not in range, the module returns $\{\{Null, Null\}, 0\}$.

```
QuadGaussRuleInfo[rule_, numer_, point_] := Module[
  {xi, eta, p1, p2, i, j, w1, w2, m, info = {{Null, Null}, 0}},
  If [Length[rule] == 2, {p1, p2} = rule, p1 = p2 = rule];
  If [p1 < 0, Return[QuadNonProductGaussRuleInfo[
    {-p1, numer}, point]];
  If [Length[point] == 2, {i, j} = point, m = point;
    j = Floor[(m-1)/p1] + 1; i = m - p1*(j-1)];
  {xi, w1} = LineGaussRuleInfo[{p1, numer}, i];
  {eta, w2} = LineGaussRuleInfo[{p2, numer}, j];
  info = {{xi, eta}, w1*w2};
  If [numer, Return[N[info]], Return[Simplify[info]]];
];
```

FIGURE 11.6. A *Mathematica* module that returns two-dimensional product Gauss rules.

Example 11.2. `{{ xi, eta }, w}=QuadGaussRuleInfo[{3, False }, {2, 3}]` returns `xi=0, eta=Sqrt[3/5]` and `w=40/81`.

Example 11.3. `{{ xi, eta }, w}=QuadGaussRuleInfo[{3, True }, {2, 3}]` returns (to 16-place precision) `xi=0., eta=0.7745966692414834` and `w=0.49382716049382713`.

§11.6. Rank Sufficiency: Linkup with Numerical Quadrature

The following material on numerical integration is transcribed here from Chapter 19 of the IFEM Notes for convenience.

One desirable attribute of the element stiffness matrix is: $\mathbf{K}^e$ should not possess any zero-energy kinematic mode other than rigid body modes.²

This can be mathematically expressed as follows. Let n_F be the number of element degrees of freedom, and n_R be the number of independent rigid body modes. Let r denote the rank of $\mathbf{K}^e$. The element is called *rank sufficient* if $r = n_F - n_R$ and *rank deficient* if $r < n_F - n_R$. In the latter case, the *rank deficiency* is defined by

$$d = (n_F - n_R) - r \quad (11.26)$$

If an isoparametric element is numerically integrated, let n_G be the number of Gauss points, while n_E denotes the order of the stress-strain matrix $\mathbf{E}$. Two additional assumptions are made:

- (i) The element shape functions satisfy completeness in the sense that the rigid body modes are exactly captured by them.
- (ii) Matrix $\mathbf{E}$ is of full rank.

Then each Gauss point adds up to n_E to the rank of $\mathbf{K}^e$, up to a maximum of $n_F - n_R$.³ Assuming that each points does add up exactly n_E , the rank of $\mathbf{K}^e$ will be

$$r = \min(n_F - n_R, n_E n_G) \quad (11.27)$$

To attain rank sufficiency, $n_E n_G$ must equal or exceed $n_F - n_R$:

$$n_E n_G \geq n_F - n_R \quad (11.28)$$

from which the appropriate Gauss integration rule can be selected.

In the axisymmetric solid problem, $n_E = 4$ because $\mathbf{E}$ is a 4×4 matrix of elastic moduli. Also $n_R = 1$. Consequently $r = \min(n_F - 1, 4n_G)$ and $4n_G \geq n_F - 1$.

² In an introductory FEM course, “should” is strengthened to “must”. In an advanced course one can find exceptions to the rule. For example, the reduced-integration Quad8 has a rank deficient stiffness but the element is nonetheless useful.

³ Note the proviso “adds up to.” There are cases when the addition comes up short of n_E . For a somewhat trivial example, think of using the same rule twice: the rank is not affected although the number of Gauss points is doubled. The reduced-integration Quad8 is a nontrivial case in point.

Example 11.4. Consider a 6-node quadratic triangle ring element. Then $n_F = 2 \times 6 = 12$. To attain the proper rank of $12 - n_R = 12 - 1 = 11$, $n_G \geq 3$. A 3-point Gauss rule, such as the midpoint rule defined in Chapter 24 of the IFEM Notes makes the element rank sufficient.

Example 11.5. Consider an 9-node biquadratic quadrilateral ring element. Then $n_F = 2 \times 9 = 18$. To attain the proper rank of $18 - n_R = 18 - 1 = 17$, $n_G \geq 5$. The 2×2 product Gauss rule is insufficient because $n_G = 4$. Hence a 3×3 rule, which yields $n_G = 9$, is required to attain rank sufficiency.

Homework Exercises for Chapter 11

Axisymmetric Solid Iso-P Elements

The following material is largely covered in Chapter 24 of the IFEM Notes, posted at <http://www.colorado.edu/engineering/CAS/courses.d/IFEM.d/IFEM.Ch24.d> but is recapitulated here for completeness.

The four simplest Gauss integration rules over a triangle of area A use 1, 3, 3 and 7 points. These rules are tabulated below. In the following expressions, $F(\zeta_1, \zeta_2, \zeta_3)$ denotes the function to be integrated over the triangle, expressed in terms of the triangular coordinates ζ_1, ζ_2 and ζ_3 , while A is the area of the element.

One point rule (exact for constant and linear polynomials over straight sided triangles):

$$\frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \quad (\text{E11.1})$$

Three-point rules (exact for constant through quadratic polynomials over straight sided triangles):

$$\frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx \frac{1}{3} F\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right) + \frac{1}{3} F\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right) + \frac{1}{3} F\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right). \quad (\text{E11.2})$$

$$\frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx \frac{1}{3} F\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{3} F\left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{3} F\left(\frac{1}{2}, 0, \frac{1}{2}\right). \quad (\text{E11.3})$$

The latter is also called the 3-midpoint rule for obvious reasons.

Seven point rule (exact for constant through cubic polynomials over straight sided triangles):

$$\begin{aligned} \frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx & w_0 F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + w_1 [F(\alpha_1, \beta_1, \beta_1) + F(\beta_1, \alpha_1, \beta_1) + F(\beta_1, \beta_1, \alpha_1)] \\ & + w_2 [F(\alpha_2, \beta_2, \beta_2) + F(\beta_2, \alpha_2, \beta_2) + F(\beta_2, \beta_2, \alpha_2)], \end{aligned} \quad (\text{E11.4})$$

where $w_0 = 9/40 = 0.225$, $w_1 = (155 - \sqrt{15})/1200 = 0.1259391805$, $w_2 = (155 + \sqrt{15})/1200 = 0.1323941528$, $\alpha_1 = (9 + 2\sqrt{15})/21 = 0.7974269853$, $\beta_1 = (6 - \sqrt{15})/21 = 0.1012865073$, $\alpha_2 = (9 - 2\sqrt{15})/21 = 0.0597158718$, $\beta_2 = (6 + \sqrt{15})/21 = 0.4701420641$.

A *Mathematica* module that implement these rules is `TrigGaussRuleInfo.nb` posted in the index of this Chapter. The use of this module is explained in Chapter 24 of the IFEM Notes.

EXERCISE 11.1 [A/C:15] Using the minimum quadrature rule necessary for exactness, verify the following polynomial integrals over straight-sided triangles, where indices i, j, k run over 1,2,3, and r is interpolated as $r = r_1\zeta_1 + r_2\zeta_2 + r_3\zeta_3$.

$$\int_{A^e} \zeta_i dA^e = \frac{1}{3} A, \quad (\text{E11.5})$$

$$\int_{A^e} \zeta_i \zeta_j dA^e = \frac{1}{12} A(1 + \delta_{ij}), \quad (\text{E11.6})$$

$$\int_{A^e} \zeta_i \zeta_j \zeta_k dA^e = \frac{1}{60} \gamma_{ijk} A, \quad (\text{E11.7})$$

$$\int_{A^e} r dA^e = \frac{1}{3} A(r_1 + r_2 + r_3) = A r_0, \quad (\text{E11.8})$$

$$\int_{A^e} \zeta_i r dA^e = \frac{1}{12} A(r_1 + r_2 + r_3 + r_i), \quad (\text{E11.9})$$

$$\int_{A^e} r^2 dA^e = \frac{1}{12} A[(r_1 + r_2 + r_3)^2 + r_1^2 + r_2^2 + r_3^2], \quad (\text{E11.10})$$

$$\int_{A^e} \zeta_i r^2 dA^e = \frac{1}{60} \gamma_{ijk} A r_j r_k. \quad (\text{E11.11})$$

In the above, $\delta_{ij} = 1$ if $i = j$ else 0 (the Kronecker delta); $\gamma_{ijk} = 6$ if $i = j = k$, $\gamma_{ijk} = 1$ if $i \neq j \neq k$, else 2. Note: you can (and should) make use of previous results for expediency; to prove for example (E8.8) substitute (E8.5) as appropriate.

EXERCISE 11.2 [A/C:15] Apply the triangle integration formulas to the non-polynomial integral

$$\int_{A^e} \frac{dA^e}{r} \quad (\text{E11.12})$$

for the two cases

- (a) $r_1 = 0, r_2 = r_3 = a$, and compare to the exact integral $2A/a$.
- (b) $r_1 = 1, r_2 = 2, r_3 = 3$ and compare to the exact integral $\log(27/16) A/a = 0.523238A/a$. (The 7 point rule should be accurate to 4 digits).

EXERCISE 11.3 [A/C:15] Repeat (a) of the previous exercise for the integrals

$$\int_{A^e} \frac{\zeta_1}{r} dA^e, \quad \int_{A^e} \frac{\zeta_2}{r} dA^e, \quad \int_{A^e} \frac{\zeta_3}{r} dA^e. \quad (\text{E11.13})$$

The exact integrals are A/a for the first one, and $\frac{1}{2}A/a$ for the other two. Why are all numerical quadrature formula exact for the latter case?

EXERCISE 11.4 [A/C:20] The isoparametric definition of the 3-node linear SOR triangle is

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \quad (\text{E11.14})$$

Using results stated in Exercise 11.1 as appropriate, compute the entries

$$K_{r1r1}, \quad K_{r1z1}, \quad K_{z1z1} \quad (\text{E11.15})$$

of the element stiffness matrix $\mathbf{K}^e$ for the geometry defined by

$$r_1 = 0, \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b. \quad (\text{E11.16})$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad (\text{E11.17})$$

Note: For integrals that contain $1/r$, use one of the 3-point rules. Partial answer (if midpoint rule used) $K_{r1r1} = \frac{1}{2}Eb$.

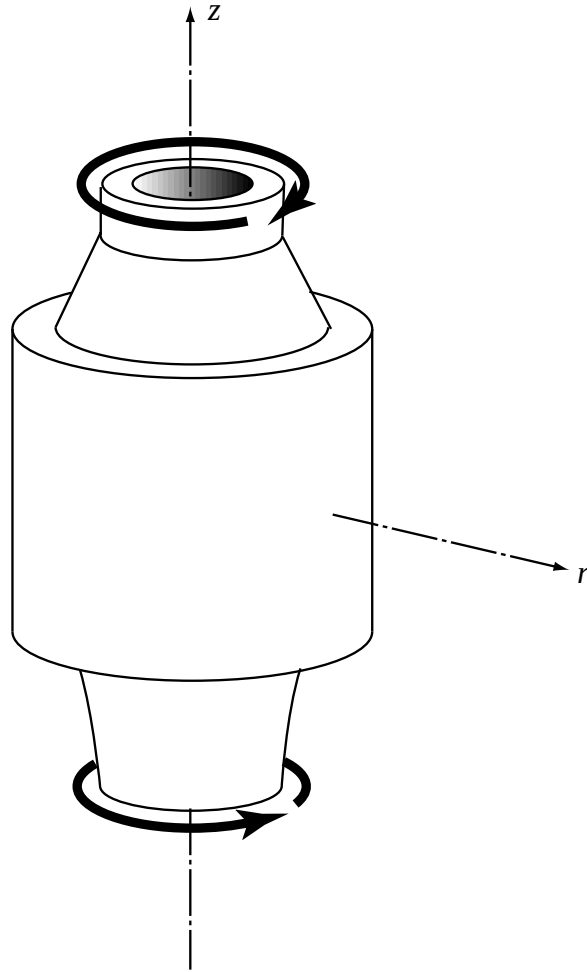


Figure E11.1. Variable-section circular shaft conveying torque: the subject of Exercise 11.11.

EXERCISE 11.5 [A/C:20] The triangle of Exercise 11.4 is subjected to the radial-centrifugal body force field

$$b_r = \rho\omega^2 r, \quad b_z = 0 \quad (\text{E11.18})$$

where ρ and ω are constant (ρ is the mass density while ω is the angular velocity.)

Compute the consistent node force vector $\mathbf{f}^e$. In doing so interpolate the radial component b_r *linearly* over the element:

$$b_r(\zeta_1, \zeta_2, \zeta_3) = b_{r1}\zeta_1 + b_{r2}\zeta_2 + b_{r3}\zeta_3, \quad (\text{E11.19})$$

where b_{ri} is $\rho\omega^2 r$ evaluated at corner i . Partial answers: $f_{r1} = \omega a^3/10$, $f_{z1} = 0$.

EXERCISE 11.6 [A/C:20] Repeat the previous Exercise for the own-weight body force field

$$b_r = 0, \quad b_z = -\rho g \quad (\text{E11.20})$$

where ρ and g are constant. Partial answer: $f_{z2} = -ga/4$.

EXERCISE 11.7 [A/C:20] For the triangle geometry of the preceding 3 exercises, find the consistent thermal forces pertaining to a uniform temperature increase ΔT , assuming an isotropic material with zero Poisson's ratio.

EXERCISE 11.8 [A/C:15] Show that $u_z = c$ (c is a constant) is the only possible rigid body mode of a SOR element. (Hint: consider the presence of the circumferential strain). Hence deduce that the correct rank of the stiffness matrix of a SOR element with n nodes and 2 DOFs per node is $2n - 1$.

EXERCISE 11.9 [A/C:15] Find the displacement fields that separately generate the following constant strain states:

$$e_{zz} = c_{zz}, \quad \text{others zero} \quad (\text{E11.21})$$

$$\gamma_{rz} = c_{rz}, \quad \text{others zero} \quad (\text{E11.22})$$

where c_{zz} and c_{rz} are constants.

EXERCISE 11.10 [A/C:15] Show that there are no displacement fields that separately generate the following constant strain states:

$$e_{rr} = c_{rr}, \quad \text{others zero} \quad (\text{E11.23})$$

$$e_{\theta\theta} = c_{\theta\theta}, \quad \text{others zero} \quad (\text{E11.24})$$

where c_{rr} and $c_{\theta\theta}$ are constants. Hint: integrate the appropriate strain-displacement relations.

EXERCISE 11.11 [A/C:25] The SOR sketched in Figure E11.1 (a circular shaft with varying cross section) is subjected to torsional loading as indicated. According to Saint-Venant's torsion theory, the displacement components for this case are entirely circumferential, that is, $u_r = u_z = 0$ and $u_\theta = u_\theta(r, z)$. The torsional shear strains

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad \gamma_{z\theta} = \frac{\partial u_\theta}{\partial z}, \quad (\text{E11.25})$$

are nonzero and functions of r, z only; all other strains (e_{rr} , e_{zz} , $e_{\theta\theta}$ and γ_{rz}) vanish. Assuming the shaft is fabricated with an isotropic material, the only nonzero stress components are the shear stresses

$$\sigma_{r\theta} = G \gamma_{r\theta}, \quad \sigma_{z\theta} = G \gamma_{z\theta}, \quad (\text{E11.26})$$

where $G = \frac{1}{2}E/(1 + \nu)$ is the shear modulus.

- Explain why this problem can be discretized by two-dimensional “ring” finite elements by laying out a mesh over the (r, z) plane ($r \geq 0$), although the element type is different from that considered previously in this Chapter. How many degrees of freedoms would these “torqued ring” elements have per node?
- If an isoparametric formulation is used for the torqued-ring elements of (a), the element stiffness matrix, on suppressing the 2π factor, is given by the usual expression

$$\mathbf{K}^e = \int_{A^e} \mathbf{B}^T \mathbf{E} \mathbf{B} r \, dA^e. \quad (\text{E11.27})$$

But how would $\mathbf{B}$ and $\mathbf{E}$ look like?

12

4- and 8-Node Iso-P Quadrilateral Ring Elements

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§12.1. Introduction

This Chapter illustrates the computer implementation of isoparametric *quadrilateral* elements for the axisymmetric problem. These are called *ring* elements. Triangles, which present some programming quirks, are not described in these Notes. For details on those the reader may consult Chapter 24 of the Introduction to Finite Elements Notes.

The Chapter described two elements, which are identified by the following type labels.

Quad4 The standard 4-node isoparametric quadrilateral. This is usually processed with a 2×2 Gauss integration rule, which represents full integration.

Quad8RI The 8-node isoparametric quadrilateral. This is often processed by Reduced Integration: a 2×2 Gauss rule, whence the label. This rule results in rank deficiency, but this is generally harmless. It can also be integrated with a 3×3 rule for safety, but performance suffers.

A third element was supposed to be described here:

Quad4SRI The 4-node isoparametric quadrilateral processed by Selective Reduced Integration (SRI). Implementation, however, has not been completed.

The element description that follows covers the computation of the element stiffness matrix, consistent node force vector for a body force field, consistent node force for surface tractions, and recovery of element stresses from displacements.

We consider the implementation of the 4-node and 8-node quadrilateral ring elements for axisymmetric solid analysis. The element cross sections are depicted in Figure 12.1.

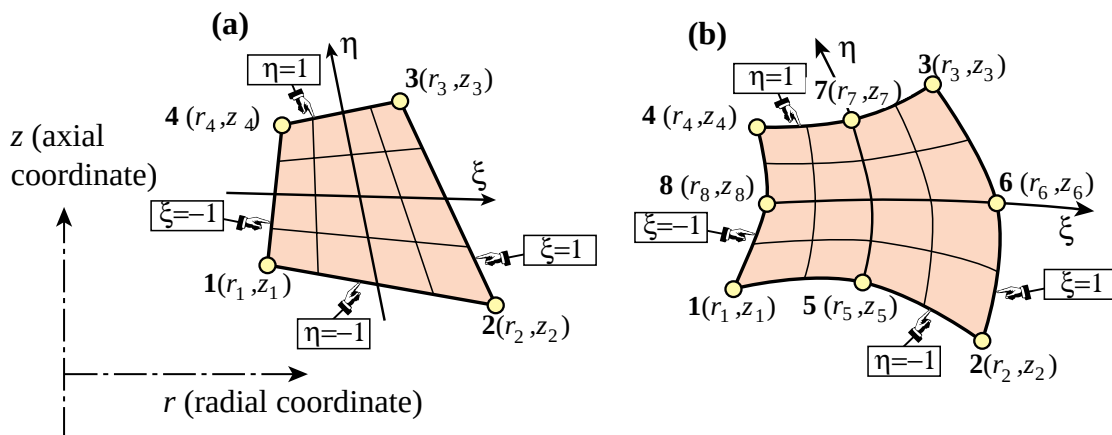


FIGURE 12.1. The 4-node and 8-node iso-P quadrilateral ring elements described in this Chapter.

For 2D and 3D elements iso-P elements it is convenient to break up the implementation into *application dependent* and *application independent* modules, as sketched in Figure 12.2. The application independent modules can be “reused” in other FEM applications, for example to form thermal, fluid or electromagnetic elements.

For the 4-node quadrilateral studied here, the subdivision of Figure 12.2 is done through the following modules:

Quad4IsoPRingStiffness -	forms K_e of standard isoP 4-node quad ring
QuadGaussRuleInfo -	returns Gauss quadrature product rules of order 1-4
IsoQuad4ShapeFunDer -	evaluates shape functions and their x/y derivatives
Quad4isoPRingForces -	forms traction force f_e of 4-node standard isoP quad ring
QuadGaussRuleInfo -	returns Gauss quadrature product rules of order 1-4
IsoQuad4ShapeFun -	evaluates shape functions
Quad4isoPRingTracForces -	forms traction force f_e of 4-node standard isoP quad ring
ring	
QuadGaussRuleInfo -	returns Gauss quadrature product rules of order 1-4
IsoQuad4ShapeFun -	evaluates shape functions
Quad4isoPRingStresses -	evaluates stresses f_e of 4-node standard isoP quad ring
QuadGaussRuleInfo -	returns Gauss quadrature product rules of order 1-4
IsoQuad4ShapeFunDer -	evaluates shape functions

See Figure 12.2.

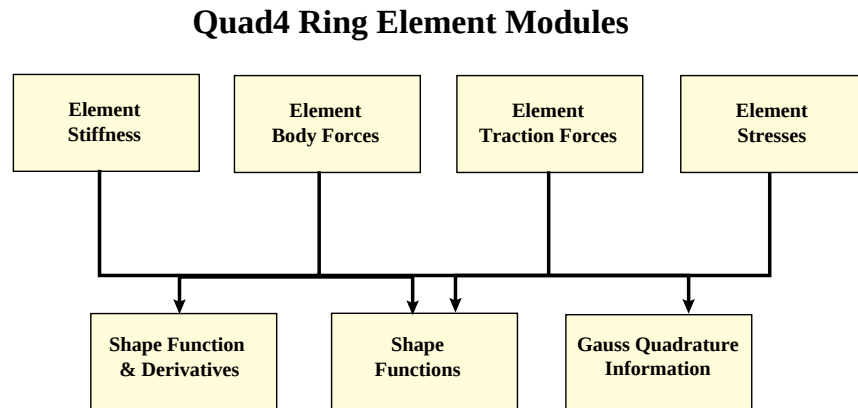


FIGURE 12.2. Organization of element modules.

These modules are presented in the following subsections, except for the Gauss quadrature information modules, which were described in the previous Chapter.

```

Quad4IsoP RingShapeFunDer[ncoor_, qcoor_, Jcons_] := Module[
{ r1, r2, r3, r4, z1, z2, z3, z4, ξ, η, Nf, dNr, dNz, A0, A1, A2, Jdet},
{{r1, z1}, {r2, z2}, {r3, z3}, {r4, z4}} = ncoor; {ξ, η} = qcoor;
Nf = {(1-ξ)*(1-η), (1+ξ)*(1-η), (1+ξ)*(1+η), (1-ξ)*(1+η)}/4;
A0 = ((r3-r1)*(z4-z2) - (r4-r2)*(z3-z1))/2;
A1 = ((r3-r4)*(z1-z2) - (r1-r2)*(z3-z4))/2;
A2 = ((r2-r3)*(z1-z4) - (r1-r4)*(z2-z3))/2;
Jdet = (A0+A1*ξ+A2*η)/4; If [Jcons, Jdet=A0/4];
dNr = {z2-z4+(z4-z3)*ξ+(z3-z2)*η, z3-z1+(z3-z4)*ξ+(z1-z4)*η,
z4-z2+(z1-z2)*ξ+(z4-z1)*η, z1-z3+(z2-z1)*ξ+(z2-z3)*η}/(8*Jdet);
dNz = {r4-r2+(r3-r4)*ξ+(r2-r3)*η, r1-r3+(r4-r3)*ξ+(r4-r1)*η,
r2-r4+(r2-r1)*ξ+(r1-r4)*η, r3-r1+(r1-r2)*ξ+(r3-r2)*η}/(8*Jdet);
Return[{Nf, dNr, dNz, Jdet}]
];

```

FIGURE 12.3. Shape function module for 4-node bilinear quadrilateral ring element.

§12.2. The 4-Node Quadrilateral Ring Element

This is the axisymmetric solid version of the well known isoparametric quadrilateral with bilinear shape functions. The element has 4 nodes and 8 displacement degrees of freedom arranged as

$$\mathbf{u}^e = [u_{r1} \ u_{z1} \ u_{r2} \ u_{z2} \ u_{r3} \ u_{z3} \ u_{r4} \ u_{z4}]^T. \quad (12.1)$$

§12.2.1. Shape Function Module

Module Quad4IsoP RingShapeFunDer, listed in Figure 12.3, computes the shape functions N_i^e , $i = 1, 2, 3, 4$ and their partial derivatives with respect to r and z at a specified point in the element. Usually this module is called at sample points of a Gauss quadrature rule, but it may also be used with symbolic inputs to get information for an arbitrary point at $\{\xi, \eta\}$. The element geometry is defined by the 8 coordinates $\{r_i, z_i\}$, $i = 1, 2, 3, 4$. These are collected in arrays

$$\mathbf{r} = [r_1 \ r_2 \ r_3 \ r_4]^T, \quad \mathbf{z} = [z_1 \ z_2 \ z_3 \ z_4]^T. \quad (12.2)$$

We will use the abbreviations $r_{ij} = r_i - r_j$ and $z_{ij} = z_i - z_j$ for coordinate differences. The shape functions and their partial derivatives with respect to the quadrilateral coordinates are collected in the arrays

$$\begin{aligned}
\mathbf{N} &= \frac{1}{4} [(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta)], \\
\mathbf{N}_{,\xi} &= \frac{\partial \mathbf{N}}{\partial \xi} = \frac{1}{4} [-1+\eta \quad 1-\eta \quad 1+\eta \quad -1-\eta], \\
\mathbf{N}_{,\eta} &= \frac{\partial \mathbf{N}}{\partial \eta} = \frac{1}{4} [-1+\xi \quad -1-\xi \quad 1+\xi \quad 1-\xi].
\end{aligned} \quad (12.3)$$

The Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{,\xi} \mathbf{r} & \mathbf{N}_{,\xi} \mathbf{z} \\ \mathbf{N}_{,\eta} \mathbf{r} & \mathbf{N}_{,\eta} \mathbf{z} \end{bmatrix}. \quad (12.4)$$

Expanding the inner products yields the explicit expressions

$$\begin{aligned} J_{11} &= \frac{1}{4}(r_{21} + r_{34} + (r_{12} + r_{34}) \eta), & J_{12} &= \frac{1}{4}(z_{21} + z_{34} + (z_{12} + z_{34}) \eta), \\ J_{21} &= \frac{1}{4}(r_{32} + r_{41} + (r_{12} + r_{34}) \xi), & J_{22} &= \frac{1}{4}(z_{32} + z_{41} + (z_{12} + z_{34}) \xi), \\ J &= \det(\mathbf{J}) = J_{11}J_{22} - J_{12}J_{21} = \frac{1}{4}(A_0 + A_1 \xi + A_2 \eta), \end{aligned} \quad (12.5)$$

in which

$$A_0 = \frac{1}{2}(r_{31}z_{42} - r_{42}z_{31}), \quad A_1 = \frac{1}{2}(r_{34}z_{12} - r_{12}z_{34}), \quad A_2 = \frac{1}{2}(r_{23}z_{14} - r_{14}z_{23}). \quad (12.6)$$

The inverse Jacobian is obtained by explicit inversion. Finally the $\{r, z\}$ derivatives produced by the chain rule emerge as the explicit formulas

$$\mathbf{N}_{,r} = \frac{\partial \mathbf{N}}{\partial r} = \frac{1}{8J} \begin{bmatrix} z_{24} + z_{43} \xi + z_{32} \eta \\ z_{31} + z_{34} \xi + z_{14} \eta \\ z_{42} + z_{12} \xi + z_{41} \eta \\ z_{13} + z_{21} \xi + z_{23} \eta \end{bmatrix}, \quad \mathbf{N}_{,z} = \frac{\partial \mathbf{N}}{\partial z} = \frac{1}{8J} \begin{bmatrix} r_{42} + r_{34} \xi + r_{23} \eta \\ r_{13} + r_{43} \xi + r_{41} \eta \\ r_{24} + r_{21} \xi + r_{14} \eta \\ r_{31} + r_{12} \xi + r_{32} \eta \end{bmatrix}. \quad (12.7)$$

The logic of Quad4IsoPRingShapeFunDer, listed in Figure 12.3, implements the foregoing equations. The module is invoked as

$$\{ \text{Nf}, \text{dNr}, \text{dNz}, \text{Jdet} \} = \text{Quad4IsoPRingShapeFunDer}[\text{ncoor}, \text{qcoor}, \text{Jcons}] \quad (12.8)$$

where the arguments are

- ncoor** Quadrilateral node coordinates arranged in two-dimensional list form: $\{ \{ r_1, z_1 \}, \{ r_2, z_2 \}, \{ r_3, z_3 \}, \{ r_4, z_4 \} \}$.
- qcoor** Quadrilateral coordinates $\{ \xi, \eta \}$ of the point at which shape functions and derivatives are to be evaluated.
- Jcons** A logical flag. If True, the Jacobian determinant J is set to its value at the element center, namely, $A_0/4$, for any $\{ \xi, \eta \}$. That setting is useful in certain research studies.

The module returns the list $\{ \text{Nf}, \text{dNr}, \text{dNz}, \text{Jdet} \}$, where

- Nf** Shape function values¹ arranged as list $\{ N_1, N_2, N_3, N_4 \}$.
- dNr** r shape function derivatives (12.7) arranged as $\{ \text{dNr1}, \text{dNr2}, \text{dNr3}, \text{dNr4} \}$.
- dNz** z shape function derivatives (12.7) arranged as $\{ \text{dNz1}, \text{dNz2}, \text{dNz3}, \text{dNz4} \}$.
- Jdet** Jacobian determinant.

§12.2.2. Element Stiffness Module

Module Quad4IsoPRingStiffness, listed in Figure 12.4, computes the stiffness matrix of a 4-noded iso-P quadrilateral ring element. The computation is carried out using numerical quadrature. It essentially follows the procedure outlined in the previous Chapter.

¹ Note that N cannot be used as name of the list of shape function values, because that symbol is reserved.

```

Quad4IsoPRingStiffness[ncoor_,Emat_,options_]:=Module[
{p=2,number=False,Jcons=False,Kfac=1,qcoor,k,
r1,r2,r3,r4,z1,z2,z3,z4,Nf,N1,N2,N3,N4,A0,Jdet,Be,
dNr1,dNr2,dNr3,dNr4,dNz1,dNz2,dNz3,dNz4,rk,w,c,Ke,
Ke0=Table[0,{8},{8}],modname="Quad4IsoPRingStiffness: "},
If [Length[options]==1,{number}=options];
If [Length[options]==2,{number,p}=options];
If [Length[options]==3,{number,p,Jcons}=options];
If [Length[options]==4,{number,p,Jcons,Kfac}=options];
If [p<1||p>5, Print[modname,"illegal p:",p]; Return[Ke0]];
{{r1,z1},{r2,z2},{r3,z3},{r4,z4}}=ncoor;
A0=((r3-r1)*(z4-z2)-(r4-r2)*(z3-z1))/2;
If [number&&(A0<=0), Print[modname,"Neg or zero area"];
Return[Ke0]]; Ke=Ke0;
For [k=1,k<=p*p,k++,
{qcoor,w}= QuadGaussRuleInfo[{p,number},k];
{Nf,{dNr1,dNr2,dNr3,dNr4},{dNz1,dNz2,dNz3,dNz4},
Jdet}=Quad4IsoPRingShapeFunDer[ncoor,qcoor,Jcons];
If [number&&(Jdet<=0), Print[modname,"Neg or zero",
" Gauss point Jacobian at k=",k]; Return[Ke0]];
{N1,N2,N3,N4}=Nf; rk=r1*N1+r2*N2+r3*N3+r4*N4;
Be={{ dNr1, 0, dNr2, 0, dNr3, 0, dNr4, 0},
{ 0,dNz1, 0,dNz2, 0,dNz3, 0,dNz4},
{N1/rk, 0,N2/rk, 0,N3/rk, 0,N4/rk, 0},
{ dNz1,dNr1, dNz2,dNr2, dNz3,dNr3, dNz4,dNr4}};
c=Kfac*w*rk*Jdet; If [number,Be=N[Be]; c=N[c]];
Ke+=c*Transpose[Be].(Emat.Be);
]; ClearAll[Ke0,Be]; Return[Ke ]];

```

FIGURE 12.4. Element stiffness formation module for 4-node iso-P quadrilateral ring.

The module is invoked as

$$\mathbf{K}_e = \text{Quad4IsoPRingStiffness}[\text{ncoor}, \text{Emat}, \text{options}] \quad (12.9)$$

The arguments are:

- ncoor** Quadrilateral node coordinates arranged in two-dimensional list form:
 $\{\{r_1, z_1\}, \{r_2, z_2\}, \{r_3, z_3\}, \{r_4, z_4\}\}.$
- Emat** The 4×4 matrix of elastic moduli:

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix}, \quad (12.10)$$

arranged as a two-dimensional list array: $\{\{E_{11}, E_{12}, E_{13}, E_{14}\}, \{E_{12}, E_{22}, E_{23}, E_{24}\}, \{E_{13}, E_{23}, E_{33}, 0\}, \{E_{14}, E_{24}, 0, E_{44}\}\}.$

Note that $E_{34} = 0$ to satisfy axisymmetric behavior assumptions. If the material is isotropic, with elastic modulus E and Poisson ratio ν ,

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\nu \end{bmatrix}. \quad (12.11)$$

options A list of processing options. This list may be either empty or contain up to 4 items. Possible configurations are $\{ \}$, $\{ \text{numer} \}$, $\{ \text{numer}, p \}$, $\{ \text{numer}, p, Jcons \}$, or $\{ \text{numer}, p, Jcons, Kfac \}$.

numer is a logical flag with value True or False. If True, the computations are forced to proceed in floating point arithmetic. For symbolic or exact arithmetic work set **numer** to False. If omitted, False is assumed.

p is an integer specifying that the Gauss product rule used in computing $\mathbf{K}^e$ is to have p points in each direction. It may be 1 through 4. For rank sufficiency, p must be 2 or higher. If p is 1 the element will be rank deficient by three. If omitted $p = 2$ is assumed.

Jcons is a logical flag with value True or False. If True the Jacobian determinant at the element center is assumed to be constant over the element, even if it has arbitrary geometry. This is useful in certain research studies. If omitted, False is assumed.

Kfac is a ring-circumference-span factor by which the stiffness matrix will be scaled. Typically $Kfac=1$ to make the ring element span one radian, $Kfac=2\pi$ to make a complete circle. If omitted, $Kfac = 1$ is assumed.

As function value the module returns

Ke a 8×8 symmetric matrix pertaining to the arrangement (12.2) of element node displacements. If an error is detected during processing, a zero matrix is returned.

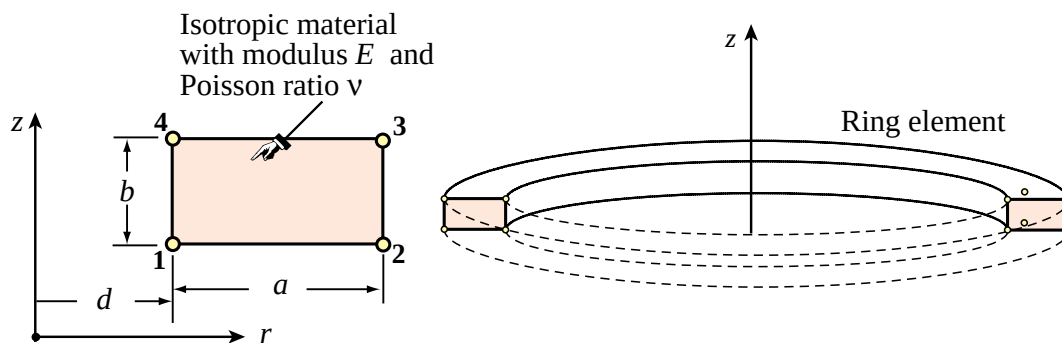


FIGURE 12.5. Test quadrilateral ring element geometry.

Example 12.1. The stiffness module is tested on the geometry identified in Figure 12.5. The cross section is a rectangle dimensioned $a \times b$ with sides parallel to the $\{r, z\}$ axes. The distance of the leftmost side to the z axis is d . The material is isotropic with modulus E and Poisson's ratio ν .

The script of Figure 12.8 computes and prints the stiffness of the test element shown in for $E = 96$, $\nu = 1/3$, $a = 4$, $b = 2$ and $d = 0$. The default $Kfac = 1$ is used. Nodes 1 and 2 sit on the z axes. The value of p is changed in a loop. The flag **numer** is set to True to use floating-point computation for speed. The computed entries of $\mathbf{K}^e$ are exact integers for all values of p :


```

ClearAll[Em,v,a,b,d,h,p,num];
Em=96; v=1/3; a=4; b=2; d=0; Kfac=2*Pi; Kfac=1;
ncoor={{d,0},{a+d,0},{a+d,b},{d,b}}; num=False;
Emat=Em/((1+v)*(1-2*v))*{{1-v,v,v,0},{v,1-v,v,0},
{v,v,1-v,0},{0,0,0,1/2-v}};
Print["Emat=",Emat//MatrixForm];
For [p=1,p<=4,p++, Print["Gauss rule p=",p];
Ke=Quad4IsoPRingStiffness[ncoor,Emat,Kfac,{num,p}];
Ke=Simplify[Ke]; Print["Ke=",Ke//MatrixForm];
Print["Eigenvalues of Ke=",Chop[Eigenvalues[N[Ke]]]];
];

```

FIGURE 12.6. Driver for exercising the Quad4IsoPRingStiffness module of Figure 12.4 using the ring element geometry shown in Figure 12.5, with $E = 96$, $\nu = 1/3$, $a = 4$, $b = 2$, $d = 0$ and four Gauss product integration rules.

$$\mathbf{K}_{1 \times 1}^e = \begin{bmatrix} 72 & 18 & 36 & -18 & -36 & -18 & 0 & 18 \\ 18 & 153 & -54 & 135 & -90 & -153 & -18 & -135 \\ 36 & -54 & 144 & -90 & 72 & 54 & -36 & 90 \\ -18 & 135 & -90 & 153 & -54 & -135 & 18 & -153 \\ -36 & -90 & 72 & -54 & 144 & 90 & 36 & 54 \\ -18 & -153 & 54 & -135 & 90 & 153 & 18 & 135 \\ 0 & -18 & -36 & 18 & 36 & 18 & 72 & -18 \\ 18 & -135 & 90 & -153 & 54 & 135 & -18 & 153 \end{bmatrix} \quad (12.12)$$

$$\mathbf{K}_{2 \times 2}^e = \begin{bmatrix} 168 & -12 & 24 & 12 & -24 & -36 & 48 & 36 \\ -12 & 108 & -24 & 84 & -72 & -102 & -36 & -90 \\ 24 & -24 & 216 & -120 & 0 & 72 & -24 & 72 \\ 12 & 84 & -120 & 300 & -72 & -282 & 36 & -102 \\ -24 & -72 & 0 & -72 & 216 & 120 & 24 & 24 \\ -36 & -102 & 72 & -282 & 120 & 300 & -12 & 84 \\ 48 & -36 & -24 & 36 & 24 & -12 & 168 & 12 \\ 36 & -90 & 72 & -102 & 24 & 84 & 12 & 108 \end{bmatrix} \quad (12.13)$$

$$\mathbf{K}_{3 \times 3}^e = \begin{bmatrix} 232 & -12 & 24 & 12 & -24 & -36 & 80 & 36 \\ -12 & 108 & -24 & 84 & -72 & -102 & -36 & -90 \\ 24 & -24 & 216 & -120 & 0 & 72 & -24 & 72 \\ 12 & 84 & -120 & 300 & -72 & -282 & 36 & -102 \\ -24 & -72 & 0 & -72 & 216 & 120 & 24 & 24 \\ -36 & -102 & 72 & -282 & 120 & 300 & -12 & 84 \\ 80 & -36 & -24 & 36 & 24 & -12 & 232 & 12 \\ 36 & -90 & 72 & -102 & 24 & 84 & 12 & 108 \end{bmatrix} \quad (12.14)$$

$$\mathbf{K}_{4 \times 4}^e = \begin{bmatrix} 280 & -12 & 24 & 12 & -24 & -36 & 104 & 36 \\ -12 & 108 & -24 & 84 & -72 & -102 & -36 & -90 \\ 24 & -24 & 216 & -120 & 0 & 72 & -24 & 72 \\ 12 & 84 & -120 & 300 & -72 & -282 & 36 & -102 \\ -24 & -72 & 0 & -72 & 216 & 120 & 24 & 24 \\ -36 & -102 & 72 & -282 & 120 & 300 & -12 & 84 \\ 104 & -36 & -24 & 36 & 24 & -12 & 280 & 12 \\ 36 & -90 & 72 & -102 & 24 & 84 & 12 & 108 \end{bmatrix} \quad (12.15)$$

As can be seen entries change substantially in going from $p = 1$ to $p = 2$. From then on only four entries,

associated with the r stiffness at nodes 1 and 4, change. The eigenvalues of these matrices are:

Rule	Eigenvalues of $\mathbf{K}^e$ for varying integration rule							
1×1	667.794	180.000	124.206	72.000	0	0	0	0
2×2	745.201	261.336	248.750	129.451	100.389	88.598	10.275	0
3×3	745.446	330.628	266.646	133.236	126.343	98.690	11.011	0
4×4	745.716	397.372	272.092	144.542	135.004	101.908	11.365	0

(12.16)

The stiffness matrix computed by the one-point rule is rank deficient by three. For $p = 2$ and up it has the correct rank of 7. The eigenvalues do not change appreciably after $p = 2$. Because the nonzero eigenvalues measure the internal energy taken up by the element in deformation eigenmodes, it can be seen that raising the order of the integration stiffens the element.

§12.2.3. Body Force Module

Module `Quad4IsoPRingBodyForces`, listed in Figure 12.7 computes the consistent force vector associated with a body force field $\bar{\mathbf{b}} = \{b_x, b_y\}$ specified over a four-node iso-P quadrilateral ring element. The field is assumed to be given per unit of volume, in radial-axial component-wise form. Two common scenarios for this kind of forcing effect are:

1. The element is subjected to a gravity acceleration field g due to self-weight in the $-z$ direction. Then $b_r = 0$ and $b_z = -\rho g$, where ρ is the mass density of the element material.
2. The element rotates at constant angular velocity ω (radians per second) around the axis of revolution z . Then $b_r = \rho \omega^2 r$ and $b_z = 0$.

The force vector is computed by Gauss numerical integration as described in the previous chapter. The module is invoked as

$$\mathbf{K}_e = \text{Quad4IsoPRingBodyForces}[\text{ncoor}, \text{bfor}, \text{options}] \quad (12.17)$$

The arguments are:

<code>ncoor</code>	Same as in <code>Quad4IsoPRingStiffness</code>
<code>bfor</code>	Specifies body force field (forces per unit of volume) over the element. Two specification forms are allowed. One-dimensional list: $\{\text{br}, \text{bz}\}$ Two-dimensional list: $\{\{\text{br1}, \text{bz1}\}, \{\text{br2}, \text{bz2}\}, \{\text{br3}, \text{bz3}\}, \{\text{br4}, \text{bz4}\}\}$ In the first form the body force field is taken to be uniform over the element, with radial component br and axial component bz . The second form assumes body forces to vary over the element. Radial and axial components are specified at the four corners; for example $\{\text{br1}, \text{bz1}\}$ are the values of b_r and b_z at corner 1. From this information the field is interpolated over the element using the iso-P bilinear shape functions.
<code>options</code>	Same as in <code>Quad4IsoPRingStiffness</code> .

As function value the module returns

```

Quad4IsoPRingBodyForces[ncoor_, bfor_, options_] := Module[
{p=2, numer=False, Jcons=False, Kfac=1, qcoor, k, m,
r1, r2, r3, r4, z1, z2, z3, z4, N1, N2, N3, N4, dNr, dNz, Jdet, Be,
br1, bz1, br2, bz2, br3, bz3, br4, bz4, brc, bzc, bk, rk, w, c, fe,
fe0=Table[0, {8}], modname="Quad4IsoPRingBodyForces: "},
If [Length[options]==1, {numer}=options];
If [Length[options]==2, {numer, p}=options];
If [Length[options]==3, {numer, p, Jcons}=options];
If [Length[options]==4, {numer, p, Jcons, Kfac}=options];
If [p<1||p>5, Print[modname, "p out of range"]; Return[fe0]];
{{r1, z1}, {r2, z2}, {r3, z3}, {r4, z4}}=ncoor;
A0=((r3-r1)*(z4-z2)-(r4-r2)*(z3-z1))/2;
If [numer&&(A0<=0), Print[modname, "Neg or zero area"];
Return[fe0]]; fe=fe0; m=Length[bfor];
If [m!=2&&m!=4, Print[modname, "Illegal bfor"]; Return[fe0]];
If [m==2, br1=br2=br3=br4=bfor[[1]]; bz1=bz2=bz3=bz4=bfor[[2]]];
If [m==4, {{br1, bz1}, {br2, bz2}, {br3, bz3}, {br4, bz4}}=bfor];
For [k=1, k<=p*p, k++,
{qcoor, w}= QuadGaussRuleInfo[{p, numer}, k];
{{N1, N2, N3, N4}, dNr, dNz, Jdet}=
Quad4IsoPRingShapeFunDer[ncoor, qcoor, Jcons];
If [numer&&(Jdet<=0), Print[modname, "Neg or zero",
" Gauss point Jacobian at k=", k]; Return[fe0]];
rk=r1*N1+r2*N2+r3*N3+r4*N4; c=Kfac*w*Jdet*rk;
brk=br1*N1+br2*N2+br3*N3+br4*N4;
bzk=bz1*N1+bz2*N2+bz3*N3+bz4*N4;
bk={N1*brk, N1*bzk, N2*brk, N2*bzk,
N3*brk, N3*bzk, N4*brk, N4*bzk};
If [numer, bk=N[bk]]; fe+=c*bk;
]; If[!numer, fe=Simplify[fe]];
Return[fe] ];

```

FIGURE 12.7. Module that computes consistent node forces for a 4-noded quadrilateral ring element given a body force field.

fe Consistent force vector arranged {fr1, fz1, fr2, fz2, fr3, fz3, fr4, fz4} to represent

$$\mathbf{f}^e = [f_{r1} \quad f_{z1} \quad f_{r2} \quad f_{z2} \quad f_{r3} \quad f_{z3} \quad f_{r4} \quad f_{z4}]^T. \quad (12.18)$$

Example 12.2. Consider again the ring element of Figure 12.5. This is now exercised for body force computation, using the script listed in Figure 12.8. These specify $a = 6$, $b = 2$, $d = 1$, two body force distributions and two integration rules: $p=1$ and $p=2$.

The uniform body force distribution $b_r = 3$ and $b_z = -1$ gives for the 1×1 and 2×2 integration rules:

$$\begin{aligned} \mathbf{f}_{1 \times 1}^e &= [36 \quad -12 \quad 36 \quad -12 \quad 36 \quad -12 \quad 36 \quad -12]^T \\ \mathbf{f}_{2 \times 2}^e &= [27 \quad -9 \quad 45 \quad -15 \quad 45 \quad -15 \quad 27 \quad -9]^T \end{aligned} \quad (12.19)$$

Note that $f_{r1} + f_{r2} + f_{r3} + f_{r4} = 144$ for both rules. Likewise for the $-z$ component. Thus the total force is conserved. The varying body force distribution $b_r = r$ and $b_z = 0$, which mimics a centrifugal force, gives for the 1×1 and 2×2 integration rules:

$$\begin{aligned} \mathbf{f}_{1 \times 1}^e &= [42 \quad 0 \quad 42 \quad 0 \quad 42 \quad 0 \quad 42 \quad 0]^T \\ \mathbf{f}_{2 \times 2}^e &= [29 \quad 0 \quad 70 \quad 0 \quad 70 \quad 0 \quad 29 \quad 0]^T \end{aligned} \quad (12.20)$$

```

ClearAll[a,b,d,h,p,number];
a=6; b=2; d=1; br=3; bz=1;
Jcons=False; number=True;
ncoor={{d,0},{a+d,0},{a+d,b},{d,b}};
For [p=1,p<=2,p++, Print["Gauss rule p=",p];
  fe=Quad4IsoPRingBodyForces[ncoor,{3,-1},{number,p}];
  Print["fe =",Partition[fe,2]//MatrixForm];
  bfor={{1,0},{6,0},{6,0},{1,0}};
  fe=Quad4IsoPRingBodyForces[ncoor,bfor,{number,p}];
  Print["fe =",Partition[fe,2]//MatrixForm];
];

```

FIGURE 12.8. Test statements to exercise body force module of Figure 12.7.

Here $f_{r1} + f_{r2} + f_{r3} + f_{r4} = 168$ for $p = 1$ but that sum is 198 for $p = 2$. The 2×2 rule captures a variable body force radial distribution better, as may be expected.

Trying with $p = 3$ or greater reproduces the results of the 2×2 product rule.

§12.2.4. Stress Recovery Module

Module Quad4IsoPRingStresses, listed in Figure 12.9, recovers stresses at the 4 corner nodes of the iso-P 4-node quadrilateral ring element, given its node displacements.

The procedure is as follows. The stresses are recovered at five sample points $k = 0, 1, 2, 3, 4$ with quadrilateral coordinates $\{\xi, \eta\} = \{0, 0\}, \{-g, -g\}, \{g, -g\}, \{g, g\}, \{-g, g\}$, in which $0 < g \leq 1$, using the direct evaluation $\bar{\sigma}_k = \mathbf{E} \mathbf{B}_k^e \mathbf{u}^e$. Here a bar over the stress symbol is used to mark a sample value. Perform a least-square bilinear fit over the 5 sample points assigning weight w_0 to sample at $\{\xi, \eta\} = \{0, 0\}$ and weight 1 to each of the samples at $\{\xi, \eta\} = \{\pm g, \pm g\}$. Evaluation of the fit at the corners $\{\xi, \eta\} = \{\pm 1, \pm 1\}$ yields

$$\begin{bmatrix} \sigma_{rr1} & \sigma_{zz1} & \sigma_{\theta\theta1} & \sigma_{rz1} \\ \sigma_{rr2} & \sigma_{zz2} & \sigma_{\theta\theta2} & \sigma_{rz2} \\ \sigma_{rr3} & \sigma_{zz3} & \sigma_{\theta\theta3} & \sigma_{rz3} \\ \sigma_{rr4} & \sigma_{zz4} & \sigma_{\theta\theta4} & \sigma_{rz4} \end{bmatrix} = \frac{1}{T_d} \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_3 \\ T_1 & T_3 & T_2 & T_3 & T_4 \\ T_1 & T_4 & T_3 & T_2 & T_3 \\ T_1 & T_3 & T_4 & T_3 & T_2 \end{bmatrix} \begin{bmatrix} \bar{\sigma}_{rr0} & \bar{\sigma}_{zz0} & \bar{\sigma}_{\theta\theta0} & \bar{\sigma}_{rz0} \\ \bar{\sigma}_{rr1} & \bar{\sigma}_{zz1} & \bar{\sigma}_{\theta\theta1} & \bar{\sigma}_{rz1} \\ \bar{\sigma}_{rr2} & \bar{\sigma}_{zz2} & \bar{\sigma}_{\theta\theta2} & \bar{\sigma}_{rz2} \\ \bar{\sigma}_{rr3} & \bar{\sigma}_{zz3} & \bar{\sigma}_{\theta\theta3} & \bar{\sigma}_{rz3} \\ \bar{\sigma}_{rr4} & \bar{\sigma}_{zz4} & \bar{\sigma}_{\theta\theta4} & \bar{\sigma}_{rz4} \end{bmatrix}, \quad (12.21)$$

in which $T_1 = 4g^2 w_0$, $T_2 = 4 + 4g^2 + w_0 + 2g(4 + w_0)$, $T_3 = 4g^2 - 4 - w_0$, $T_4 = 4 + 4g^2 + w_0 - 2g(4 + w_0)$ and $T_d = 4g^2(4 + w_0)$. The default values used in the least-square fit are $w_0 = 0$ and $g = 1/\sqrt{3}$, in which case $\{\xi, \eta\} = \{\pm g, \pm g\}$ are located at the sample points of the 2×2 Gauss product rule.

The module is invoked as

$$\mathbf{K}_e = \text{Quad4IsoPRingStresses}[\text{ncoor}, \text{Emat}, \text{ue}, \text{options}] \quad (12.22)$$

The arguments are:

ncoor	Same as in Quad4IsoPRingStiffness
Emat	Same as in Quad4IsoPRingStiffness

ue The element node displacements arranged as a one-dimensional list: {ur1,uz1,ur2,uz2,ur3,uz3,ur4,uz4} representing the displacement vector

$$\mathbf{u}^e = [u_{r1} \ u_{z1} \ u_{r2} \ u_{z2} \ u_{r3} \ u_{z3} \ u_{r4} \ u_{z4}]^T. \quad (12.23)$$

options A list of processing options. This list may be either empty or contain up to 4 items. Possible configurations are {}, {number}, {number,g} or {number,g,w0}.

number is a logical flag with value True or False. If True, the computations are forced to proceed in floating point arithmetic. For symbolic or exact arithmetic work set number to False. If omitted, False is assumed.

g Defines location of 4 sample points within element from which stresses are extrapolated to the corners according to (12.21). If omitted the default $g = 1/\sqrt{3}$ is assumed.

w0 Weight used in the least-square extrapolator (12.21). If omitted the default $w_0 = 0$ is assumed.

As function value the module returns

sige computed corner stresses stored in a 4-entry, two-dimensional list:
 {{sigrr1,sigzz1,sigtt1,sigrz1}, {sigrr2,sigzz2,sigtt2,sigrz2},
 {sigrr3,sigzz3,sigtt3,sigrz3}, {sigrr4,sigzz4,sigtt4,sigrz4}}
 to represent the array shown on the left hand side of (12.23).

```
ClearAll[Em,v,a,b,d,err,ezz,grz,ur,uz,r,z];
Em=2500; v=1/4;
{err,ezz,err,grz}={3/80,-1/40,3/80,4/50};
ncoor={{d,0},{a+d,0},{a+d,b},{d,b}}; num=False;
Emat=Em/((1+v)*(1-2*v))*{{1+v,v,v,0},{v,1+v,v,0},
{v,v,1+v,0},{0,0,0,1/2-v}};
{err,ezz,err,grz}={3/80,-1/40,3/80,4/50};
ur[r_,z_]:=err*r; uz[r_,z_]:=ezz*z+grz*r;
ue=Table[{0,0},{4}];
For [n=1,n<=4,n++,{rn,zn}=ncoor[[n]];
ue[[n]]={ur[rn,zn],uz[rn,zn]}];
ue=Flatten[ue]; Print["ue=",ue];
sige=Quad4IsoPRingStresses[ncoor,Emat,ue,{}];
Print["Corner stresses=",sige//MatrixForm];
```

FIGURE 12.10. Test statements for stress recovery module Quad4IsoPRingStresses.

Example 12.3. The stress recovery module is tested by the statements listed in Figure 12.10. The technique used is to generate the element node displacements by evaluating a test displacement field

$$u_r(r, z) = e_{rr}r, \quad u_z(r, z) = e_{zz}r + \gamma_{rz}z \quad (12.24)$$

in which $\{e_{rr}, e_{zz}, \gamma_{rz}\}$ are specified strains assumed constant over the element. Note that the hoop strain is $e_{\theta\theta} = u_r/r = e_{rr}$. The geometry is that of the rectangular cross-section element of Figure 12.5.

```

Quad4IsoPRingStresses[ncoor_, Emat_, ue_, options_] :=
Module[{number=False, g=1/Sqrt[3], Jcons=False, w0=0,
  eps=10.^(-9), r1, r2, r3, r4, z1, z2, z3, z4, Nf, N1, N2, N3, N4,
  dNr1, dNr2, dNr3, dNr4, dNz1, dNz2, dNz3, dNz4,
  T1, T2, T3, T4, Td, Tg4, Jdet, qcoor, w, c, Be,
  gctab={{0,0}}, k, kg, rk, sigg, sige, udis=ue,
  modname="Quad4IsoPRingStresses: "},
  If [Length[options]==1, {number}=options];
  If [Length[options]==2, {number, g}=options];
  If [Length[options]==3, {number, g, w0}=options];
  If [Head[g]==Symbol||g>0, Td=4*g^2*(4+w0);
    T1=4*g^2*w0; T2=4+4*g^2+w0+2*g*(4+w0);
    T3=-4+4*g^2-w0; T4=4+4*g^2+w0-2*g*(4+w0);
    Tg4={{T1, T2, T3, T4, T3}, {T1, T3, T2, T3, T4},
      {T1, T4, T3, T2, T3}, {T1, T3, T4, T3, T2}}/Td;
    gctab={{0,0}, {-1,-1}, {1,-1}, {1,1}, {-1,1}}*g];
  kg=Length[gctab]; sigg=Table[{0,0,0,0}, {kg}];
  If [number, gctab=N[gctab]; Tg4=N[Tg4]; udis=N[ue] ];
  {{r1, z1}, {r2, z2}, {r3, z3}, {r4, z4}}=ncoor;
  For [k=1, k<=kg, k++, qcoor=gctab[[k]];
    {Nf, {dNr1, dNr2, dNr3, dNr4}, {dNz1, dNz2, dNz3, dNz4},
      Jdet}=Quad4IsoPRingShapeFunDer[ncoor, qcoor, Jcons];
    {N1, N2, N3, N4}=Nf; rk=r1*N1+r2*N2+r3*N3+r4*N4;
    Be={{ dNr1, 0, dNr2, 0, dNr3, 0, dNr4, 0},
      { 0, dNz1, 0, dNz2, 0, dNz3, 0, dNz4},
      {N1/rk, 0, N2/rk, 0, N3/rk, 0, N4/rk, 0},
      { dNz1, dNr1, dNz2, dNr2, dNz3, dNr3, dNz4, dNr4}};
    If [number, Be=N[Be]]; sigg[[k]]=Emat.(Be.udis);
  ];
  If [kg==1, sige=Table[sigg[[1]], {4}], sige=Tg4.sigg];
  If [number, sige=Chop[sige, eps]];
  If [!number, sige=Simplify[sige]]; Return[sige] ];

```

FIGURE 12.9. Module for recovery of Quad4 ring element corner stresses from displacements.

The displacement field (12.24) is evaluated at the corner nodes to construct the node displacement vector, which is then fed to the stress recovery module. Dimensions a , b and d are kept arbitrary as symbolic variables. Numeric data: $e_{rr} = e_{\theta\theta} = 3/80$, $e_{zz} = -1/40$, $\gamma_{rz} = 4/50$, $E = 2500$ and $\nu = 1/4$. The associated displacement field as per (12.24) is

$$\mathbf{u}^e = \begin{bmatrix} \frac{3d}{80} & \frac{2d}{25} & \frac{3(a+d)}{80} & \frac{2(a+d)}{25} & \frac{3(a+d)}{80} & \frac{b}{40} + \frac{2(a+d)}{25} & \frac{3d}{80} & -\frac{b}{40} + \frac{2d}{25} \end{bmatrix} \quad (12.25)$$

The computed corner stresses returned by the module (note that logical flag `number` is `False` since that is the default) are

$$\sigma^e = \begin{bmatrix} 200 & -50 & 200 & 80 \\ 200 & -50 & 200 & 80 \\ 200 & -50 & 200 & 80 \\ 200 & -50 & 200 & 80 \end{bmatrix} \quad (12.26)$$

which may be verify to be correct.

§12.3. The 8-Node Quadrilateral Ring Element

This is the axisymmetric solid version of the isoparametric quadrilateral with serendipity shape functions. The element has 8 nodes and 16 displacement degrees of freedom arranged as

$$\mathbf{u}^e = [u_{r1} \ u_{z1} \ u_{r2} \ u_{z2} \ u_{r3} \ u_{z3} \ \dots \ u_{r8} \ u_{z8}]^T. \quad (12.27)$$

```

Quad8IsoPRingShapeFunDer[ncoor_, qcoor_, Jcons_] := Module[
{r1, r2, r3, r4, r5, r6, r7, r8, z1, z2, z3, z4, z5, z6, z7, z8,
ξ, η, rv, zv, A0, dNξ, dNη, N1B, N2B, N3B, N4B, J11, J12, J21, J22,
Nf, dNr, dNz, Jdet}, {ξ, η} = qcoor;
{{r1, z1}, {r2, z2}, {r3, z3}, {r4, z4}, {r5, z5}, {r6, z6}, {r7, z7},
{r8, z8}} = ncoor; A0 = ((r5 - r7) * (z6 - z8) - (r6 - r8) * (z5 - z7)) / 4;
N1B = (1 - ξ) * (1 - η) / 4; N2B = (1 + ξ) * (1 - η) / 4;
N3B = (1 + ξ) * (1 + η) / 4; N4B = (1 - ξ) * (1 + η) / 4;
Nf = {-N1B * (1 + ξ + η), -N2B * (1 - ξ + η), -N3B * (1 - ξ - η),
-N4B * (1 + ξ - η), 2 * N1B * (1 + ξ), 2 * N3B * (1 - η),
2 * N3B * (1 - ξ), 2 * N4B * (1 - η)};
dNξ = {(1 - η) * (2 * ξ + η), (1 - η) * (2 * ξ - η), (1 + η) * (2 * ξ + η),
(1 + η) * (2 * ξ - η), 4 * ξ * (η - 1), 2 * (1 - η^2),
-4 * ξ * (1 + η), -2 * (1 - η^2)} / 4;
dNη = {(1 - ξ) * (ξ + 2 * η), -(1 + ξ) * (ξ - 2 * η), (1 + ξ) * (ξ + 2 * η),
-(1 - ξ) * (ξ - 2 * η), -2 * (1 - ξ^2), -4 * (1 + ξ) * η,
2 * (1 - ξ^2), -4 * (1 - ξ) * η} / 4;
rv = {r1, r2, r3, r4, r5, r6, r7, r8}; zv = {z1, z2, z3, z4, z5, z6, z7, z8};
J11 = dNξ.rv; J12 = dNξ.zv; J21 = dNη.rv; J22 = dNη.zv;
Jdet = Simplify[J11 * J22 - J12 * J21]; If [Jcons, Jdet = A0];
dNr = (J22 * dNξ - J12 * dNη) / Jdet;
dNz = (-J21 * dNξ + J11 * dNη) / Jdet;
Return[{Nf, dNr, dNz, Jdet}];

```

FIGURE 12.11. Shape function module for 8-node bilinear quadrilateral ring element.

§12.3.1. Shape Function Module

Module Quad8IsoPRingShapeFunDer, listed in Figure 12.11, computes the shape functions N_i^e , $i = 1, 2, \dots, 8$ and their partial derivatives with respect to r and z at a specified point in the element. Usually this module is called at sample points of a Gauss quadrature rule, but it may also be used with symbolic inputs to get information for an arbitrary point at $\{\xi, \eta\}$. The element geometry is defined by the 16 coordinates $\{r_i, z_i\}$, $i = 1, 2, \dots, 8$. These are collected in the arrays

$$\mathbf{r} = [r_1 \ r_2 \ \dots \ r_8]^T, \quad \mathbf{z} = [z_1 \ z_2 \ \dots \ z_8]^T. \quad (12.28)$$

We will use the abbreviations $r_{ij} = r_i - r_j$ and $z_{ij} = z_i - z_j$ for coordinate differences. The shape functions and their partial derivatives with respect to the quadrilateral coordinates are collected in the following arrays. Using the abbreviations $N_1^B = \frac{1}{4}(1 - \xi)(1 - \eta)$, $N_2^B = \frac{1}{4}(1 + \xi)(1 - \eta)$, $N_3^B = \frac{1}{4}(1 + \xi)(1 + \eta)$ and $N_4^B = \frac{1}{4}(1 - \xi)(1 + \eta)$ for the shape functions of the 4-noded bilinear quadrilateral, we have

$$\mathbf{N} = \begin{bmatrix} -N_1^B(1 + \xi + \eta) \\ -N_2^B(1 - \xi + \eta) \\ -N_3^B(1 - \xi - \eta) \\ -N_4^B(1 + \xi - \eta) \\ 2N_1^B(1 + \xi) \\ 2N_2^B(1 - \eta) \\ 2N_3^B(1 - \xi) \\ 2N_4^B(1 - \eta) \end{bmatrix}, \quad \mathbf{N}_{,\xi} = \frac{1}{4} \begin{bmatrix} (1 - \eta)(2\xi + \eta) \\ (1 - \eta)(2\xi - \eta) \\ (1 + \eta)(2\xi + \eta) \\ (1 + \eta)(2\xi - \eta) \\ 2\xi(\eta - 1) \\ 2(1 - \eta^2) \\ -2\xi(1 + \eta) \\ -2(1 - \eta^2) \end{bmatrix}, \quad \mathbf{N}_{,\eta} = \frac{1}{4} \begin{bmatrix} (1 - \xi)(\xi + 2\eta) \\ -(1 + \xi)(\xi - 2\eta) \\ (1 + \xi)(\xi + 2\eta) \\ -(1 - \xi)(\xi - 2\eta) \\ -2(1 - \xi^2) \\ -2(1 + \xi)\eta \\ 2(1 - \xi^2) \\ -2(1 - \xi)\eta \end{bmatrix}. \quad (12.29)$$

The Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{,\xi} \mathbf{r} & \mathbf{N}_{,\xi} \mathbf{z} \\ \mathbf{N}_{,\eta} \mathbf{r} & \mathbf{N}_{,\eta} \mathbf{z} \end{bmatrix}. \quad (12.30)$$

with determinant $J = \det(\mathbf{J}) = J_{11} J_{22} - J_{12} J_{21}$. Finally the $\{r, z\}$ partials are obtained from

$$\begin{bmatrix} \mathbf{N}_{,r} \\ \mathbf{N}_{,z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial r} \\ \frac{\partial \mathbf{N}}{\partial z} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \mathbf{N}_{,\xi} \\ \mathbf{N}_{,\eta} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{bmatrix} \mathbf{N}_{,\xi} \\ \mathbf{N}_{,\eta} \end{bmatrix}. \quad (12.31)$$

or $\mathbf{N}_{,r} = (J_{11}\mathbf{N}_{,\xi} - J_{12}\mathbf{N}_{,\eta})/J$ and $\mathbf{N}_{,z} = (-J_{21}\mathbf{N}_{,\xi} + J_{22}\mathbf{N}_{,\eta})/J$. Unlike the 4-node quadrilateral, explicit expressions for $\mathbf{N}_{,r}$ and $\mathbf{N}_{,z}$ are difficult to work out because of the increased complexity of the polynomials. The module listed in Figure 12.11 does not attempt to do so. The module is invoked as

$$\{\text{Nf}, \text{dNr}, \text{dNz}, \text{Jdet}\} = \text{Quad4IsoP RingShapeFunDer}[\text{ncoor}, \text{qcoor}, \text{Jcons}] \quad (12.32)$$

where the arguments are

- ncoor** Quadrilateral node coordinates arranged in two-dimensional list form: $\{\{r1, z1\}, \{r2, z2\}, \{r3, z3\}, \dots, \{r8, z8\}\}$.
- qcoor** Quadrilateral coordinates $\{\xi, \eta\}$ of the point at which shape functions and derivatives are to be evaluated.
- Jcons** A logical flag. If True, the Jacobian determinant J is set to its value at the element center: $A_0 = (r_{57}z_{68} - r_{68}z_{57})/4$, for any $\{\xi, \eta\}$. That option is useful in certain research studies.

The module returns the list $\{\text{Nf}, \text{dNr}, \text{dNz}, \text{Jdet}\}$, where

- Nf** Shape function values² arranged as the list $\{\text{N1}, \text{N2}, \text{N3}, \dots, \text{N8}\}$.
- dNr** r shape function derivatives (12.31) arranged as the list $\{\text{dNr1}, \text{dNr2}, \text{dNr3}, \dots, \text{dNr8}\}$.
- dNz** z shape function derivatives (12.31) arranged as the list $\{\text{dNz1}, \text{dNz2}, \text{dNz3}, \dots, \text{dNz8}\}$.
- Jdet** Jacobian determinant.

§12.3.2. Element Stiffness Module

Module `Quad8IsoP RingStiffness`, listed in Figure 12.12, computes the stiffness matrix of a 8-noded iso-P quadrilateral ring element. The computation is carried out using numerical quadrature. The module is invoked as

$$\text{Ke} = \text{Quad8IsoP RingStiffness}[\text{ncoor}, \text{Emat}, \text{options}] \quad (12.33)$$

² Note that N cannot be used as name of the list of shape function values, because that symbol is reserved.


```

Quad8IsoPRingStiffness[ncoor_,Emat_,options_]:=Module[
{p=2,numer=False,Jcons=False,Kfac=1,qcoor,k,
r1,r2,r3,r4,r5,r6,r7,r8,z1,z2,z3,z4,z5,z6,z7,z8,
Nf,N1,N2,N3,N4,N5,N6,N7,N8,
dNr1,dNr2,dNr3,dNr4,dNr5,dNr6,dNr7,dNr8,
dNz1,dNz2,dNz3,dNz4,dNz5,dNz6,dNz7,dNz8,
rk,w,c,A0,Jdet,Be,Ke,Ke0=Table[0,{16},{16}]},
modname="Quad8IsoPRingStiffness: ",Ke=Ke0;
If [Length[options]==1,{numer}=options];
If [Length[options]==2,{numer,p}=options];
If [Length[options]==3,{numer,p,Jcons}=options];
If [Length[options]==4,{numer,p,Jcons,Kfac}=options];
If [p<1||p>5, Print[modname,"illegal p:",p]; Return[Ke0]];
{{r1,z1},{r2,z2},{r3,z3},{r4,z4},
{r5,z5},{r6,y6},{r7,z7},{r8,z8}}=ncoor;
A0=((r5-r7)*(z6-z8)-(r6-r8)*(z5-z7))/4;
If [numer&&(A0<=0), Print[modname,"Neg or zero area"];
Return[Ke0]];
For [k=1,k<=p*p,k++,
{qcoor,w}= QuadGaussRuleInfo[{p,numer},k];
{{N1,N2,N3,N4,N5,N6,N7,N8},
{dNr1,dNr2,dNr3,dNr4,dNr5,dNr6,dNr7,dNr8},
{dNz1,dNz2,dNz3,dNz4,dNz5,dNz6,dNz7,dNz8},
Jdet}=Quad8IsoPRingShapeFunDer[ncoor,qcoor,Jcons];
If [numer&&(Jdet<=0), Print[modname,"Neg or zero",
"Gauss point Jacobian at k=",k]; Return[Ke0]];
rk=r1*N1+r2*N2+r3*N3+r4*N4+r5*N5+r6*N6+r7*N7+r8*N8;
Be={{ dNr1, 0, dNr2, 0, dNr3, 0, dNr4, 0,
dNr5, 0, dNr6, 0, dNr7, 0, dNr8, 0},
{ 0,dNz1, 0,dNz2, 0,dNz3, 0,dNz4,
0,dNz5, 0,dNz6, 0,dNz7, 0,dNz8},
{N1/rk, 0,N2/rk, 0,N3/rk, 0,N4/rk, 0,
N5/rk, 0,N6/rk, 0,N7/rk, 0,N8/rk, 0},
{ dNz1,dNr1, dNz2,dNr2, dNz3,dNr3, dNz4,dNr4,
dNz5,dNr5, dNz6,dNr6, dNz7,dNr7, dNz8,dNr8}};
c=Kfac*w*rk*Jdet; If[!numer, Be=Simplify[Be]];
If [numer,Be=N[Be]; c=N[c]];
Ke+=c*Transpose[Be].(Emat.Be);
]; Return[Ke] ];

```

FIGURE 12.12. Element stiffness formation module for 8-node iso-P quadrilateral ring.

The arguments are:

ncoor	Quadrilateral node coordinates arranged in two-dimensional list form: { {r1,z1},{r2,z2},{r3,z3}, ... {r8,z8} }.
Emat	Same as for Quad4IsoPRingStiffness.
options	Same as for Quad4IsoPRingStiffness.

As function value the module returns

Ke	a 16×16 symmetric matrix pertaining to a node by node arrangement of element node displacements. If an error is detected during processing, a zero matrix is returned.

Example 12.4. The stiffness module is tested on the geometry identified in Figure 12.5. The cross section is a rectangle dimensioned $a \times b$ with sides parallel to the $\{r, z\}$ axes. The distance of the leftmost side to the z

```

ClearAll[Em,v,a,b,d,h,p,num];
Em=96; v=1/3; a=4; b=2; d=0; Kfac=2*Pi; Kfac=1;
ncoor={{d,0},{a+d,0},{a+d,b},{d,b}}; num=False;
Emat=Em/((1+v)*(1-2*v))*{{1-v,v,v,0},{v,1-v,v,0},
{v,v,1-v,0},{0,0,0,1/2-v}};
Print["Emat=",Emat//MatrixForm];
For [p=1,p<=4,p++, Print["Gauss rule p=",p];
Ke=Quad4IsoPRingStiffness[ncoor,Emat,Kfac,{num,p}];
Ke=Simplify[Ke]; Print["Ke=",Ke//MatrixForm];
Print["Eigenvalues of Ke=",Chop[Eigenvalues[N[Ke]]]];
];

```

FIGURE 12.13. Driver for exercising the Quad8IsoPRingStiffness module of Figure 12.12 using the ring element geometry shown in Figure 12.5, with $E = 96$, $\nu = 1/3$, $a = 4$, $b = 2$, $d = 0$ and four Gauss product integration rules.

axis is d . The material is isotropic with modulus E and Poisson's ratio ν .

The script of Figure 12.8 computes and prints the stiffness of the test element shown in for $E = 96$, $\nu = 1/3$, $a = 4$, $b = 2$, $d = 0$. The default $Kfac = 1$ is used. Nodes 1 and 2 sit on the z axes. The value of p is changed in a loop. The flag `num` is set to `True` to use floating-point computation for speed. The computed entries of $\mathbf{K}^e$ are exact integers for all values of p :

The eigenvalues of these matrices are:

Rule	Eigenvalues of $\mathbf{K}^e$ for varying integration rule							
1×1	667.794	180.000	124.206	72.000	0	0	0	0
2×2	745.201	261.336	248.750	129.451	100.389	88.598	10.275	0
3×3	745.446	330.628	266.646	133.236	126.343	98.690	11.011	0
4×4	745.716	397.372	272.092	144.542	135.004	101.908	11.365	0

(12.34)

The stiffness matrix computed by the one-point rule is rank deficient by 11. For $p = 2$ it is rank deficient by one, but the element is useful³ since the spurious mode is not usually propagated over the mesh. For $p = 3$ and higher the element has full rank of 15. The eigenvalues do not change appreciably after $p = 2$.

§12.3.3. Body Force Module

Module `Quad8IsoPRingBodyForces`, listed in Figure 12.14 computes the consistent force vector associated with a body force field $\bar{\mathbf{b}} = \{b_x, b_y\}$ specified over an 8-node iso-P quadrilateral ring element. The field is assumed to be given per unit of volume, in radial-axial component-wise form. The force vector is computed by Gauss numerical integration as described in the previous chapter. The module is invoked as

$$\mathbf{K}^e = \text{Quad8IsoPRingBodyForces}[\text{ncoor}, \text{bfor}, \text{options}] \quad (12.35)$$

The arguments are:

`ncoor` Same as in `Quad8IsoPRingStiffness`

³ It will be seen in the benchmarks of Chapter 14 that the 8-node quadrilateral integrated with the 2×2 Gauss rule outperforms the fully integrated version, especially for near-incompressible material behavior.

- bfor** Specifies body force field (forces per unit of volume) over the element. Three specification formats are allowed.
- One-dimensional list: { br , bz }
- Two-dimensional list with corner values only: { { br1 , bz1 } , { br2 , bz2 } ,
... { br4 , bz4 } }
- Two-dimensional list with values at corners and midnodes: { { br1 , bz1 } , ...
{ br8 , bz8 } }
- In the first form the body force field is taken to be uniform over the element, with radial component br and axial component bz.
- The second and third forms assume body forces to vary over the element. If only the corner values are given, the value at midnodes is determined from the adjacent corner nodes by averaging. From this information the field is interpolated over the element using the 8-node shape functions.
- options** Same as in Quad8IsoPRingStiffness

As function value the module returns

- fe** Consistent force vector arranged { fr1 , fz1 , fr2 , fz2 , fr3 , fz3 , fr4 , fz4 } to represent

$$\mathbf{f}^e = [f_{r1} \quad f_{z1} \quad f_{r2} \quad f_{z2} \quad f_{r3} \quad f_{z3} \quad \dots \quad f_{r8} \quad f_{z8}]^T . \quad (12.36)$$

```
ClearAll[a,b,d,p];
a=3; b=2; d=1;
ncoor={{d,0},{a+d,0},{a+d,b},{d,b},{a/2+d,0},{a+d,b/2},{a/2+d,b},
{d,b/2}};
For [p=1,p<=3,p++, For [case=1,case<=2,case++,
If [case==1, bfor={36,-18}];
If [case==2, bfor=Table[{60*ncoor[[i,1]],0},{i,8}]]];
fe=Quad8IsoPRingBodyForces[ncoor,bfor,{True,p}];
fe=Simplify[Chop[fe]];
Print["fe=",Transpose[Partition[fe,2]]//MatrixForm];
frsum=Sum[fe[[2*i-1]],{i,8}]; fzsum=Sum[fe[[2*i]],{i,8}];
Print["frsum=",frsum," fzsum=",fzsum];
];
```

FIGURE 12.15. Test statements to exercise body force module of Figure 12.7.

Example 12.5. To be added later.

```

Quad8IsoPRingBodyForces[ncoor_, bfor_, options_] := Module[
  {p=2, numer=False, Jcons=False, Kfac=1, qcoor, k, m, mOK,
   r1, r2, r3, r4, r5, r6, r7, r8, z1, z2, z3, z4, z5, z6, z7, z8,
   Nf, N1, N2, N3, N4, N5, N6, N7, N8, dNr, dNz,
   br1, br2, br3, br4, br5, br6, br7, br8,
   bz1, bz2, bz3, bz4, bz5, bz6, bz7, bz8,
   rk, w, c, A0, Jdet, fe, fe0=Table[0, {16}]},
  modname="Quad8IsoPRingBodyForces: ", fe=fe0;
  If [Length[options]==1, {numer}=options];
  If [Length[options]==2, {numer, p}=options];
  If [Length[options]==3, {numer, p, Jcons}=options];
  If [Length[options]==4, {numer, p, Jcons, Kfac}=options];
  If [p<1||p>5, Print[modname, "illegal p:", p]; Return[fe0]];
  {{r1, z1}, {r2, z2}, {r3, z3}, {r4, z4},
   {r5, z5}, {r6, z6}, {r7, z7}, {r8, z8}}=ncoor;
  A0=((r5-r7)*(z6-z8)-(r6-r8)*(z5-z7))/4;
  If [numer&&(A0<=0), Print[modname, "Neg or zero area"];
    Return[fe0]]; m=Length[bfor]; mOK=MemberQ[{2, 4, 8}, m];
  If [!mOK, Print[modname, " Illegal bfor"]; Return[fe0]];
  If [m==2, br1=br2=br3=br4=br5=br6=br7=br8=bfor[[1]];
    bz1=bz2=bz3=bz4=bz5=bz6=bz7=bz8=bfor[[2]]];
  If [m==4, {{br1, bz1}, {br2, bz2}, {br3, bz3}, {br4, bz4}}=bfor;
    {br5, bz5, br6, bz6, br7, bz7, br8, bz8}={br1+br2,
    bz1+bz2, br2+br3, bz2+bz3, br3+br4, bz3+bz4,
    br4+br1, bz4+bz1}/2];
  If [m==8, {{br1, bz1}, {br2, bz2}, {br3, bz3}, {br4, bz4},
    {br5, bz5}, {br6, bz6}, {br7, bz7}, {br8, bz8}}=bfor];
  For [k=1, k<=p*p, k++,
    {qcoor, w}= QuadGaussRuleInfo[{p, numer}, k];
    {N1, N2, N3, N4, N5, N6, N7, N8}, dNr, dNz, Jdet}=
      Quad8IsoPRingShapeFunDer[ncoor, qcoor, Jcons];
    If [numer&&(Jdet<=0), Print[modname, "Neg or zero",
      " Gauss point Jacobian at k=", k]; Return[fe0]];
    rk=r1*N1+r2*N2+r3*N3+r4*N4+r5*N5+r6*N6+r7*N7+r8*N8;
    brk=br1*N1+br2*N2+br3*N3+br4*N4+br5*N5+br6*N6+br7*N7+br8*N8;
    bzk=bz1*N1+bz2*N2+bz3*N3+bz4*N4+bz5*N5+bz6*N6+bz7*N7+bz8*N8;
    bk={N1*brk, N1*bzk, N2*brk, N2*bzk, N3*brk, N3*bzk, N4*brk, N4*bzk,
      N5*brk, N5*bzk, N6*brk, N6*bzk, N7*brk, N7*bzk, N8*brk, N8*bzk};
    c=Kfac*w*Jdet*rk; If [numer, bk=N[bk]; c=N[c]]; fe+=c*bk;
  ]; Return[fe] ];

```

FIGURE 12.14. Module that computes consistent node forces for a 8-noded quadrilateral ring element given a body force field.

§12.3.4. Stress Recovery Module

Module Quad8IsoPRingStresses, listed in Figure 12.16, recovers stresses at the 4 corner nodes and 4 midpoints of the iso-P 8-node quadrilateral ring element, given its node displacements.

The procedure is similar to that used for the 4-node quadrilateral explained in §12.2.4. The stresses are recovered at five sample points $k = 0, 1, 2, 3, 4$ with quadrilateral coordinates $\{\xi, \eta\} = \{0, 0\}$, $\{-g, -g\}$, $\{g, -g\}$, $\{g, g\}$, $\{-g, g\}$, in which $0 < g \leq 1$, using the direct evaluation $\bar{\sigma}_k = \mathbf{E} \mathbf{B}_k^e \mathbf{u}^e$. (A bar over the stress symbol is used to mark a sample value.) Perform a least-square bilinear fit over the 5 sample points assigning weight $0 \leq w_0$ to sample at $\{\xi, \eta\} = \{0, 0\}$ and weight 1 to each

```

Quad8IsoPRingStresses[ncoor_, Emat_, ue_, options_] :=
Module[{number=False, g=1/Sqrt[3], Jcons=False, w0=0,
  eps=10.^(-9), r1, r2, r3, r4, r5, r6, r7, r8, z1, z2, z3, z4,
  z5, z6, z7, z8, Nf, N1, N2, N3, N4, N5, N6, N7, N8,
  dNr1, dNr2, dNr3, dNr4, dNr5, dNr6, dNr7, dNr8,
  dNz1, dNz2, dNz3, dNz4, dNz5, dNz6, dNz7, dNz8,
  T1, T2, T3, T4, T5, T6, Td, Tg8, Jdet, qcoor, w, c, Be,
  gctab={{0,0}}, k, kg, rk, sigg, sige, udis=ue,
  modname="Quad8IsoPRingStresses: "},
If [Length[options]==1, {number}=options];
If [Length[options]==2, {number, g}=options];
If [Length[options]==3, {number, g, w0}=options];
If [Head[g]==Symbol || g>0, Td=4*g^2*(4+w0);
  T1=4*g^2*w0; T2=4+4*g^2+w0+2*g*(4+w0);
  T3=-4+4*g^2-w0; T4=4+4*g^2+w0-2*g*(4+w0);
  T5=g*(4+4*g+w0); T6=g*(-4+4*g-w0);
  Tg8={{T1, T2, T3, T4, T5}, {T1, T3, T2, T3, T4},
    {T1, T4, T3, T2, T3}, {T1, T3, T4, T3, T2},
    {T1, T5, T5, T6, T6}, {T1, T6, T5, T5, T6},
    {T1, T6, T6, T5, T5}, {T1, T5, T6, T6, T5}}/Td;
  gctab={{0,0}, {-1,-1}, {1,-1}, {1,1}, {-1,1}}*g];
kg=Length[gctab]; sigg=Table[{0,0,0,0}, {kg}];
If [number, gctab=N[gctab]; Tg8=N[Tg8]; udis=N[ue]];
{{r1, z1}, {r2, z2}, {r3, z3}, {r4, z4},
  {r5, z5}, {r6, z6}, {r7, z7}, {r8, z8}}=ncoor;
For [k=1, k<=kg, k++, qcoor=gctab[[k]];
  {{N1, N2, N3, N4, N5, N6, N7, N8},
  {dNr1, dNr2, dNr3, dNr4, dNr5, dNr6, dNr7, dNr8},
  {dNz1, dNz2, dNz3, dNz4, dNz5, dNz6, dNz7, dNz8},
  Jdet}=Quad8IsoPRingShapeFunDer[ncoor, qcoor, Jcons];
  rk=r1*N1+r2*N2+r3*N3+r4*N4+r5*N5+r6*N6+r7*N7+r8*N8;
  Be={{ dNr1, 0, dNr2, 0, dNr3, 0, dNr4, 0,
    dNr5, 0, dNr6, 0, dNr7, 0, dNr8, 0},
    { 0, dNz1, 0, dNz2, 0, dNz3, 0, dNz4,
    0, dNz5, 0, dNz6, 0, dNz7, 0, dNz8},
    {N1/rk, 0, N2/rk, 0, N3/rk, 0, N4/rk, 0,
    N5/rk, 0, N6/rk, 0, N7/rk, 0, N8/rk, 0},
    { dNz1, dNr1, dNz2, dNr2, dNz3, dNr3, dNz4, dNr4,
    dNz5, dNr5, dNz6, dNr6, dNz7, dNr7, dNz8, dNr8}};
  If [number, Be=N[Be]]; sigg[[k]]=Emat.(Be.udis)
  ];
If [kg==1, sige=Table[sigg[[1]], {4}], sige=Tg8.sigg];
If [number, sige=Chop[sige, eps]];
If [!number, sige=Simplify[sige]]; Return[sige] ];

```

FIGURE 12.16. Module for recovery of Quad8 ring element corner stresses from displacements.

of the samples at $\{\xi, \eta\} = \{\pm g, \pm g\}$. Evaluation of the fit at the corner and midpoint nodes yields

$$\begin{bmatrix} \sigma_{rr1} & \sigma_{zz1} & \sigma_{\theta\theta1} & \sigma_{rz1} \\ \sigma_{rr2} & \sigma_{zz2} & \sigma_{\theta\theta2} & \sigma_{rz2} \\ \sigma_{rr3} & \sigma_{zz3} & \sigma_{\theta\theta3} & \sigma_{rz3} \\ \sigma_{rr4} & \sigma_{zz4} & \sigma_{\theta\theta4} & \sigma_{rz4} \\ \sigma_{rr5} & \sigma_{zz5} & \sigma_{\theta\theta5} & \sigma_{rz5} \\ \sigma_{rr6} & \sigma_{zz6} & \sigma_{\theta\theta6} & \sigma_{rz6} \\ \sigma_{rr7} & \sigma_{zz7} & \sigma_{\theta\theta7} & \sigma_{rz7} \\ \sigma_{rr8} & \sigma_{zz8} & \sigma_{\theta\theta8} & \sigma_{rz8} \end{bmatrix} = \frac{1}{T_d} \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 \\ T_1 & T_3 & T_2 & T_3 & T_4 \\ T_1 & T_4 & T_3 & T_2 & T_3 \\ T_1 & T_3 & T_4 & T_3 & T_2 \\ T_1 & T_5 & T_5 & T_6 & T_6 \\ T_1 & T_6 & T_5 & T_5 & T_6 \\ T_1 & T_6 & T_6 & T_5 & T_5 \\ T_1 & T_5 & T_6 & T_6 & T_5 \end{bmatrix} \begin{bmatrix} \bar{\sigma}_{rr0} & \bar{\sigma}_{zz0} & \bar{\sigma}_{\theta\theta0} & \bar{\sigma}_{rz0} \\ \bar{\sigma}_{rr1} & \bar{\sigma}_{zz1} & \bar{\sigma}_{\theta\theta1} & \bar{\sigma}_{rz1} \\ \bar{\sigma}_{rr2} & \bar{\sigma}_{zz2} & \bar{\sigma}_{\theta\theta2} & \bar{\sigma}_{rz2} \\ \bar{\sigma}_{rr3} & \bar{\sigma}_{zz3} & \bar{\sigma}_{\theta\theta3} & \bar{\sigma}_{rz3} \\ \bar{\sigma}_{rr4} & \bar{\sigma}_{zz4} & \bar{\sigma}_{\theta\theta4} & \bar{\sigma}_{rz4} \end{bmatrix}, \quad (12.37)$$

in which T_1, T_2, T_3, T_4 and T_d are the same as in the 4-node quadrilateral module whereas $T_5 = g(4 + 4g + w_0)$ and $T_6 = g(-4 + 4g - w_0)$. The default values used in the least-square fit are $w_0 = 0$ and $g = 1/\sqrt{3}$, in which case $\{\xi, \eta\} = \{\pm g, \pm g\}$ are located at the sample points of the 2×2 Gauss product rule.

```
ClearAll[Em,v,a,b,d,err,ezz,grz,ur,uz,r,z];
Em=2500; v=1/4; d=1; a=3; b=2;
{err,ezz,grz}={3/80,-1/40,3/80,4/50};
ncoor={{d,0},{a+d,0},{a+d,b},{d,b},{a/2+d,0},{a+d,b/2},{a/2+d,b},
{d,b/2}};
Emat=Em/((1+v)*(1-2*v))*{{1+v,v,v,0},{v,1+v,v,0},
{v,v,1+v,0},{0,0,0,1/2-v}};
{err,ezz,grz}={3/80,-1/40,3/80,4/50};
ur[r_,z_]:=err*r; uz[r_,z_]:=ezz*z+grz*r;
ue=Table[{0,0},{8}];
For [n=1,n<=8,n++, {rn,zn}=ncoor[[n]];
ue[[n]]={ur[rn,zn],uz[rn,zn]}];
ue=Flatten[ue]; Print["ue=",ue];
sige=Quad8IsoPRingStresses[ncoor,Emat,ue,{True}];
Print["Corner stresses=",sige//MatrixForm];
```

FIGURE 12.17. Test statements for stress recovery module Quad8IsoPRingStresses.

The module is invoked as

$$\text{Ke} = \text{Quad8IsoPRingStresses}[\text{ncoor}, \text{Emat}, \text{ue}, \text{options}] \quad (12.38)$$

The arguments are:

ncoor Node coordinates: same as in Quad8IsoPRingStiffness
Emat Elasticity matrix: same as in Quad8IsoPRingStiffness
ue The element node displacements arranged as a one-dimensional list: $\{ur_1, uz_1, ur_2, uz_2, ur_3, uz_3, \dots, ur_8, uz_8\}$ representing the displacement vector

$$\mathbf{u}^e = [u_{r1} \ u_{z1} \ u_{r2} \ u_{z2} \ u_{r3} \ u_{z3} \ \dots \ u_{r8} \ u_{z8}]^T. \quad (12.39)$$

options Same as in Quad4IsoPRingStresses. The same defaults for omitted values apply.

As function value the module returns

sige computed corner stresses stored in a 8-entry, two-dimensional list:
 $\{\{ \text{sigrr1}, \text{sigzz1}, \text{sigtt1}, \text{sigrz1} \}, \{ \text{sigrr2}, \text{sigzz2}, \text{sigtt2}, \text{sigrz2} \},$
 $\dots \{ \text{sigrr8}, \text{sigzz8}, \text{sigtt8}, \text{sigrz8} \} \}$ to represent the array shown on
 the left hand side of (12.37).

Example 12.6. To be added later.

Homework Exercises for Chapter 12
4- and 8-Node Iso-P Quadrilateral Ring Elements

No Exercises constructed for this Chapter yet. The elements are used in Exercises 12.1 through 12.3.

13

A Complete Axisymmetric FEM Program

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This chapter describes a complete axisymmetric FEM program for static analysis of axisymmetric solids with the quadrilateral ring elements described in the previous Chapter. The program is implemented in the *Mathematica* Notebook QuadSOR.nb, which is made available on the course web site for assigned computer exercises.

Unlike the previous chapter, the description is top down, i.e. starts from the main driver programs. Three benchmark examples that illustrate the operation of the code as well as the operation of the code and the level of accuracy attainable with 4-node versus 8-node configurations, are given in the next Chapter.

§13.1. FEM Analysis Stages

As in all FEM programs, the analysis of a structure by the Direct Stiffness Method involves three major stages: (I) preprocessing or model definition, (II) processing, and (III) postprocessing. The overall program organization is shown in Figure 13.1.

The preprocessing portion involves:

- I.1 Model definition by direct setting of the data structures.
- I.2 Printing data structures (optional).
- I.3 Plot of the FEM mesh, including nodes and element labels (optional).

The processing stage is done by an *analysis driver* module that performs the following steps:

- II.1 Assembly of the master stiffness matrix and force vector, using the element library.
- II.2 Application of displacement BC. This is done by the equation modification method.
- II.3 Solution of the modified equations for displacements. The built in *Mathematica* function `LinearSolve` is used.

Upon executing these three processing steps, the node displacements are available. Several post-processing steps follow:

- III.1 Recovery of forces including reactions, carried out element by element.
- III.2 Computation of element stresses and interelement averaging to get nodal stresses.
- III.3 Printing computed results, including displacements, forces and stresses. If an analytical solution is available, might be printed here for comparison purposes.
- III.4 Contour plot of stress fields (optional).

Steps I.1, I.2, I.3, III.3 and III.4 are done directly by a *driver script* written by the analyst. Steps II.1, II.2, II.3, III.1 and III.2 are done by the *analysis driver* program supplied as part of the code.

The *driver script*, shown on top of Figure 13.1, is written by the analyst to carry out the analysis. In *Mathematica* the driver script is placed on its own Notebook cell. The script is problem dependent: each different problem requires a new one to be written.

A driver script is divided into three sections that are roughly associated with the foregoing steps:

Input data	{ Preprocessing phase: problem definition data Processing phase: problem solving data Postprocessing phase: result display data }

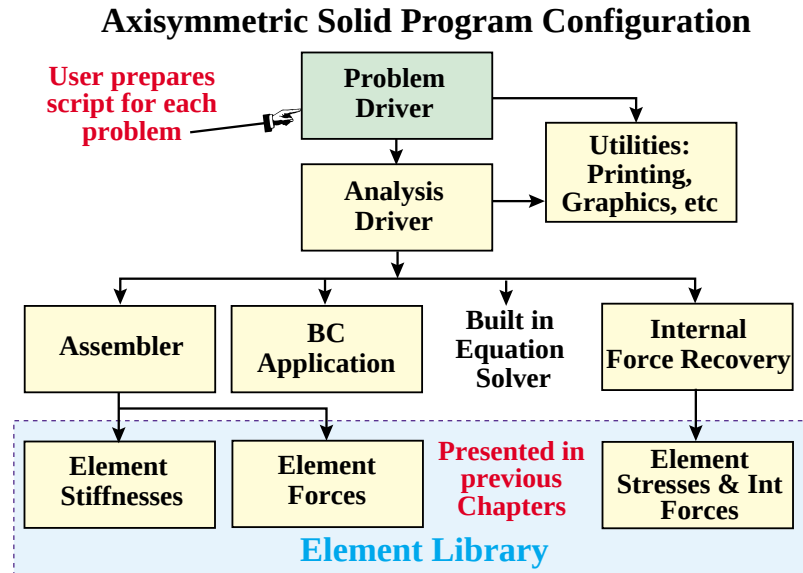


FIGURE 13.1. Organization of the axisymmetric solid program QuadSOR.

These are described in the following sections.

§13.2. Problem Definition

The problem-definition data may be broken down into three sets, which are listed below by order of appearance:

Model data	{ Geometry data: node coordinates Element data: type, connectivity, material, body and surface traction forces. Degree of freedom data: node forces and support BCs

Note that the element data is split into five subsets: type, connectivity, material, body force and surface traction forces, each of which has its own data structure.¹ The degree of freedom data is split into two subsets: tags and values. In addition there are generic processing options, which are conveniently collected in a separate data set.

The input to the axisymmetric solid program QuadSOR consists of nine data structures, which are called NodeCoordinates, ElemTypes, ElemNodes, ElemMaterials, ElemBodyForces, ElemTractionForces, FreedomTags, FreedomValues and DefaultOptions. These data structures are described in the following subsections. For specific examples, reference is made to the benchmark problem and FEM discretization shown in Figures 13.2 and 13.3.

¹ For axisymmetric solid models, the fabrication data that appears, for example, in plane stress programs is empty.

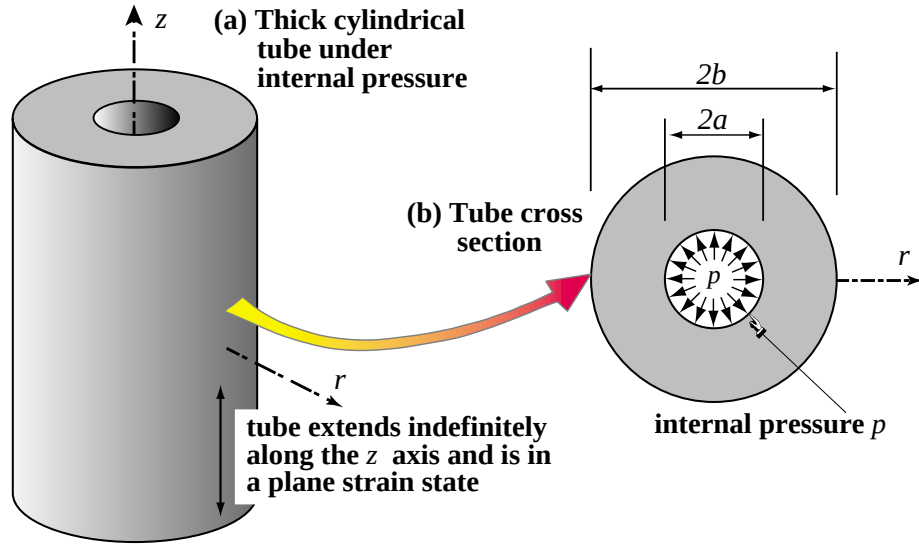


FIGURE 13.2. Benchmark problem: a thick cylinder under internal pressure.

§13.2.1. Benchmark Problem

The problem illustrated in Figure 13.2 is a standard benchmark for FEM discretizations of axisymmetric solids. A thick circular tube of internal radius a and external radius b is subject to internal pressure p . The z axis runs along the hole center. The tube extends indefinitely along the z direction and is considered to be in a plane strain state (meaning that both the axial displacement u_z and axial strain e_{zz} vanish.) The material is isotropic with elastic modulus E and Poisson ratio ν .

The tube is cut with a normal-to- z slice of width d as depicted in Figure 13.3(a). A two element discretization using two 4-node quadrilateral ring elements is constructed as shown in Figure 13.3(b).

The *Mathematica* driver script for this benchmark problem is listed in Figure 13.4. It is explained in detail in following subsections.

§13.2.2. Node Coordinates

The geometry data is specified through `NodeCoordinates`. This is a node-by-node coordinate array configured as a two-dimensional list:

$$\text{NodeCoordinates} = \{ \{r_1, z_1\}, \{r_2, z_2\}, \dots, \{r_N, z_N\} \}; \quad (13.1)$$

where N is the number of nodes. Coordinate values should be floating point numbers;² use the `N` function to insure that property if necessary. Nodes are numbered consecutively, starting from 1. No gaps in node numbers are permitted.

For the 2-element model of Figure 13.3, with $a = 4$, $b = 10$ and $d = 2$ it is recommended to use a parametric form:

² The QuadSOR program can also be used to process small discretizations symbolically or in exact arithmetic for special studies. But for most problems the numeric floating-point form is most appropriate as well as efficient, and that is the use emphasized here.

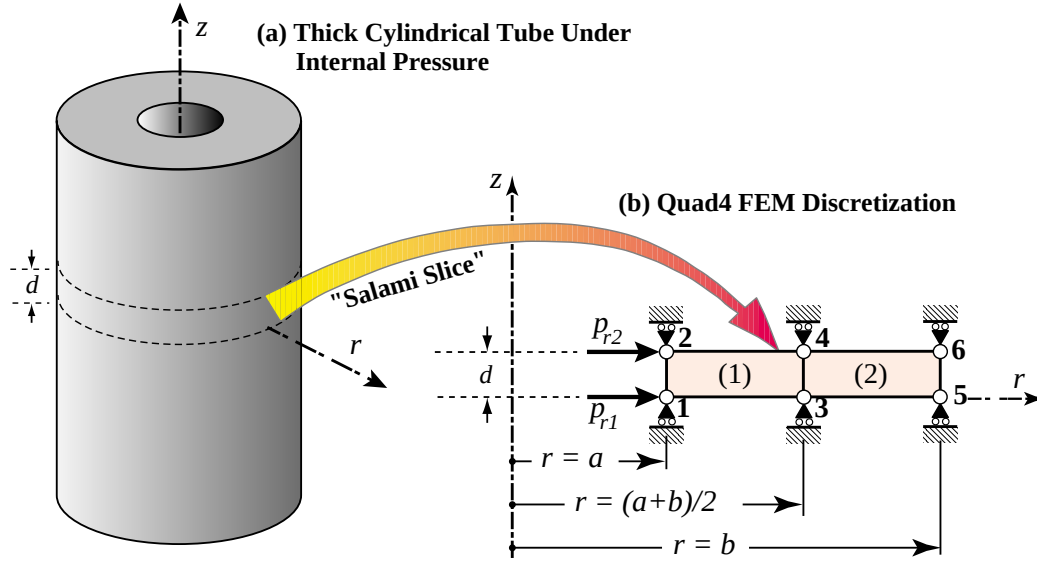


FIGURE 13.3. A slice of width d is cut from the thick tube of Figure 13.1 and used to build a 2-element discretization with 4-node quadrilateral ring elements.

```
a=4; b=10; d=2;
```

```
NodeCoordinates=N[{{a,0},{a,d},{(a+b)/2,0},{(a+b)/2,d},{b,0},{b,d}}];
```

The advantage of parametrizing `NodeCoordinates` is that other dimension values can be rapidly tested without the need of retyping the array each time. Such changes are error prone.

Remark 13.1. For regular discretizations, node coordinates may be constructed through mesh generation. This may be advantageous for systematically exploring mesh refinement. A module provided in `QuadSOR.nb` is `GenQuad4NodeCoordinates`, which builds nodes over a regular discretization of a four-sided region $ABCD$. This is invoked as

```
NodeCoordinates = GenQuad4NodeCoordinates[MeshCorners,Ner,Nez];
```

 (13.2)

Here

MeshCorners Array $\{\{r_A, z_A\}, \{r_B, z_B\}, \{r_C, z_C\}, \{r_D, z_D\}\}$ of $ABCD$ region corner coordinates, traversed counterclockwise $A \rightarrow B \rightarrow C \rightarrow D$

Ner number of elements in the radial direction, which is assumed to be $A \rightarrow B$

Nez number of elements in the axial direction, which is assumed to be $A \rightarrow D$.

The function returns `NodeCoordinates` in the format (13.1).

The generation is done by the standard isoparametric mapping method with equal increments in the zonal ξ and η . For the discretization of Figure 13.3(b) use

```
a=4; b=10; d=2;
```

```
MeshCorners=N[{{a,0},{b,0},{b,d},{0,d}}];
```

```
NodeCoordinates=GenQuad4NodeCoordinates[MeshCorners,2,1];
```

To generate nodes for a regular mesh of 8-noded quadrilaterals use

```
NodeCoordinates = GenQuad8NodeCoordinates[MeshCorners,Ner,Nez];
```

 (13.3)

```

ClearAll[Em,v,th,a,b,d,p,number];
Em=1000.; v=0; etype="Quad4"; number=True;
Kfac=1; a=4; b=10; d=2; p=10; aspect=d/(b-a);

(* Define FEM model *)

NodeCoordinates=N[{{a,0},{a,d},{(a+b)/2,0},{(a+b)/2,d},{b,0},{b,d}}];
ElemNodes={{1,3,4,2},{3,5,6,4}};
numnod=Length[NodeCoordinates]; numele=Length[ElemNodes];
ElemType=Table[etype,{numele}];
ElemMaterial=Table[N[{Em,v}],{numele}];
FreedomValues=Table[{0,0},{numnod}];
FreedomTags=Table[{0,1},{numnod}]; pfor=N[Kfac*p*a*d];
FreedomValues[[1]]=FreedomValues[[2]]={pfor/2,0};
ElemBodyForces={}; ElemTractionForces={};
DefaultOptions={number};
PrintRingNodeCoordinates[NodeCoordinates,"Node Coordinates",{2,4}];
PrintRingElementTypeNodes[ElemType,ElemNodes,"Element type & nodes",{}}];
PrintRingElementMaterials[ElemMaterial,"Material data",{}}];
PrintRingElemBodyForces[ElemBodyForces,"Body forces",{}}];
PrintRingElemTracForces[ElemTractionForces,"Traction forces",{}}];
PrintRingFreedomActivity[FreedomTags,FreedomValues,"BC data",{}}];
Plot2DElementsAndNodes[NodeCoordinates,ElemNodes,aspect,
  "Pressurized thick cylinder mesh",True,True];

(* Solve problem and print results *)

{NodeDisplacements,NodeForces,NodeStresses}=RingAnalysisDriver[
  NodeCoordinates,ElemType,ElemNodes,
  ElemMaterial,ElemBodyForces,ElemTractionForces,
  FreedomTags,FreedomValues,DefaultOptions];
PrintRingAnalysisSolution[NodeDisplacements,NodeForces,
  NodeStresses,"Computed solution",{6,6}];
{ExactNodeDisplacements,ExactNodeStresses}=
  ExactSolution[NodeCoordinates,{a,b},{Em,v,p},p,
  "PressThickCylinder",number];
PrintRingNodeDispStresses[ExactNodeDisplacements,
  ExactNodeStresses,"Exact (Lame) solution",{}}];

(* Plot Node Stress Distributions *)

legend={{(a+2*b)/3,0.8*d};
ContourPlotStresses[NodeCoordinates,ElemNodes,NodeStresses,
  {True,False,True,False},True,{},legend,aspect];

(* Through-wall plots comparing FEM vs exact solutions *)

pwhat={"ur","σrr","σzz","σθθ"}; If [v==0,pwhat={"ur","σrr","σθθ"}];
For [ip=1,ip<=Length[pwhat],ip++, what=pwhat[[ip]];
  RadialPlotFEMvsExact[etype,NodeCoordinates,NodeDisplacements,
  NodeStresses,{a,b,h},{Em,v,p},p,2,
  "PressThickCylinder",what,number] ];

```

FIGURE 13.4. Driver script for example problem of Figure 13.3.

The arguments have same meaning as in (13.2). Here N_{er} and N_{ez} refer to the number of Quad8 elements in the radial and axial direction, respectively. As a result about 3 times more nodes will be generated compared to (13.2) if N_{er} and N_{ez} are the same.

§13.2.3. Element Types

The element type is a label that specifies the model to be used in the discretization. Those labels

are placed in a one-dimensional list:

$$\text{ElemType} = \{ \text{type}^{(1)}, \text{type}^{(2)}, \dots, \text{type}^{N_e} \}; \quad (13.4)$$

Here type^e is the type descriptor of the e -th element, specified as a character string. The two element types available in the axisymmetric program are

- "Quad4" 4-node iso-P quadrilateral ring element with 2 x 2 Gauss quadrature
- "Quad8" 8-node iso-P quadrilateral ring element with 2 x 2 Gauss quadrature. Although rank deficient by one, this usually causes no problems as spurious modes are suppressed by boundary conditions and do not propagate through the mesh.

Some minor variations are possible:

- "Quad4.1" 4-node iso-P quadrilateral ring element with 1 x 1 Gauss quadrature. This element is strongly rank deficient and should be avoided except in some research studies.
- "Quad8.3" 8-node iso-P quadrilateral ring element with 3 x 3 Gauss quadrature. Although this is rank sufficient, its performance is generally inferior to that of the reduced integrated 8-node quadrilateral, especially in near-incompressible situations.

For the discretization of Figure 13.3(b), use

```
ElemType=Table["Quad4",{2}];
```

which is the same as $\text{ElemType}=\{ \text{"Quad4"}, \text{"Quad4"} \};$. A parametrized form of the above is

```
ElemType=Table["Quad4",{numele}];
```

in which `numele` is a variable that contains the number of elements. This can be extracted, for example, as `numele=Length[ElemNodes]`, where `ElemNodes` is described below, if `ElemNodes` is defined before `ElemType`, as is often the case. The advantage of the foregoing parametrized form is that the statement does not have to be modified if the number of element changes, or if a mesh generation utility is used.

§13.2.4. Element Connectivity

Element connectivity information specifies how elements are connected.³ This information is stored in `ElemNodes`, which is a list of element nodelists:

$$\text{ElemNodes} = \{ \{ \text{enl}^{(1)} \}, \{ \text{enl}^{(2)} \}, \dots, \{ \text{enl}^{N_e} \} \}; \quad (13.5)$$

where enl^e denotes the lists of nodes of the e^{th} element given by global node numbers, and N_e is the total number of elements.

Element boundaries must be traversed counterclockwise (CCW) starting at any corner. Numbering elements with midnodes, such as the 8-node quadrilateral, requires some care: first the corners are listed CCW, followed by the midpoints (first midpoint is the one that follows the first corner when going CCW). When elements have an interior node, as in the 9-node biquadratic quadrilateral (not implemented in QuadSOR) that node goes last.

For the discretization of Figure 13.3(b):

```
ElemNodes = { { 1,3,4,2 }, { 3,5,6,4 } };
```

³ Some FEM programs call this the "topology data," which sounds more impressive.

Remark 13.2. This data structure may also be generated if the mesh is topologically regular. A companion module of `GenQuad4NodeCoordinates` that generates `ElemNodes` for a 4-node quadrilateral mesh is `GenerateQuad4ElemNodes`. This is invoked as

$$\text{ElemNodes} = \text{GenQuad4ElemNodes}[\text{Ner}, \text{Nez}] \quad (13.6)$$

in which the two arguments must agree with the `Ner` and `Nez` used in `GenQuad4NodeCoordinates`; see (13.2). For the discretization of Figure 13.3(b), use

$$\text{ElemNodes} = \text{GenQuad4ElemNodes}[2, 1];$$

Module `GenQuad8ElemNodes` may be used to generate a regular mesh of 8-node quadrilaterals. It is invoked as

$$\text{ElemNodes} = \text{GenQuad8ElemNodes}[\text{Ner}, \text{Nez}] \quad (13.7)$$

in which the two arguments must agree with the `Ner` and `Nez` used in `GenQuad8NodeCoordinates`; see (13.3).

§13.2.5. Material Properties

Array `ElemMaterial` lists the elastic modulus E and the Poisson's ratio ν for each element:

$$\text{ElemMaterial} = \{ \{ E^{(1)}, \nu^{(1)} \}, \{ E^{(2)}, \nu^{(2)} \}, \dots, \{ E^{N_e}, \nu^{N_e} \} \} \quad (13.8)$$

In the frequent case in which all elements have the same E and ν , this list can be easily generated by a `Table` command. For the discretization of Figure 13.3(b), assuming $E = 1000$ and $\nu = 0$:

$$\text{Em}=1000; \nu=0; \text{ElemMaterial} = \text{Table}[\text{N}[\{\text{Em}, \nu\}], \{\text{numele}\}]$$

in which `numele=Length[ElemNodes]` is the number of elements (2 for this mesh). Again it is advantageous to declare the numerical values separately for ease of modification.

Remark 13.3. In its present version, `QuadSOR` only accepts isotropic materials. The element modules, however, have been written for arbitrary anisotropic material. The limitation resides in the `ElemMaterial` data structure. Extending the input data to accept more general materials would not be too difficult.

§13.2.6. Element Body Forces

Array `ElemBodyForces` lists the value of the body force field element by element, specified as forces per unit volume. This list can have two forms. The simplest form gives, for each element, the value of the body force radial and axial components b_r and b_z at the quadrilateral center (the center coordinates are the average of the node coordinates):

$$\text{ElemBodyForces} = \{ \{ \text{bz}_0^{(1)}, \text{br}_0^{(1)} \}, \{ \text{bz}_0^{(2)}, \text{br}_0^{(2)} \}, \dots, \{ \text{br}_0^{N_e}, \text{bz}_0^{N_e} \} \} \quad (13.9)$$

For the problem of Figure 13.2, the body forces are zero, so one could just say

$$\text{ElemBodyForces} = \{ \{ 0, 0 \}, \{ 0, 0 \} \}$$

or in parametrized form:

$$\text{ElemBodyForces} = \text{Table}[\{ 0, 0 \}, \{\text{numele}\}]$$

in which `numele = Length[ElemNodes]` is the number of elements. An even more compact form is to set `ElemBodyForces` to the empty list:

```
ElemBodyForces = { };
```

This has the advantage that the consistent node force computations are skipped.

To give a less trivial example, assume that for the mesh of Figure 13.3(b), $b_r = 6r$ (linear in r and $b_z = -3$ (constant). Then

```
ElemBodyForces = { {33,-3},{51,-3} }
```

Where do values 33 and 51 come from? The center of element (1) is at $r_0 = (4+7+7+4)/4 = 5.5$ and so $b_{r0}^{(1)} = 6 \times 5.5 = 33$, whereas for element (2), $b_{r0}^{(2)} = 6 \times 8.5 = 51$.

The second form of `ElemBodyForces` lists, also element by element, the value of b_r and b_z at each element node, in one to one correspondence with the element node lists provided in `ElemNodes`. This is more easily visualized for the example mesh of Figure 13.3(b). Assuming again the force distribution $b_r = 6r$ and $b_z = -3$,

```
ElemBodyForces = { { {24,-3},{42,-3},{42,-3},{24,-3} },
                    { {42,-3},{60,-3},{60,-3},{42,-3} } };
```

Remark 13.4. If body forces are functions of the position coordinates $\{r, z\}$, (for example, in the case of a centrifugal force), it might be convenient to calculate the entries of `ElemBodyForces` in terms of the data in `ElemNodes` and `NodeCoordinates` instead of entering numerical values directly. This requires some programming but it is less error prone than hand computation. For example, suppose that $b_r = 3r - 2z$ and $b_z = 6r + 4z$ over a mesh of 4-noded quadrilaterals. Using the first `ElemBodyForces` format:

```
ElemBodyForces = Table[{0,0},{numele}];
For [e=1,e<=numele,e++, {n1,n2,n3,n4}=ElemNodes[[e]];
    {r0,z0}=(NodeCoordinates[[n1]]+NodeCoordinates[[n2]]+
            NodeCoordinates[[n3]]+NodeCoordinates[[n4]])/4;
    ElemBodyForces[[e]]={3*r0-2*z0,6*r0+4*z0}];
```

§13.2.7. Element Traction Forces

Array `ElemTractionForces` specifies values of surface tractions, such as pressures, that are to be converted to consistent node forces. This conversion capability is not yet implemented in QuadSOR. Consequently this array must be set to the empty list:

```
ElemTractionForces = { };
```

§13.2.8. Freedom Tags

Array `FreedomTags` labels each nodal degree of freedom as to whether the load or the displacement is specified. The configuration of this list is similar to that of `NodeCoordinates`:

$$\text{FreedomTags} = \{ \{ \text{tag}_{r1}, \text{tag}_{z1} \}, \{ \text{tag}_{r2}, \text{tag}_{z2} \}, \dots, \{ \text{tag}_{rN}, \text{tag}_{zN} \} \} \quad (13.10)$$

The tag value is 0 if the force is specified and 1 if the displacement is specified. When there are a lot of nodes and comparatively few displacement BCs, a quick way to build this list is to start from all zeros, and then insert the boundary conditions appropriately.

For the discretization of Figure 13.3(b), all nodes happen to have the same displacement boundary condition: free to move along the radial direction while unable to move axially. The tag for each node is $\{0, 1\}$, and one Table statement suffices:

```
FreedomTags=Table[{0,1},{6}];
```

Instead of entering the number of nodes (6) directly, however, parametrization is recommended:

```
FreedomTags=Table[{0,1},{numnod}];
```

where numnod is the number of nodes as extracted from the geometry definition; that is, numnod = Length[NodeCoordinates].

§13.2.9. Freedom Values

Array FreedomValues has the same node by node configuration as that of FreedomTags:

$$\text{FreedomValues} = \{ \{ \text{val}_{r1}, \text{val}_{z1} \}, \{ \text{val}_{r2}, \text{val}_{z2} \}, \dots \{ \text{val}_{rN}, \text{val}_{zN} \} \} \quad (13.11)$$

This array lists the specified values of the applied node force component if the corresponding freedom tag is zero, and of the prescribed displacement component if the tag is one. Often most of the list entries are zero.

For the mesh of Figure 13.3(b), only two force values are nonzero: the radial forces on nodes 1 and 2 due to the internal pressure p . It is convenient to let the script calculate the node force values from the data: $f_{r1} = f_{r2} = \frac{1}{2} p a d = \frac{1}{2} 10 \times 4 \times 2 = 40$:

```
a=4; d=2; p=10; pfor=p*a*d;
numnod = Length[NodeCoordinates];
FreedomValues = Table[{0,0},{numnod}];
FreedomValues[[1]] = FreedomValues[[2]]={pfor/2,0};
```

Here the circumferential span factor K_{fac} , introduced in Chapter 11, is assumed to be 1. If this factor, called Kfac in the QuadSOR program, is not one, the foregoing computation of the total pressure force on face 1–2 should be changed to $pfor = Kfac * p * a * d$. In fact this is the way implemented in the driver script of Figure 13.4; note that Kfac is set to 1 at the script top.

§13.2.10. Element Processing Options

Array DefaultOptions declares certain behavioral options for the processing phase. The most important is numer. This is a logical flag that specifies that computations are to be carried out in floating-point arithmetic if True, which significantly speeds up processing. Setting numer to False specifies exact or symbolic arithmetic, which is only advisable for certain research studies that carry free parameters along. Accordingly the recommended setting is

$$\text{numer=True; DefaultOptions= \{ numer \};} \quad (13.12)$$

Although more default options may be given following numer, it is not recommended to play with those.

```

Node Coordinates
node      r-coor      z-coor
  1        4.0000      0.0000
  2        4.0000      2.0000
  3        7.0000      0.0000
  4        7.0000      2.0000
  5       10.0000      0.0000
  6       10.0000      2.0000

Element type & nodes
elem      type      node-list
  1      Quad4      {1, 3, 4, 2}
  2      Quad4      {3, 5, 6, 4}

Material data
elem      material: {E,ν}
  1          {1000., 0.}
  2          {1000., 0.}

No body forces specified
No traction forces specified

BC data
node      r-tag      z-tag      r-value      z-value
  1         0         1        40.00        0.00
  2         0         1        40.00        0.00
  3         0         1         0.00        0.00
  4         0         1         0.00        0.00
  5         0         1         0.00        0.00
  6         0         1         0.00        0.00

```

FIGURE 13.5. Model definition data structures printed by the script of Figure 13.4.

§13.2.11. Printing Model Definition Data

The model definition data structures may be printed by the statements shown in the driver script of Figure 13.4, which are

```

PrintRingNodeCoordinates[NodeCoordinates,"Node Coordinates",{2,4}];
PrintRingElementTypeNodes[ElemType,ElemNodes,"Element type & nodes",{1}];
PrintRingElementMaterials[ElemMaterial,"Material data",{1}];
PrintRingElemBodyForces[ElemBodyForces,"Body forces",{1}];
PrintRingElemTracForces[ElemTractionForces,"Traction forces",{1}];
PrintRingFreedomActivity[FreedomTags,FreedomValues,"BC data",{1}];

```

The resulting printed output is shown in Figure 13.5. Such statements could be commented out to reduce output volume once the model definition is considered bug free.

§13.2.12. Mesh Display

A mesh plot may be produced by the statement:

```
Plot2DElementsAndNodes[NodeCoordinates, ElemNodes, aspect, title, True, True];
```

(13.13)

Here *aspect* is the plot frame aspect ratio (*z* dimension over *r* dimension), *title* is a character string specifying a plot title, and the last two *True* argument values ask that node labels and element labels, respectively, be displayed.

The mesh plot produced by

```
Plot2DElementsAndNodes[NodeCoordinates,ElemNodes,aspect,
  "Pressurized thick cylinder mesh",True,True];
```

that appears in the driver script of Figure 13.4 is shown in Figure 13.6.

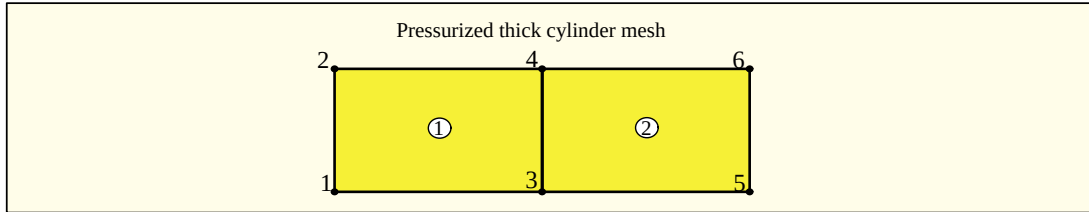


FIGURE 13.6. Mesh plot produced by the script of Figure 13.4.

§13.3. Processing

The static solution is carried out by calling the analysis driver module `RingAnalysisDriver`. The function call is

```
{NodeDisplacements,NodeForces,NodeStresses}=RingAnalysisDriver[
  NodeCoordinates,ElemType,ElemNodes,
  ElemMaterial,ElemBodyForces,ElemTractionForces,
  FreedomTags,FreedomValues,DefaultOptions];
```

The arguments of this call: `NodeCoordinates`, `ElemType`, `ElemNodes`, `ElemMaterial`, `ElemBodyForces`, `ElemTractionForces`, `FreedomTags`, `FreedomValues` and `DefaultOptions`, have been described in the foregoing section.

Assuming no processing errors occur, `RingAnalysisDriver` returns the following arrays:

<code>NodeDisplacements</code>	The computed node displacements, arranged as $\{\{ur1,uz1\}, \{ur2,uz2\}, \dots \{urN,uzN\}\}$.
<code>NodeForces</code>	The computed node forces, including recovered reactions, arranged as $\{\{fr1,fz1\}, \{fr2,fz2\}, \dots \{frN,fzN\}\}$.
<code>NodeStresses</code>	Recovered stresses at node locations, arranged as $\{\{\sigma_{rr1},\sigma_{zz1},\sigma_{\theta\theta1},\sigma_{rz1}\}, \dots \{\sigma_{rrN},\sigma_{zzN},\sigma_{\theta\theta N},\sigma_{rzN}\}\}$.

§13.4. Postprocessing

Postprocessing mean activities undertaken on return from the analysis driver. They include printout of node displacements, node forces and node stresses, as well as contour plots of stress distributions.

§13.4.1. Printing Analysis Results

The displacements, forces and stresses return by the analysis driver may be printed by saying

```
PrintRingAnalysisSolution[NodeDisplacements,NodeForces,
  NodeStresses,"Computed solution",{ }];
```

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-xy	r-force	z-force
1	0.0532	0.0000	-4.5536	0.0000	12.3907	0.0000	40.0000	0.0000
2	0.0532	0.0000	-4.5536	0.0000	12.3907	0.0000	40.0000	0.0000
3	0.0395	0.0000	-2.6527	0.0000	5.3211	0.0000	0.0000	0.0000
4	0.0395	0.0000	-2.6527	0.0000	5.3211	0.0000	0.0000	0.0000
5	0.0373	0.0000	-0.7518	0.0000	3.6338	0.0000	0.0000	0.0000
6	0.0373	0.0000	-0.7518	0.0000	3.6338	0.0000	0.0000	0.0000
Exact (Lame) solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-xy		
1	0.0552	0.0000	-10.0000	0.0000	13.8095	0.0000		
2	0.0552	0.0000	-10.0000	0.0000	13.8095	0.0000		
3	0.0405	0.0000	-1.9825	0.0000	5.7920	0.0000		
4	0.0405	0.0000	-1.9825	0.0000	5.7920	0.0000		
5	0.0381	0.0000	0.0000	0.0000	3.8095	0.0000		
6	0.0381	0.0000	0.0000	0.0000	3.8095	0.0000		

FIGURE 13.7. Computed solution results produced by script of Figure 13.4.

Here "Computed solution" may be replaced with another title string. The last argument may be used to control the number of places printed after the decimal point. See Figure 13.7 for an example.

If an analytical solution is available as in the case of the pressurized thick cylinder, it may be evaluated and printed at this point to facilitate comparison by inspection. Cells 11–13 of the posted notebook provide a template on how to do that.

§13.4.2. Stress Field Contour Plots

To do contour plots of stress fields you may use the call to module `ContourPlotStresses`, which is invoked as

<pre>ContourPlotStresses [NodeCoordinates, ElemNodes, NodeStresses, whichones, showrange, plotopt, legend, aspect];</pre>	(13.14)

The arguments beyond `NodeCoordinates`, `ElemNodes` and `NodeStresses` are:

- whichones** A 4-logical-flag list that specifies which stress components are to be plotted. Better understood by example: `whichones={ True, False, True, False }` requests that σ_{rr} and $\sigma_{\theta\theta}$, which are entries 1 and 3 of the stress vector, be plotted.
- showrange** A logical flag. If `True`, a line giving the stress plot range and the increment between contour lines is printed before the plot.
- plotopt** A list of six logical flags passed to the plotter for things such as "draw node locations" and so on. If set to `{ }`, the `ContourPlotStresses` default options are assumed; this is recommended unless one wants to play with the appearance.
- legend** If given as an empty list, that is, `legend={ }`, no legend table is produced. If `legend={ rleg, zleg }` a contourline legend box will appear drawn with the upper left corner at that position. The implementation of this feature is presently crude; thus `legend={ }`, which skips the legend display, is recommended unless you want to play with settings.
- aspect** The axial-to-radial (z dimension over r dimension) plot aspect ratio.

See Figure 13.8 for the plots produced by the driver script using

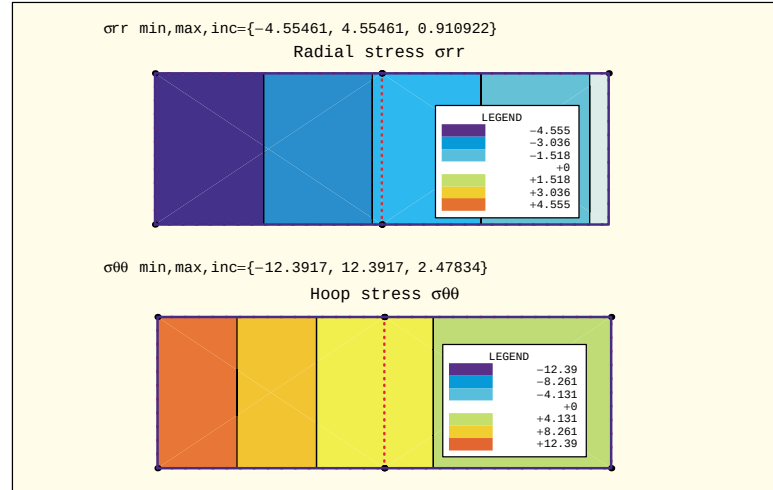


FIGURE 13.8. Stress contour plots results produced by script of Figure 13.4.

```

legend={(a+2*b)/3,0.8*d};
ContourPlotStresses[NodeCoordinates,ElemNodes,NodeStresses,
  {True,False,True,False},True,{},legend,aspect];

```

here $\text{aspect}=d/(b-a)$ is defined at script start, since that value is used also for the mesh plot.

§13.4.3. Exact vs. FEM Radial Plots

The driver script produces 3 plots comparing FEM results versus the exact solution for radial displacements u_r as well as radial and hoop stresses $\{\sigma_{rr}, \sigma_{\theta\theta}\}$.⁴ The plots are produced by

```

pwhat={"ur","srr","szz","sθθ"}; If [ν==0,pwhat={"ur","srr","sθθ"}];
For [ip=1,ip<=Length[pwhat],ip++, what=pwhat[[ip]];
  RadialPlotFEMvsExact[etype,NodeCoordinates,NodeDisplacements,
    NodeStresses, {a,b,h},{Em,ν,r},p,2,
    "PressThickCylinder",what,numer] ];

```

The three plots produced by this loop are collected in Figure 13.9.

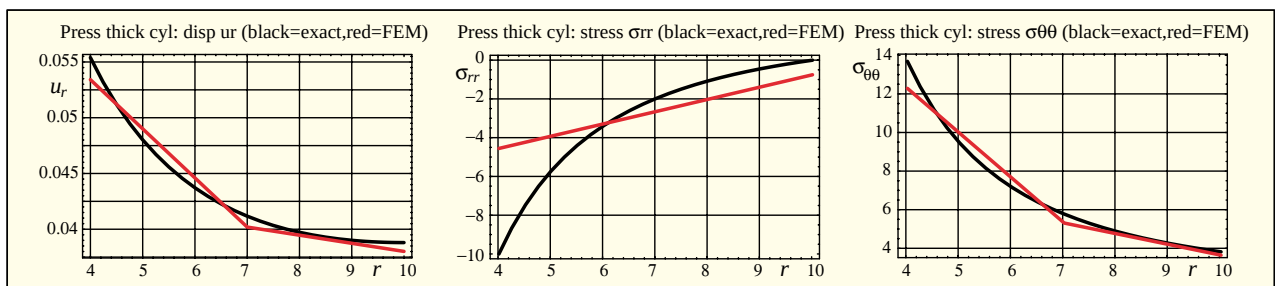


FIGURE 13.9. Comparison of exact vs. FEM results produced by driver script of 13.4.

⁴ Note that 2 components that are identically zero: σ_{zz} and σ_{rz} , are skipped. Component σ_{zz} would be nonzero if $\nu > 0$.

14

Axisymmetric Solid Benchmark Problems

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This chapter presents three benchmark examples that are often used to assess the performance of FEM models for axisymmetric solid analysis.

They were previously included as part of Chapter 13, but have been split to a new Chapter on account of length.

§14.1. Benchmark 1: Internally Pressurized Thick Cylinder

The first benchmark problem is that used in the previous Chapter to illustrate driver script preparation for the QuadSOR code. It is illustrated in Figure 14.1, which reproduces Figure 13.1 for convenience. The study in the present Section covers: effect of mesh refinement in the radial direction, relative performance of 4-node versus 8-node quadrilateral elements, and influence of Poisson ratio ν when passing from $\nu = 0$ to near the incompressible limit $\nu \approx \frac{1}{2}$.

§14.1.1. Problem Description

Restating the problem: a cylindrical hollow tube of inner radius a and outer radius b is subjected to internal pressure p . The tube and its cross section are shown in Figure 14.1. The tube extends indefinitely along the z axis and is in a plane strain state along that direction. The material is isotropic with elastic modulus E and Poisson's ratio ν . A “slice” of thickness d is extracted and discretized as shown in Figure 14.2 using N_{er} quadrilateral ring elements along the radial direction r and one along the axial direction z (In that Figure, N_{er} are 4 and 2 for the 4-node and 8-node quadrilateral meshes, respectively.) Nodes move in the radial direction only, which results in the support conditions drawn in Figure 14.2(b,c).

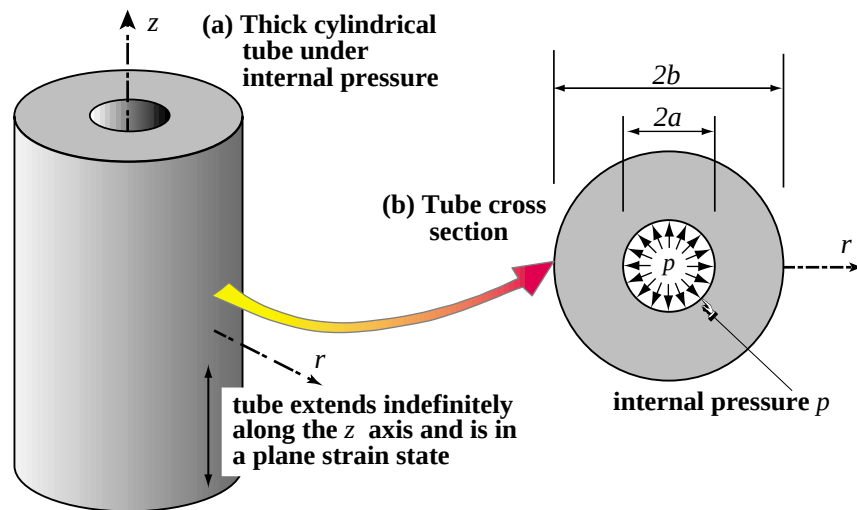


FIGURE 14.1. Pressurized thick cylinder benchmark problem. Reproduced from Figure 13.1 for convenience.

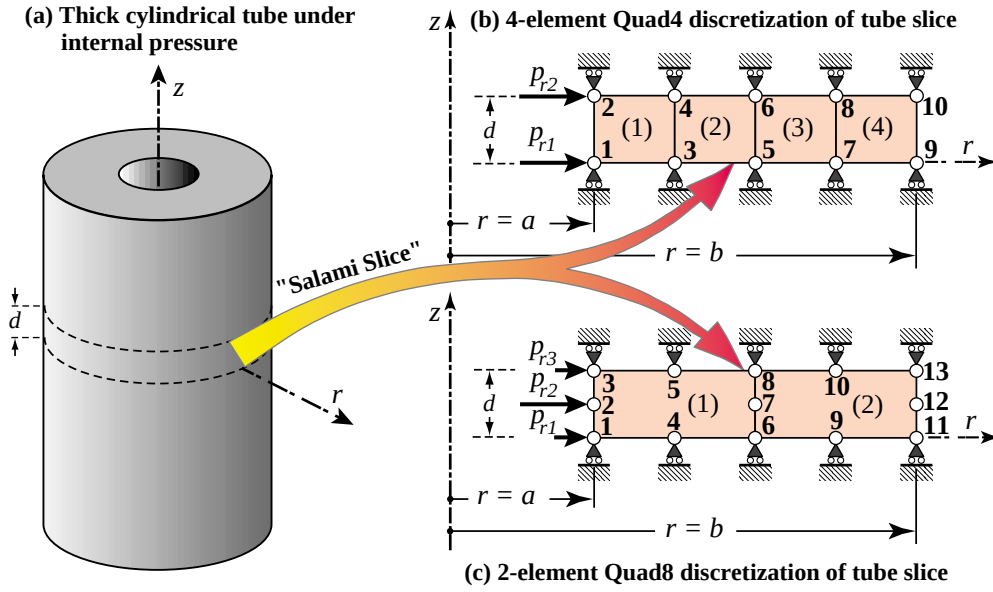


FIGURE 14.2. Two example FEM discretizations for the pressurized thick cylinder benchmark.

§14.1.2. Exact Solution

The exact stress distribution¹ across the wall is

$$\begin{aligned} \sigma_{rr} &= p \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2}\right), & \sigma_{zz} &= p \frac{2a^2\nu}{b^2 - a^2}, \\ \sigma_{\theta\theta} &= p \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2}\right), & \text{others zero.} \end{aligned} \quad (14.1)$$

Note that $\sigma_{zz} = 0$ if $\nu = 0$ and that $\sigma_{rr} + \sigma_{\theta\theta}$ does not depend on r . Transforming this stress field to strains through $\mathbf{e} = \mathbf{E}^{-1}\boldsymbol{\sigma}$ gives the exact strain field, which can be verified to satisfy $e_{zz} = 0$. The hoop strain thus obtained is $e_{\theta\theta} = pa^2(1 + \nu)(b^2 + r^2(1 - 2\nu))/(E(b^2 - a^2)r^2)$, which multiplied by r yields the exact radial displacement

$$u_r = p \frac{a^2(1 + \nu)(b^2 + r^2(1 - 2\nu))}{E(b^2 - a^2)r}. \quad (14.2)$$

§14.1.3. Driver Script

The *Mathematica* driver script listed in Figure 14.3 accepts both 4-node and 8-node quad elements. Note that geometric, constitutive and discretization properties are declared at the top to make parametrization simpler. To set the element type it is sufficient to declare `etype` in the second line of the script to be either "Quad4" or "Quad8". The number of elements along the radial and axial directions: N_{er} and N_{ez} , are parametrized by the values assigned to `Ner` and `Nez`, respectively. The latter is assumed to be 1 since the solution only depends on r .

¹ Taken from S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, 2nd ed., 1951, Chapter 4, for a condition of plane strain in the z direction. The solution is due to G. Lamé, *Leçons sur la Théorie Mathématique de l'Elasticité des Corps Solides*, Paris, Bachelier, 1852.

```

ClearAll[Em,v,th,a,b,d,p,Ner,Nez];
Em=1000.; v=0.0; etype="Quad4"; numer=True; Ner=4; Nez=1;
Kfac=1; a=4; b=10; d=2; aspect=d/(b-a); p=10;

(* Define FEM model *)

MeshCorners=N[{a,0},{b,0},{b,d},{a,d}];
If [etype=="Quad4",
  NodeCoordinates=GenQuad4NodeCoordinates[MeshCorners,Ner,Nez];
  ElemNodes=GenQuad4ElemNodes[Ner,Nez]];
If [etype=="Quad8",
  NodeCoordinates=GenQuad8NodeCoordinates[MeshCorners,Ner,Nez];
  ElemNodes=GenQuad8ElemNodes[Ner,Nez]];
numnod=Length[NodeCoordinates]; numele=Length[ElemNodes];
ElemType=Table[N[{Em,v}],{numele}];
ElemMaterial=Table[N[{Em,v}],{numele}];
FreedomTags=Table[{0,1},{numnod}];
FreedomValues=Table[{0,0},{numnod}]; pfor=N[Kfac*p*a*d];
If [etype=="Quad4",
  FreedomValues[[1]]=FreedomValues[[2]]=[pfor/2,0]];
If [etype=="Quad8",
  FreedomValues[[1]]=FreedomValues[[3]]=[pfor/6,0];
  FreedomValues[[2]]=[2*pfor/3,0]];
ElemBodyForces= ElemTractionForces={}; DefaultOptions={numer};
(* Problem data print statements removed *)
Plot2DElementsAndNodes[NodeCoordinates,ElemNodes,aspect,
  "Press thick cylinder",True,True];

(* Solve problem and print results *)

{NodeDisplacements,NodeForces,NodeStresses}=RingAnalysisDriver[
  NodeCoordinates,ElemType,ElemNodes,
  ElemMaterial,ElemBodyForces,ElemTractionForces,
  FreedomTags,FreedomValues,DefaultOptions];
PrintRingAnalysisSolution[NodeDisplacements,NodeForces,
  NodeStresses,"Computed solution",{}}];
{ExactNodeDisplacements,ExactNodeStresses}=
  ExactSolution[NodeCoordinates,{a,b},{Em,v,p},p,
  "PressThickCylinder",numer];
PrintRingNodeDispStresses[ExactNodeDisplacements,
  ExactNodeStresses,"Exact (Lame) solution",{}}];

(* Contour plots of stress distributions *)

legend={({a+b)/2,0.75*d}; whichones={True,True,True,False};
If [v==0, whichones={True,False,True,False}];
ContourPlotStresses[NodeCoordinates,ElemNodes,NodeStresses,
  whichones,True,{},legend,aspect];

(* Radial plots comparing FEM vs exact solutions *)

pwhat={"ur","σrr","σzz","σθθ"};
For [ip=1,ip<=Length[pwhat],ip++, what=pwhat[[ip]];
  RadialPlotFEMvsExact[etype,NodeCoordinates,NodeDisplacements,
    NodeStresses,{a,b},{Em,v,p},p,{Ner,Nez},
    "PressThickCylinder",what,1,numer] ];

```

FIGURE 14.3. Driver script for pressurized thick cylinder benchmark. This one accepts both Quad4 and Quad8 elements, as well as (through mesh generation) arbitrary number of elements in the radial direction.

If the element type is "Quad4" a regular mesh is generated by modules `GenQuad4NodeCoordinates` and `GenQuad4ElemNodes`, described in the previous Chapter. If the element type is Quad8 the generation modules invoked are `GenQuad8NodeCoordinates` and `GenQuad8ElemNodes`. the element type is "Quad8". Pressure lumping to the nodes on the inner radius $r = a$ depends on element type. The total pressure force is $p_{for} = K_{fac} p a d$, in which $K_{fac} = 1$ for QuadSOR because it uses a one-radian circumferential ring span. That value is stored in `pfor`. If the element type is "Quad4", the two innermost nodes 1 and 2 receive $\frac{1}{2} p_{for}$ each. If the element type is "Quad8" the three innermost nodes 1, 2 and 3 receive $\frac{1}{6} p_{for}$, $\frac{2}{3} p_{for}$ and $\frac{1}{6} p_{for}$, respectively, in accordance with consistent node force lumping.

§14.1.4. Results For Zero Poisson Ratio

The results presented here were computed for $a = 4$, $b = 10$, $d = 2$, $p = 10$, $E = 1000$, two Poisson ratio values, two element types, and two radial discretizations for each type.

The script of Figure 14.3 specifically sets `etype="Quad4"`, `Ner=4` and $\nu = 0$. The mesh is actually that pictured in Figure 14.2(b). Computed and exact nodal values are tabulated in Figure 14.4. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over the wall $a \leq r \leq b$ with the exact solution in Figure 14.5(a).

As can be seen u_r and $\sigma_{\theta\theta}$ are satisfactorily predicted. The hole-edge radial stress, however, is significantly underestimated: $\sigma_{rr} = -6.604$ compared to the exact $\sigma_{rr} = -p = -10$. This is a consequence of the impossibility of doing interelement stress averaging at that high-stress-gradient edge. For this low-order model the variation of σ_{rr} in the r direction is limited to be constant within the element. Thus $\sigma_{rr} = -6.604$ is more representative of the stress at the center of element (1). The σ_{rr} agreement at other nodes is reasonable given the coarseness of the mesh. The higher accuracy of the hoop stress $\sigma_{\theta\theta}$ is incidental, reflecting an idiosyncrasy of axisymmetric solids: the hoop strain $\epsilon_{\theta\theta} = u_r/r$ is not obtained through displacement differentiation. It thus attains the same accuracy as u_r . And for zero ν , $\sigma_{\theta\theta} = E \epsilon_{\theta\theta} = E u_r/r$.

Increasing `Ner` to 16 gives a solution that is compared with the exact one in Figure 14.5(b). Both u_r and $\sigma_{\theta\theta}$ are correct to at least 3 places. The only visible flaw is the discrepancy of σ_{rr} at the hole boundary, where it gives $\sigma_{rr} = -8.965$ instead of $-p = -10$. Again this is a consequence of lack of interelement averaging. Away from the hole σ_{rr} agrees with the exact solution to plot accuracy.

The analysis is then redone with the 8-node quadrilateral Quad8 with reduced (2×2) integration. To make a fair comparison with Quad4, the Quad8 meshes contain half the elements: 2 and 8, respectively, which results in a similar number of nodes.

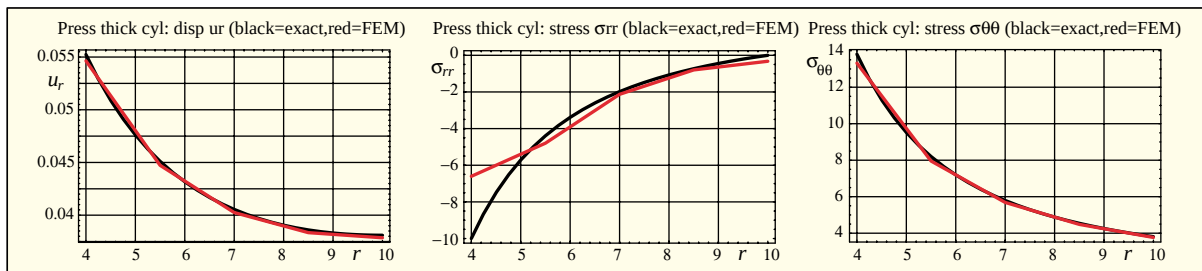
The results of running Quad8 with `Ner=2` are tabulated in Figure 14.6. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over $a \leq r \leq b$ with the exact solution in Figure 14.7(a). This element is supposed to be nodally exact for one-dimensional problems, and indeed the computed and exact u_r may be verified to agree numerically to 15 places at all nodes. The hoop stress should be also nodally exact since $\sigma_{\theta\theta} = E u_r/r$, but the extrapolation from Gauss points introduces discrepancies. The computed radial stress σ_{rr} is as good as can be expected from a linear variation over the element.

Running Quad8 with `Ner=8` give the results plotted in Figure 14.7(b). Again the displacements are nodally exact. Both σ_{rr} and $\sigma_{\theta\theta}$ agree everywhere with the exact solution at plot accuracy.

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz	r-force	z-force
1	0.0546	0.0000	-6.6040	0.0000	13.3193	0.0000	40.0000	0.0000
2	0.0546	0.0000	-6.6040	0.0000	13.3193	0.0000	40.0000	0.0000
3	0.0447	0.0000	-4.7954	0.0000	7.9555	0.0000	0.0000	0.0000
4	0.0447	0.0000	-4.7954	0.0000	7.9555	0.0000	0.0000	0.0000
5	0.0402	0.0000	-2.1293	0.0000	5.6858	0.0000	0.0000	0.0000
6	0.0402	0.0000	-2.1293	0.0000	5.6858	0.0000	0.0000	0.0000
7	0.0383	0.0000	-0.7984	0.0000	4.4821	0.0000	0.0000	0.0000
8	0.0383	0.0000	-0.7984	0.0000	4.4821	0.0000	0.0000	0.0000
9	0.0379	0.0000	-0.3250	0.0000	3.7674	0.0000	0.0000	0.0000
10	0.0379	0.0000	-0.3250	0.0000	3.7674	0.0000	0.0000	0.0000
Exact (Lame) solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz		
1	0.0552	0.0000	-10.0000	0.0000	13.8095	0.0000		
2	0.0552	0.0000	-10.0000	0.0000	13.8095	0.0000		
3	0.0451	0.0000	-4.3920	0.0000	8.2015	0.0000		
4	0.0451	0.0000	-4.3920	0.0000	8.2015	0.0000		
5	0.0405	0.0000	-1.9825	0.0000	5.7920	0.0000		
6	0.0405	0.0000	-1.9825	0.0000	5.7920	0.0000		
7	0.0386	0.0000	-0.7316	0.0000	4.5411	0.0000		
8	0.0386	0.0000	-0.7316	0.0000	4.5411	0.0000		
9	0.0381	0.0000	0.0000	0.0000	3.8095	0.0000		
10	0.0381	0.0000	0.0000	0.0000	3.8095	0.0000		

FIGURE 14.4. Pressurized thick cylinder benchmark: computed and exact nodal solution for 4-element Quad4 model with Poisson ratio $\nu = 0$. The mesh is that shown in Figure 14.2(b).

(a) 4 x 1 Mesh of Quad4 Elements, $\nu = 0$:



(b) 16 x 1 Mesh of Quad4 Elements, $\nu = 0$:

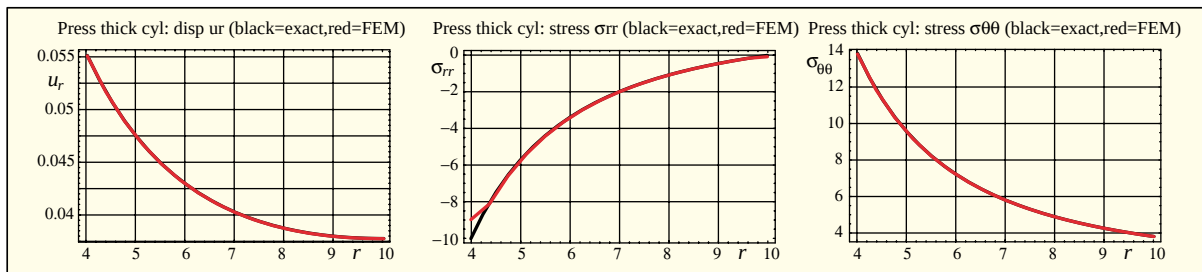


FIGURE 14.5. Pressurized thick cylinder benchmark: r plots of computed versus exact u_r , σ_{rr} and $\sigma_{\theta\theta}$ for Quad4 meshes with $\nu = 0$: (a) 4-element mesh, (b) 16-element mesh.

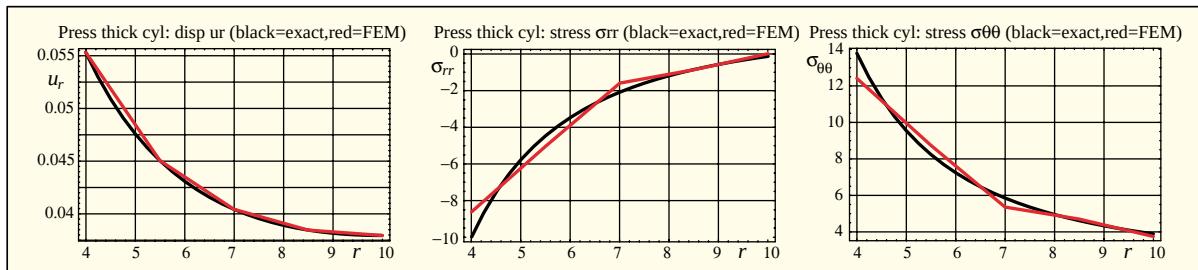
§14.1.5. Results For Near Incompressible Material

If Poisson ratio is increased over zero, Quad4 results gradually lose accuracy if the number of elements is kept the same. In the limit $\nu \rightarrow \frac{1}{2}$, the material approaches incompressibility, and the computed solution deterioration accelerates. This phenomenon is known as *volumetric locking* in the FEM literature. To illustrate that degradation, the 4-element Quad4 mesh is run with $\nu = 0.499$,

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz	r-force	z-force
1	0.0552	0.0000	-8.6085	0.0000	12.4181	0.0000	13.3333	0.0000
2	0.0552	0.0000	-8.6085	0.0000	12.4181	0.0000	53.3333	0.0000
3	0.0552	0.0000	-8.6085	0.0000	12.4181	0.0000	13.3333	0.0000
4	0.0451	0.0000	-4.8980	0.0000	8.7075	0.0000	0.0000	0.0000
5	0.0451	0.0000	-4.8980	0.0000	8.7075	0.0000	0.0000	0.0000
6	0.0405	0.0000	-1.4820	0.0000	5.2916	0.0000	0.0000	0.0000
7	0.0405	0.0000	-1.4820	0.0000	5.2916	0.0000	0.0000	0.0000
8	0.0405	0.0000	-1.4820	0.0000	5.2916	0.0000	0.0000	0.0000
9	0.0386	0.0000	-0.8163	0.0000	4.6259	0.0000	0.0000	0.0000
10	0.0386	0.0000	-0.8163	0.0000	4.6259	0.0000	0.0000	0.0000
11	0.0381	0.0000	0.1441	0.0000	3.6655	0.0000	0.0000	0.0000
12	0.0381	0.0000	0.1441	0.0000	3.6655	0.0000	0.0000	0.0000
13	0.0381	0.0000	0.1441	0.0000	3.6655	0.0000	0.0000	0.0000
Exact (Lame) solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz		
1	0.0552	0.0000	-10.0000	0.0000	13.8095	0.0000		
2	0.0552	0.0000	-10.0000	0.0000	13.8095	0.0000		
3	0.0552	0.0000	-10.0000	0.0000	13.8095	0.0000		
4	0.0451	0.0000	-4.3920	0.0000	8.2015	0.0000		
5	0.0451	0.0000	-4.3920	0.0000	8.2015	0.0000		
6	0.0405	0.0000	-1.9825	0.0000	5.7920	0.0000		
7	0.0405	0.0000	-1.9825	0.0000	5.7920	0.0000		
8	0.0405	0.0000	-1.9825	0.0000	5.7920	0.0000		
9	0.0386	0.0000	-0.7316	0.0000	4.5411	0.0000		
10	0.0386	0.0000	-0.7316	0.0000	4.5411	0.0000		
11	0.0381	0.0000	0.0000	0.0000	3.8095	0.0000		
12	0.0381	0.0000	0.0000	0.0000	3.8095	0.0000		
13	0.0381	0.0000	0.0000	0.0000	3.8095	0.0000		

FIGURE 14.6. Pressurized thick cylinder benchmark: computed and exact nodal solution for 2-element Quad8 model with Poisson ratio $\nu = 0$. The mesh is that shown in Figure 14.2(c).

(a) 2 x 1 Mesh of Quad8 Elements, $\nu = 0$:



(b) 8 x 1 Mesh of Quad8 Elements, $\nu = 0$:

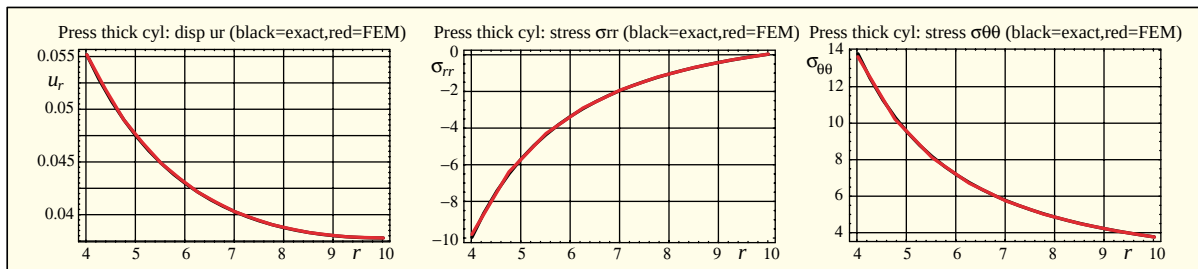


FIGURE 14.7. Pressurized thick cylinder benchmark: r plots of computed versus exact u_r , σ_{rr} and $\sigma_{\theta\theta}$ for Quad8 meshes with $\nu = 0$: (a) 2-element mesh, (b) 8-element mesh.

which exemplifies near-incompressible behavior.²

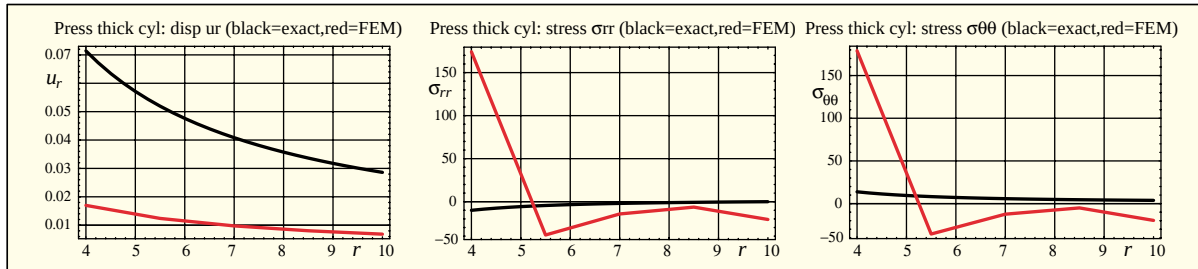
² Setting $\nu = \frac{1}{2}$ exactly makes the elasticity matrix $\mathbf{E}$, as well as $\mathbf{K}$, “blow up” and no solution would be obtained.

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz	r-force	z-force
1	0.0170	0.0000	174.2420	176.2850	179.0340	0.0000	40.0000	-207.7300
2	0.0170	0.0000	174.2420	176.2850	179.0340	0.0000	40.0000	207.7300
3	0.0124	0.0000	-38.6784	-37.0706	-35.6114	0.0000	0.0000	55.3264
4	0.0124	0.0000	-38.6784	-37.0706	-35.6114	0.0000	0.0000	-55.3264
5	0.0097	0.0000	-14.2501	-13.2835	-12.3703	0.0000	0.0000	21.8783
6	0.0097	0.0000	-14.2501	-13.2835	-12.3703	0.0000	0.0000	-21.8783
7	0.0080	0.0000	-6.2986	-5.6522	-5.0285	0.0000	0.0000	7.4675
8	0.0080	0.0000	-6.2986	-5.6522	-5.0285	0.0000	0.0000	-7.4675
9	0.0068	0.0000	-20.6648	-20.1323	-19.6805	0.0000	0.0000	43.2181
10	0.0068	0.0000	-20.6648	-20.1323	-19.6805	0.0000	0.0000	-43.2181

Exact (Lame) solution						
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz
1	0.0714	0.0000	-10.0000	1.9010	13.8095	0.0000
2	0.0714	0.0000	-10.0000	1.9010	13.8095	0.0000
3	0.0519	0.0000	-4.3920	1.9010	8.2015	0.0000
4	0.0519	0.0000	-4.3920	1.9010	8.2015	0.0000
5	0.0408	0.0000	-1.9825	1.9010	5.7920	0.0000
6	0.0408	0.0000	-1.9825	1.9010	5.7920	0.0000
7	0.0336	0.0000	-0.7316	1.9010	4.5411	0.0000
8	0.0336	0.0000	-0.7316	1.9010	4.5411	0.0000
9	0.0286	0.0000	0.0000	1.9010	3.8095	0.0000
10	0.0286	0.0000	0.0000	1.9010	3.8095	0.0000

FIGURE 14.8. Pressurized thick cylinder benchmark: computed and exact nodal solution for 4-element Quad4 mesh with Poisson ratio $\nu = 0.499$. The mesh is that shown in Figure 14.2(b).

(a) 4 x 1 Mesh of Quad4 Elements, $\nu = 0.499$:



(b) 16 x 1 Mesh of Quad4 Elements, $\nu = 0.499$:

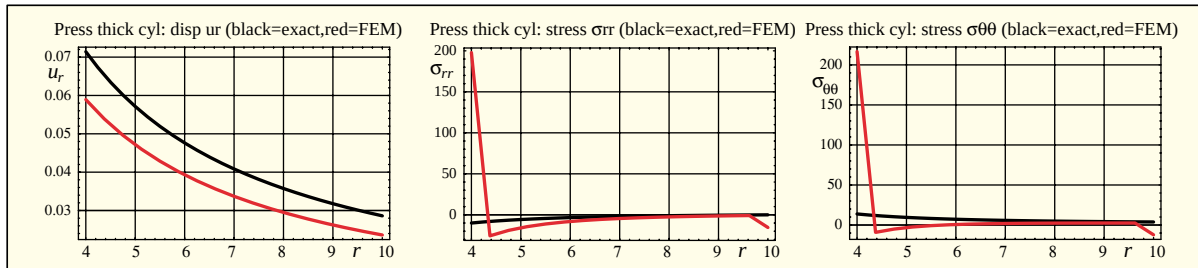


FIGURE 14.9. Pressurized thick cylinder benchmark: r plots of computed versus exact u_r , σ_{rr} and $\sigma_{\theta\theta}$ for Quad4 meshes with $\nu = 0.499$: (a) 4-element mesh, (b) 16-element mesh.

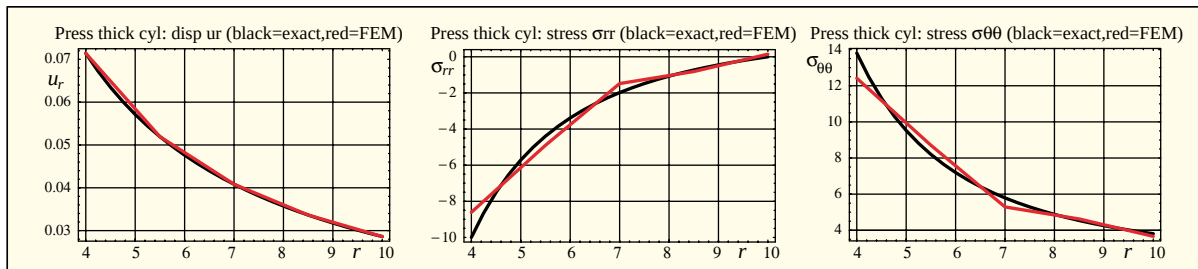
Computed and exact nodal values are tabulated in Figure 14.8. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over $a \leq r \leq b$ with the exact solution in Figure 14.9(a). Serious deficiencies can be observed. The radial displacement u_r has the right profile but is only about 20% of the correct values; for example at $r = a$ the computed value is 0.0170 versus 0.0714. All stress components violently oscillate as one approaches the inner boundary, and the values taken there are nonsensical. For example $\sigma_{rr} \approx 174$ at $r = a$ whereas it

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz	r-force	z-force
1	0.0714	0.0000	-8.6085	1.9010	12.4181	0.0000	13.3333	-3.8019
2	0.0714	0.0000	-8.6085	1.9010	12.4181	0.0000	53.3333	0.0000
3	0.0714	0.0000	-8.6085	1.9010	12.4181	0.0000	13.3333	3.8019
4	0.0519	0.0000	-4.8980	1.9010	8.7075	0.0000	0.0000	-20.9105
5	0.0519	0.0000	-4.8980	1.9010	8.7075	0.0000	0.0000	20.9105
6	0.0408	0.0000	-1.4820	1.9010	5.2916	0.0000	0.0000	-13.3067
7	0.0408	0.0000	-1.4820	1.9010	5.2916	0.0000	0.0000	0.0000
8	0.0408	0.0000	-1.4820	1.9010	5.2916	0.0000	0.0000	13.3067
9	0.0336	0.0000	-0.8163	1.9010	4.6259	0.0000	0.0000	-32.3162
10	0.0336	0.0000	-0.8163	1.9010	4.6259	0.0000	0.0000	32.3162
11	0.0286	0.0000	0.1441	1.9010	3.6655	0.0000	0.0000	-9.5048
12	0.0286	0.0000	0.1441	1.9010	3.6655	0.0000	0.0000	0.0000
13	0.0286	0.0000	0.1441	1.9010	3.6655	0.0000	0.0000	9.5048

Exact (Lame) solution						
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz
1	0.0714	0.0000	-10.0000	1.9010	13.8095	0.0000
2	0.0714	0.0000	-10.0000	1.9010	13.8095	0.0000
3	0.0714	0.0000	-10.0000	1.9010	13.8095	0.0000
4	0.0519	0.0000	-4.3920	1.9010	8.2015	0.0000
5	0.0519	0.0000	-4.3920	1.9010	8.2015	0.0000
6	0.0408	0.0000	-1.9825	1.9010	5.7920	0.0000
7	0.0408	0.0000	-1.9825	1.9010	5.7920	0.0000
8	0.0408	0.0000	-1.9825	1.9010	5.7920	0.0000
9	0.0336	0.0000	-0.7316	1.9010	4.5411	0.0000
10	0.0336	0.0000	-0.7316	1.9010	4.5411	0.0000
11	0.0286	0.0000	0.0000	1.9010	3.8095	0.0000
12	0.0286	0.0000	0.0000	1.9010	3.8095	0.0000
13	0.0286	0.0000	0.0000	1.9010	3.8095	0.0000

FIGURE 14.10. Pressurized thick cylinder benchmark: computed and exact nodal solution for 2-element Quad8 mesh with Poisson ratio $\nu = 0.499$. The mesh is that shown in Figure 14.2(b).

(a) 2 x 1 Mesh of Quad8 Elements, $\nu = 0.499$:



(b) 8 x 1 Mesh of Quad8 Elements, $\nu = 0.499$:

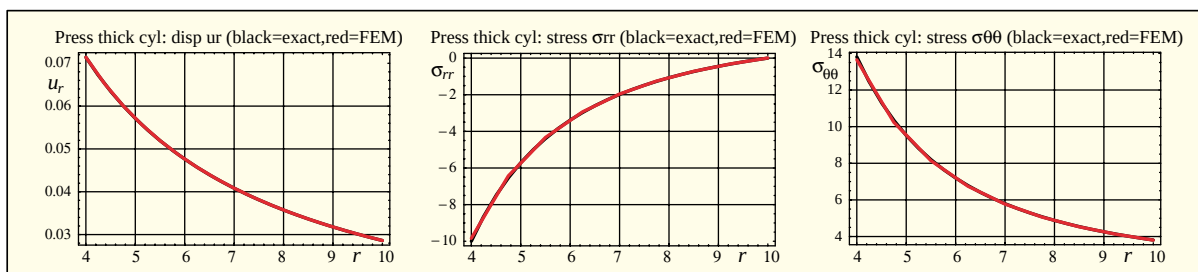


FIGURE 14.11. Pressurized thick cylinder benchmark: r plots of computed versus exact u_r , σ_{rr} and $\sigma_{\theta\theta}$ for Quad8 meshes with $\nu = 0.499$: (a) 2-element mesh, (b) 8-element mesh.

should be $-p = -10$, so it even has the wrong sign.

The number of elements is then increased to 16. Results are compared with the exact solution in Figure 14.9(b). The displacements u_r are more reasonable although still visibly off. The violent

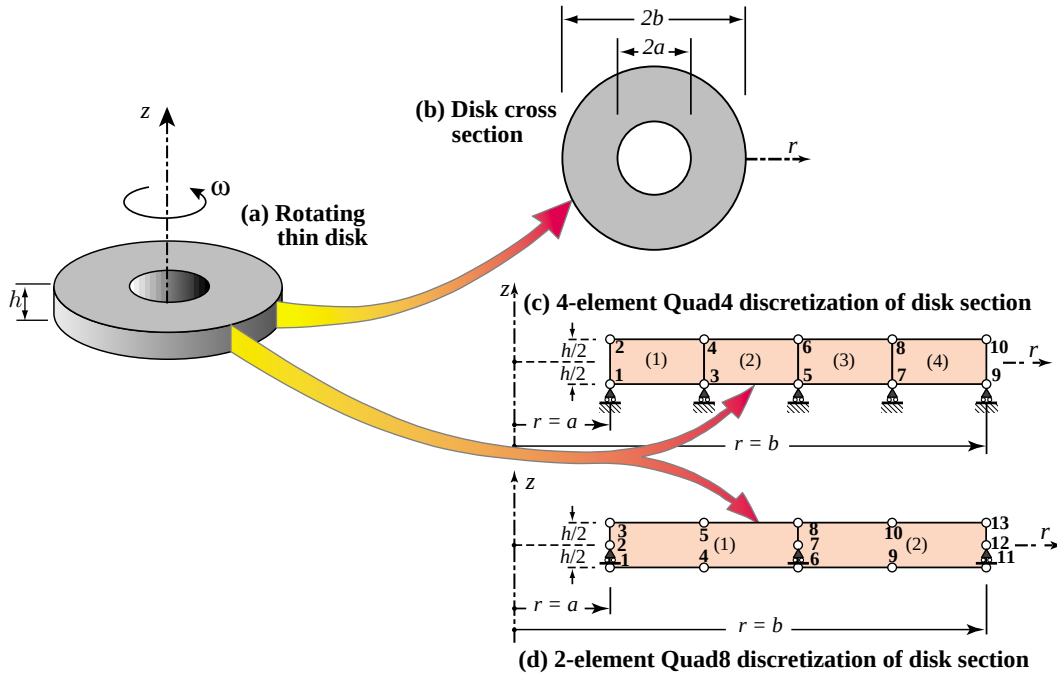


FIGURE 14.12. Rotating thin disk benchmark problem.

stress oscillation moves closer to the inner boundary, and results there are even worse than with the 4-element mesh. A minor stress oscillation can be observed at the outer boundary.

Changing the FEM model to Quad8 with reduced integration makes a big difference. For a 2-element mesh like that shown in Figure 14.2(c), computed and exact nodal values are tabulated in Figure 14.10. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over $a \leq r \leq b$ with the exact solution in Figure 14.11(a). As can be seen the model retains nodal exactness for displacements. Neither volumetric locking nor stress oscillations are observed, and the stresses are well predicted everywhere.

The number of elements is then increased to 8. Results are compared with the exact solution in Figure 14.11(b). The agreement with the exact solution is excellent.

In summary, the Quad4 model is useless for near-incompressible material in this benchmark and, in general, when modeling bulky axisymmetric solids.³ On the other hand, the reduced integration Quad8 can be strongly recommended for those problems.

Remark 14.1. If Quad8 is processed by a 3×3 integration Gauss rule, which represents full integration, volumetric locking reappears. So going from Quad4 to Quad8 is not sufficient: the integration rule makes a significant difference for near-incompressible behavior.

§14.2. Benchmark Example 2: Rotating Thin Disk

The second benchmark problem is a hollow, thin circular disk of thickness h , inner radius a and outer radius b , which spins about the z axis with constant angular frequency ω . The material is

³ The effect of volumetric locking is not so pronounced in the other two benchmarks because the plane strain condition is replaced by one of plane stress.

isotropic with elastic modulus E and Poisson's ratio ν and mass density ρ . The r axis is placed in the disk midplane. See Figure 14.12(a,b).

The FEM discretizations pictured in Figure 14.12(c,d) mimic those used in the pressurized thick cylinder benchmark, as shown in Figure 14.2(b,c). The number of elements in the radial direction is 4 and 16 for Quad4 and 2 and 8 for Quad8, respectively. Only one element is used in the axial direction. Nodes are allowed to move radially. Unlike the previous benchmark, however, movement in the axial (z) direction is permitted to allow for disk thickness contraction due to Poisson ratio. This motion is accomodated by constraining nodes in one of the constant- z surfaces to be on rollers as shown in Figure 14.12(c,d). All other nodes are left free. The only load is a centrifugal body force acting along r : $b_r = \rho \omega^2 r$ while $b_z = 0$.

§14.2.1. Exact Solution

The exact stress distribution for a condition of plane stress in the z direction is ⁴

$$\begin{aligned}\sigma_{rr} &= \rho \omega^2 r \frac{3+\nu}{8} \left(b^2 + a^2 - \frac{a^2 b^2}{r^2} - r^2 \right), \\ \sigma_{\theta\theta} &= \rho \omega^2 r \frac{3+\nu}{8} \left(b^2 + a^2 + \frac{a^2 b^2}{r^2} - \frac{(1+3\nu)r^2}{(3+\nu)} \right),\end{aligned}\tag{14.3}$$

others zero. Recovering strains from (14.3) and integrating yields the displacements

$$\begin{aligned}u_r &= \rho \omega^2 \frac{a^2(3+\nu)(r^2(1-\nu) + b^2(1+\nu)) + r^2(1-\nu)(b^2(3+\nu) - r^2(1+\nu))}{8 E r}, \\ u_z &= \rho \omega^2 z \nu \frac{(1-\nu-2\nu^2)(2r^2(1+\nu) - a^2(3+\nu) - b^2(3+\nu))}{4 E (1+3\nu)}.\end{aligned}\tag{14.4}$$

Notice that $u_z = 0$ at $z = 0$, which removes the axial rigid body motion. If $\nu = 0$, $u_z = 0$ as may be expected, whereas if $\nu = \frac{1}{2}$, $u_z = \frac{1}{2} \rho \omega^2 z$.

§14.2.2. Driver Script

The *Mathematica* driver script listed in Figure 14.13 accepts both 4-node and 8-node quad elements. Note that geometric, constitutive and discretization properties are declared at the top to make parametrization simpler. To set the element type it is sufficient to declare `etype` in the second line of the script to be either "Quad4" or "Quad8". The number of elements along the radial and axial directions: N_{er} and N_{ez} , are parametrized by the values assigned to `Ner` and `Nez`, respectively. The latter is assumed to be 1.

If the element type is "Quad4" a regular mesh is generated by modules `GenQuad4NodeCoordinates` and `GenQuad4ElemNodes`, described in the previous Chapter. If the element type is Quad8 the generation modules invoked are `GenQuad8NodeCoordinates` and `GenQuad8ElemNodes`. the element type is "Quad8". Pressure lumping to the nodes on the inner radius $r = a$ depends on element type. The body force field is specified at the 4 element corner nodes, regardless of whether the element type is Quad4 or Quad8.

⁴ Taken from S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, 2nd ed., 1951, Chapter 4. If $a \rightarrow 0$ the solution has a removable singularity at $r = 0$.

```

ClearAll[Em,v,a,b,h,Kfac, $\rho$ ,|,Ner,Nez,number];
Em=1000.; v=N[1/3]; Ner=4; Nez=1; etype="Quad4";
Kfac=1; a=4; b=10; h=1; aspect=h/(b-a);  $\rho$ =3.0;  $\omega$ =0.5;
number=True;

(* Define FEM model *)

MeshCorners=N[{a,0},{b,0},{b,h},{a,h}];
If [etype=="Quad4",
  NodeCoordinates=GenQuad4NodeCoordinates[MeshCorners,Ner,Nez];
  ElemNodes= GenQuad4ElemNodes[Ner,Nez]];
If [etype=="Quad8",
  NodeCoordinates=GenQuad8NodeCoordinates[MeshCorners,Ner,Nez];
  ElemNodes= GenQuad8ElemNodes[Ner,Nez]];
numnod=Length[NodeCoordinates]; numele=Length[ElemNodes];
ElemType= Table[etype,{numele}];
ElemMaterial= Table[{Em,v},{numele}];
ElemBodyForces=Table[{0,0},{numele}];
For [e=1,e<=numele,e++, enl=ElemNodes[[e]];
  ncoor=Table[NodeCoordinates[[enl[[i]]]],{i,4}];
  {{r1,z1},{r2,z2},{r3,z3},{r4,z4}}=ncoor;
  ElemBodyForces[[e]]= $\rho$ *| ^2*{{r1,0},{r2,0},{r3,0},{r4,0}}];
FreedomTags=FreedomValues=Table[{0,0},{numnod}];
If [etype=="Quad4",
  For [n=1,n<=numnod-Nez,n=n+Nez+1, FreedomTags[[n]]={0,1}]];
If [etype=="Quad8",
  For [n=1,n<=numnod-2*Nez,n=n+3*Nez+2, FreedomTags[[n+1]]={0,1}]];
ElemTractionForces={}; DefaultOptions={True};
(* Model definition print statements removed to shorten script *)
Plot2DElementsAndNodes[NodeCoordinates,ElemNodes,aspect,
  "Rotating disk mesh",True,True];

(* Solve problem and print results *)

{NodeDisplacements,NodeForces,NodeStresses}=RingAnalysisDriver[
  NodeCoordinates,ElemType,ElemNodes,
  ElemMaterial,ElemBodyForces,ElemTractionForces,
  FreedomTags,FreedomValues,DefaultOptions];
PrintRingAnalysisSolution[NodeDisplacements,NodeForces,
  NodeStresses,"Computed solution",{}}];
{ExactNodeDisplacements,ExactNodeStresses}=
  ExactSolution[NodeCoordinates,{a,b,h},{Em,v, $\rho$  | ,
  "RotatingThinDisk",number];
PrintRingNodeDispStresses[ExactNodeDisplacements,
  ExactNodeStresses,"Exact solution",{}}];

(* Contour plots of stress distributions *)

legend={{(a+b)/2,0.75*h}; whichones={True,False,True,False};
ContourPlotStresses[NodeCoordinates,ElemNodes,NodeStresses,
  whichones,True,{},legend,aspect];

(* Radial plots comparing FEM vs exact solutions *)

pwhat={"ur"," $\sigma_{rr}$ "," $\sigma_{zz}$ "," $\sigma_{\theta\theta}$ "};
For [ip=1,ip<=Length[pwhat],ip++, what=pwhat[[ip]];
  RadialPlotFEMvsExact[etype,NodeCoordinates,NodeDisplacements,
  NodeStresses,{a,b,h},{Em,v, $\rho$ },|,{Ner,Nez},
  "RotatingThinDisk",what,0,number] ];

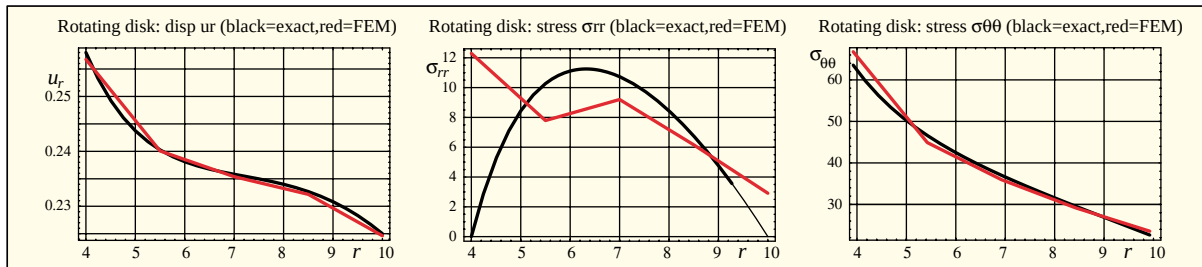
```

FIGURE 14.13. Script for rotating disk benchmark problem.

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma-θθ	sigma-rz	r-force	z-force
1	0.2567	0.0000	12.2963	2.3994	67.8220	-0.7835	5.7305	-0.5185
2	0.2547	-0.0243	12.6637	2.3936	67.4372	0.7073	5.7305	0.0000
3	0.2401	0.0000	7.7905	-0.6290	45.3527	-0.5037	17.2266	1.3833
4	0.2387	-0.0183	8.1077	-0.5589	45.2458	0.5598	17.2266	0.0000
5	0.2354	0.0000	9.2003	-0.6720	36.1805	-0.4128	27.7734	-1.3919
6	0.2343	-0.0158	9.1420	-0.7594	35.9764	0.4104	27.7734	0.0000
7	0.2322	0.0000	6.1607	0.0342	29.2155	-0.4599	40.8516	0.6446
8	0.2309	-0.0118	6.0606	-0.0697	29.0039	0.4653	40.8516	0.0000
9	0.2246	0.0000	2.9120	0.3871	23.4411	-0.4071	25.4180	-0.1175
10	0.2235	-0.0084	2.9721	0.3768	23.3500	0.4331	25.4180	0.0000
Exact (plane stress) solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma-θθ	sigma-rz		
1	0.2580	0.0000	0.0000	0.0000	64.5000	0.0000		
2	0.2580	-0.0048	0.0000	0.0000	64.5000	0.0000		
3	0.2403	0.0000	10.2679	0.0000	47.1071	0.0000		
4	0.2403	-0.0043	10.2679	0.0000	47.1071	0.0000		
5	0.2358	0.0000	10.7334	0.0000	37.2666	0.0000		
6	0.2358	-0.0036	10.7334	0.0000	37.2666	0.0000		
7	0.2327	0.0000	6.7515	0.0000	29.6235	0.0000		
8	0.2327	-0.0027	6.7515	0.0000	29.6235	0.0000		
9	0.2250	0.0000	0.0000	0.0000	22.5000	0.0000		
10	0.2250	-0.0017	0.0000	0.0000	22.5000	0.0000		

FIGURE 14.14. Rotating disk benchmark: computed results and exact solution for 4-element Quad4 mesh, with Poisson ratio $\nu = 1/3$.

(a) 4 x 1 Mesh of Quad4 Elements, $\nu = 1/3$:



(b) 16 x 1 Mesh of Quad4 Elements, $\nu = 1/3$:

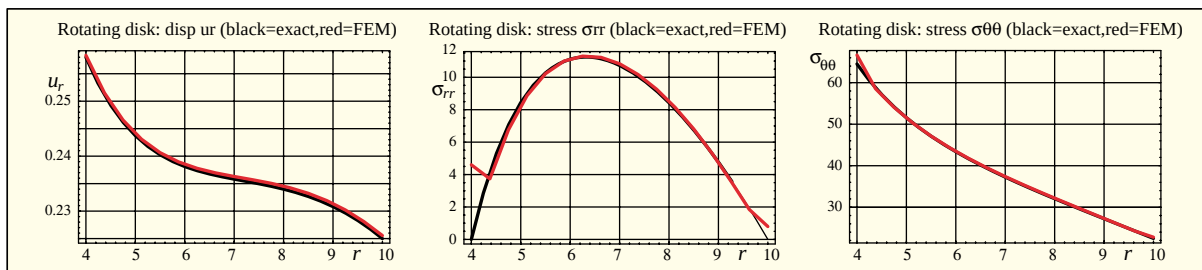


FIGURE 14.15. Rotating disk benchmark: radial plots of u_r , σ_{rr} and $\sigma_{\theta\theta}$ for 4-element and 16-element Quad4, with Poisson ratio $\nu = 1/3$.

§14.2.3. Numerical Results

The results presented here were computed for $a = 4$, $b = 10$, $h = 1$, $p = 10$, $E = 1000$, $\nu = 1/3$, two element types, and two radial discretizations for each type.

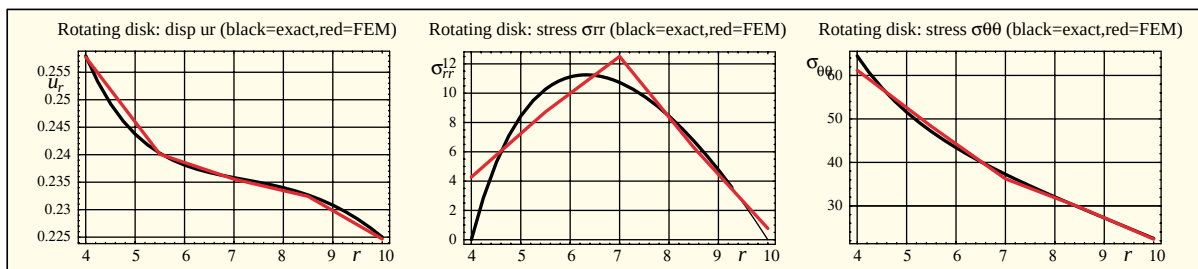
The script of Figure 14.13 specifically sets `etype="Quad4"`, `Ner=4` and $\nu = 1/3$. The mesh is actually that pictured in Figure 14.12(c). Computed and exact nodal values are tabulated in Figure

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz	r-force	z-force
1	0.2577	0.0108	4.2728	0.0000	61.2000	0.0000	-6.8437	0.0000
2	0.2579	0.0000	4.2728	0.0000	61.2000	0.0000	19.1250	0.0000
3	0.2577	-0.0108	4.2728	0.0000	61.2000	0.0000	-6.8437	0.0000
4	0.2401	0.0096	8.7239	0.0000	48.2944	0.0000	23.2500	0.0000
5	0.2401	-0.0096	8.7239	0.0000	48.2944	0.0000	23.2500	0.0000
6	0.2356	0.0080	12.5046	0.0000	36.2225	0.0000	-20.0625	0.0000
7	0.2358	0.0000	12.5046	0.0000	36.2225	0.0000	75.7500	0.0000
8	0.2356	-0.0080	12.5046	0.0000	36.2225	0.0000	-20.0625	0.0000
9	0.2324	0.0061	6.3024	0.0000	29.7056	0.0000	54.7500	0.0000
10	0.2324	-0.0061	6.3024	0.0000	29.7056	0.0000	54.7500	0.0000
11	0.2247	0.0037	0.7706	0.0000	22.3550	0.0000	-12.0937	0.0000
12	0.2251	0.0000	0.7706	0.0000	22.3550	0.0000	61.1250	0.0000
13	0.2247	-0.0037	0.7706	0.0000	22.3550	0.0000	-12.0937	0.0000

Exact (plane stress) solution						
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz
1	0.2580	0.0000	0.0000	0.0000	64.5000	0.0000
2	0.2580	-0.0024	0.0000	0.0000	64.5000	0.0000
3	0.2580	0.0048	0.0000	0.0000	64.5000	0.0000
4	0.2403	0.0000	10.2679	0.0000	47.1071	0.0000
5	0.2403	-0.0043	10.2679	0.0000	47.1071	0.0000
6	0.2358	0.0000	10.7334	0.0000	37.2666	0.0000
7	0.2358	-0.0018	10.7334	0.0000	37.2666	0.0000
8	0.2358	0.0036	10.7334	0.0000	37.2666	0.0000
9	0.2327	0.0000	6.7515	0.0000	29.6235	0.0000
10	0.2327	-0.0027	6.7515	0.0000	29.6235	0.0000
11	0.2250	0.0000	0.0000	0.0000	22.5000	0.0000
12	0.2250	-0.0008	0.0000	0.0000	22.5000	0.0000
13	0.2250	-0.0017	0.0000	0.0000	22.5000	0.0000

FIGURE 14.16. Rotating disk benchmark: computed results and exact solution for 2-element Quad8 mesh, with Poisson ratio $\nu = 1/3$.

(a) 2 x 1 Mesh of Quad8 Elements, $\nu = 1/3$:



(b) 8 x 1 Mesh of Quad8 Elements, $\nu = 1/3$:

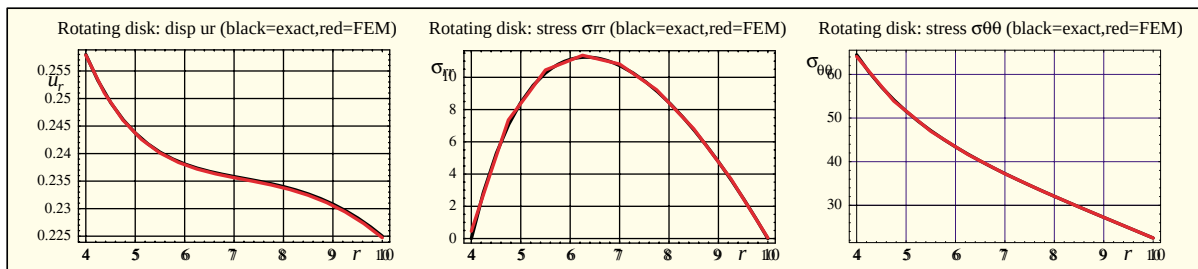


FIGURE 14.17. Rotating disk benchmark: radial plots of u_r , σ_{rr} and $\sigma_{\theta\theta}$ for 2-element and 8-element Quad8 meshes, with Poisson ratio $\nu = 1/3$.

14.14. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over $a \leq r \leq b$ with the exact solution in Figure 14.15(a).

As can be seen u_r and $\sigma_{\theta\theta}$ are satisfactorily predicted. The radial stress, however, is way off,

especially at the inner and outer boundaries, at which it should be zero. This is again a consequence of the impossibility of doing interelement stress averaging there. For this low-order model the variation of σ_{rr} in the r direction is limited to be constant within the element.

Increasing N_{er} to 16 gives a solution that is compared with the exact one in Figure 14.15(b). Both u_r and $\sigma_{\theta\theta}$ are correct to at least 3 places. The notable flaw is the discrepancy of σ_{rr} at both boundaries. Again this is a consequence of lack of interelement averaging. Away from the hole σ_{rr} agrees with the exact solution to satisfactory accuracy.

The analysis is then redone with the 8-node quadrilateral Quad8 with reduced (2×2) integration. To make a fair comparison with Quad4, the Quad8 meshes contain half the elements: 2 and 8, respectively, which results in a similar number of nodes.

The results of running Quad8 with $N_{er}=2$ are tabulated in Figure 14.16. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over $a \leq r \leq b$ with the exact solution in Figure 14.17(a). This element is supposed to be nodally exact for one-dimensional problems, and indeed the computed and exact u_r may be verified to agree numerically to 15 places at all nodes. The hoop stress should be also nodally exact since $\sigma_{\theta\theta} = E u_r / r$, but the extrapolation from Gauss points introduces discrepancies. The computed radial stress σ_{rr} is as good as can be expected from a linear variation over the element.

Running Quad8 with $N_{er}=8$ give the results plotted in Figure 14.17(b). Again the displacements are nodally exact. Both σ_{rr} and $\sigma_{\theta\theta}$ agree everywhere with the exact solution at plot accuracy.

Rerunning these cases for a Poisson ratio close to $\frac{1}{2}$ shows deterioration of Quad4 accuracy, but not so dramatic as that experienced in the pressurized thick cylinder benchmark. This indicates that volumetric locking is less of a problem in this case, as could be expected since the plane stress condition allows lateral expansion and contraction.

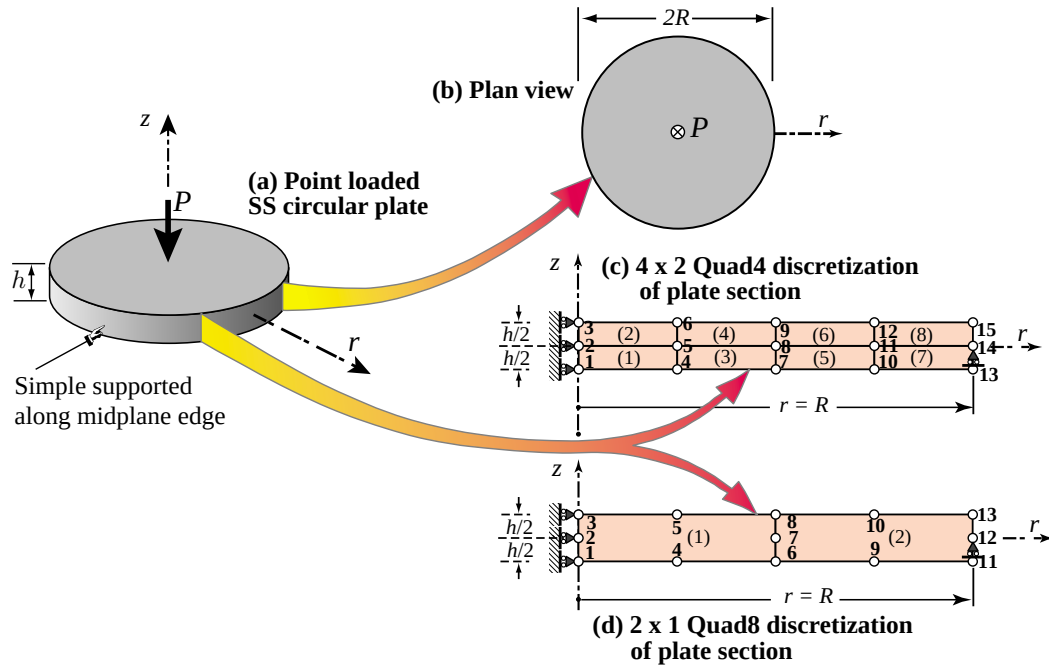


FIGURE 14.18. Point-loaded circular plate bending benchmark problem.

§14.3. Benchmark 3: Point Loaded SS Circular Plate Bending

The third benchmark problem is a simply-supported (SS) circular plate bent by a lateral point load. The plate has radius R and thickness h . The point load of magnitude P acts downward at the plate center. The material is isotropic with elastic modulus E and Poisson's ratio ν . See Figure 14.18(a,b) for the problem definition.

Two FEM discretizations are pictured in Figure 14.18(c,d).

For the Quad4 element type 4×2 and 16×2 discretizations are used, whereas for Quad8 the meshes are 2×1 and 8×1 . The reason for selecting 2 elements along z for the Quad4 discretization is that it provides nodes at the midplane $z = 0$, allowing a midplane-symmetric specification of the simple supported BC at $r = R$.

For the Quad8 discretization one element along z is sufficient since several midnodes are available on the midplane. Nodes are allowed to move in the z direction except those on the midplane $z = 0$ at $r = R$. The 3 nodes at $r = 0$ must be constrained against radial motion. The resulting support conditions are shown in Figure 14.18(c,d). The central point load P is divided by 2π since $K_{fac} = 1$ and then appropriately lumped to the 3 nodes on the z axis.

§14.3.1. Exact Solution

The exact solution⁵ for a Kirchhoff plate model of this problem⁶ gives the bending moments and associated stresses as

$$\begin{aligned} M_{rr} &= \frac{P}{4\pi} (1 + \nu) \log \frac{R}{r}, & \sigma_{rr} &= 12M_{rr}z/h^3, \\ M_{\theta\theta} &= \frac{P}{4\pi} \left[(1 + \nu) \log \frac{R}{r} + 1 - \nu \right], & \sigma_{\theta\theta} &= 12M_{\theta\theta}z/h^3, \end{aligned} \quad (14.5)$$

Other stress components are zero. Moments and stresses given by (14.5) become infinite at $r = 0$ under the point load, so when comparing to a FEM solution r is limited to $r \leq r_{truc} = R/1000$ to avoid blow-ups. The stresses at the upper and lower plate surfaces are $\sigma_{rr} = \pm 12M_{rr}(h/2)/h^3 = \pm 6M_{rr}/h^2$ and $\sigma_{\theta\theta} = \pm 12M_{\theta\theta}(h/2)/h^3 = \pm 6M_{\theta\theta}/h^2$. The transverse displacement is

$$u_z = -\frac{P}{16\pi D} \left[\frac{3 + \nu}{1 + \nu} (R^2 - r^2) + 2r^2 \log \frac{r}{R} \right], \quad (14.6)$$

where $D = Eh^3/(12(1 - \nu^2))$ is the plate rigidity. Note that u_z does not depend on z , which follows from the Kirchhoff thin plate theory assumptions. Also u_z is finite at $r = 0$ because $\lim_{r \rightarrow 0} r^2 \log(r/R) = 0$. The radial displacement is given by

$$u_r = -z \frac{\partial u_z}{\partial r} = \frac{P}{8\pi D} \left(\frac{3 + \nu}{1 + \nu} - 1 - 2 \log \frac{r}{R} \right) r z, \quad (14.7)$$

which also vanishes at $r = 0$.

§14.3.2. Driver Script

The driver script for the point-loaded circular plate benchmark is listed in Figure 14.19. This is similar to the previous scripts except for the use of 2 elements along the thickness (the axial or z direction) if the element type is Quad4. The central force is scaled by $K_{fac}/(2\pi) = 1/(2\pi)$ and distributed to the 3 nodes at $r = 0$.

⁵ Taken from S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, 2nd ed., 1959, Ch. 2.

⁶ The Kirchhoff model assumes that the plate is thin in the sense that $h \ll R$, but not so thin as to become a membrane.

```

ClearAll[Em,v,a,b,h,Kfac,ρ,|,Ner,Nez,number];
Em=1000.; v=N[1/3]; Ner=4; Nez=1; etype="Quad4";
Kfac=1; a=4; b=10; h=1; aspect=h/(b-a); ρ=3.0; | =0.5;
number=True;

(* Define FEM model *)

MeshCorners=N[{{a,0},{b,0},{b,h},{a,h}}];
If [etype=="Quad4",
  NodeCoordinates=GenQuad4NodeCoordinates[MeshCorners,Ner,Nez];
  ElemNodes= GenQuad4ElemNodes[Ner,Nez];
If [etype=="Quad8",
  NodeCoordinates=GenQuad8NodeCoordinates[MeshCorners,Ner,Nez];
  ElemNodes= GenQuad8ElemNodes[Ner,Nez];
numnod=Length[NodeCoordinates]; numele=Length[ElemNodes];
ElemType= Table[etype,{numele}];
ElemMaterial= Table[{Em,v},{numele}];
ElemBodyForces=Table[{0,0},{numele}];
For [e=1,e<=numele,e++, enl=ElemNodes[[e]];
  ncoor=Table[NodeCoordinates[[enl[[i]]]],{i,4}];
  {{r1,z1},{r2,z2},{r3,z3},{r4,z4}}=ncoor;
  ElemBodyForces[[e]]=ρ*| ^2*{{r1,0},{r2,0},{r3,0},{r4,0}}];
FreedomTags=FreedomValues=Table[{0,0},{numnod}];
If [etype=="Quad4",
  For [n=1,n<=numnod-Nez,n=n+Nez+1, FreedomTags[[n]]={0,1}];
If [etype=="Quad8",
  For [n=1,n<=numnod-2*Nez,n=n+3*Nez+2, FreedomTags[[n+1]]={0,1}];
ElemTractionForces={}; DefaultOptions={True};
(* Print model definition statements removed to shorten script *)
Plot2DElementsAndNodes[NodeCoordinates,ElemNodes,aspect,
  "Rotating disk mesh",True,True];

(* Solve problem and print results *)

{NodeDisplacements,NodeForces,NodeStresses}=RingAnalysisDriver[
  NodeCoordinates,ElemType,ElemNodes,
  ElemMaterial,ElemBodyForces,ElemTractionForces,
  FreedomTags,FreedomValues,DefaultOptions];
PrintRingAnalysisSolution[NodeDisplacements,NodeForces,
  NodeStresses,"Computed solution",{)];
{ExactNodeDisplacements,ExactNodeStresses}=
  ExactSolution[NodeCoordinates,{a,b,h},{Em,v,ρ},|,
  "RotatingThinDisk",number];
PrintRingNodeDispStresses[ExactNodeDisplacements,
  ExactNodeStresses,"Exact solution",{)];

(* Contour plot of stress distributions *)

legend={{(a+b)/2,0.75*h}; whichones={True,False,True,False};
ContourPlotStresses[NodeCoordinates,ElemNodes,NodeStresses,
  whichones,True,{],legend,aspect];

(* Radial plots comparing FEM vs exact solution *)

pwhat={"ur","σrr","σzz","σθθ"};
For [ip=1,ip<=Length[pwhat],ip++, what=pwhat[[ip]];
  RadialPlotFEMvsExact[etype,NodeCoordinates,NodeDisplacements,
  NodeStresses,{a,b,h},{Em,v,ρ},|,{Ner,Nez},
  "RotatingThinDisk",what,0,number] ];

```

FIGURE 14.19. Script for point loaded circular plate benchmark problem.

§14.3.3. Numerical Results

The numerical results presented here were computed for $R = 10$, $h = 1$, $E = 1000$, $\nu = 1/3$ and $P = 10$, two element types, and two discretizations for each type.

The script of Figure 14.19 specifically sets `etype="Quad4"`, `Ner=4`, `Nez=2` and $\nu = 1/3$. The mesh is actually that pictured in Figure 14.18(c). Computed and exact nodal values are tabulated in Figure 14.20. The values shown as “exact” for stresses σ_{rr} and $\sigma_{\theta\theta}$ are actually those evaluated from the Kirchhoff solution (14.6) at $r = R/1000 = 1/100$, since those equations have logarithmic singularities as $r \rightarrow 0$. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over $a \leq r \leq b$ with the exact solution in Figure 14.21(a). To avoid plot blow-ups the “exact” solutions are again truncated to that small radius.

The radial displacement has the right shape but is underpredicted. This is a mild case of the so-called “shear locking”: a significant amount of element energy is spent in shear, resulting in overstiffness. The effect would get worse if the thickness-to-radius ratio (1/10 for this benchmark) is decreased. Considering the coarse mesh the stress predictions are OK sufficiently away from the plate center, say, for $r > 2$. Evidently the FEM solution has trouble capturing the singularity; for that a refined mesh near the center would be required.

Increasing `Ner` to 16 gives a solution that is compared with the exact one in Figure 14.21(b). Shear locking is alleviated but the transverse displacement is still somewhat underpredicted. The stress distribution away from the center are significantly improved, but the singularity is still poorly captured.

The analysis is then redone with the 8-node quadrilateral `Quad8` with reduced (2×2) integration. To make a fair comparison with `Quad4`, the `Quad8` meshes contain half the elements: 2 and 8, respectively, in the radial direction, and only one in the axial direction.

The results of running `Quad8` with `Ner=2` are tabulated in Figure 14.22. Radial displacements u_r , radial stresses σ_{rr} and hoop stresses $\sigma_{\theta\theta}$ are graphically compared over $a \leq r \leq b$ with the exact solution in Figure 14.23(a). It can be seen that the transverse displacement u_z is well captured (within about 1%) since this element does not suffer from shear locking. The stress distribution is fine away from the center. Capturing the singularity is obviously difficult with 2 elements and a linear stress variation radially, but the model does a good fitting job.

Running `Quad8` with `Ner=8` give the results plotted in Figure 14.23(b). The fit to both displacements and stresses is good, even fairly close to the singularity. No trace of shear locking is observed.

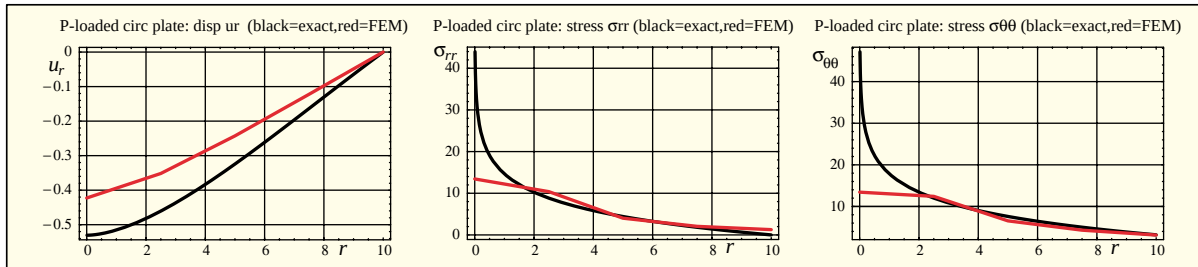
Rerunning these models with $\nu = 0.499$ (results not shown here) degrades the `Quad4` results further (some volumetric locking adds to the shear locking) but has little effect on the `Quad8` discretization.

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz	r-force	z-force
1	0.0000	-0.4227	13.4338	4.1445	13.4338	10.6640	-3.9528	-0.3979
2	0.0000	-0.4251	0.0000	0.0000	0.0000	10.8184	0.0000	-0.7958
3	0.0000	-0.4227	-13.4338	-4.1445	-13.4338	10.6640	3.9528	-0.3979
4	0.0189	-0.3516	10.3857	4.8514	12.4267	-0.7101	0.0000	0.0000
5	0.0000	-0.3530	0.0000	0.0000	0.0000	-0.5780	0.0000	0.0000
6	-0.0189	-0.3516	-10.3857	-4.8514	-12.4267	-0.7101	0.0000	0.0000
7	0.0231	-0.2428	4.0131	2.2300	6.5383	-0.1441	0.0000	0.0000
8	0.0000	-0.2434	0.0000	0.0000	0.0000	-0.0730	0.0000	0.0000
9	-0.0231	-0.2428	-4.0131	-2.2300	-6.5383	-0.1441	0.0000	0.0000
10	0.0244	-0.1222	2.0584	1.2624	4.2986	-0.0619	0.0000	0.0000
11	0.0000	-0.1226	0.0000	0.0000	0.0000	-0.0325	0.0000	0.0000
12	-0.0244	-0.1222	-2.0584	-1.2624	-4.2986	-0.0619	0.0000	0.0000
13	0.0242	0.0003	1.2697	0.9626	3.1253	0.2516	0.0000	0.0000
14	0.0000	0.0000	0.0000	0.0000	0.0000	0.2782	0.0000	1.5915
15	-0.0242	0.0003	-1.2697	-0.9626	-3.1253	0.2516	0.0000	0.0000

Thin plate (Kirchhoff) solution						
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz
1	0.0000	-0.5305	43.9761	0.0000	47.1592	0.0000
2	0.0000	-0.5305	0.0000	0.0000	0.0000	0.0000
3	0.0000	-0.5305	-43.9761	0.0000	-47.1592	0.0000
4	0.0227	-0.4606	8.8254	0.0000	12.0085	0.0000
5	0.0000	-0.4606	0.0000	0.0000	0.0000	0.0000
6	-0.0227	-0.4606	-8.8254	0.0000	-12.0085	0.0000
7	0.0306	-0.3243	4.4127	0.0000	7.5958	0.0000
8	0.0000	-0.3243	0.0000	0.0000	0.0000	0.0000
9	-0.0306	-0.3243	-4.4127	0.0000	-7.5958	0.0000
10	0.0330	-0.1634	1.8314	0.0000	5.0145	0.0000
11	0.0000	-0.1634	0.0000	0.0000	0.0000	0.0000
12	-0.0330	-0.1634	-1.8314	0.0000	-5.0145	0.0000
13	0.0318	0.0000	0.0000	0.0000	3.1831	0.0000
14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15	-0.0318	0.0000	0.0000	0.0000	-3.1831	0.0000

FIGURE 14.20. Point loaded circular plate benchmark: computed results and exact solution for 4×2 Quad4 mesh, with Poisson ratio $\nu = 1/3$.

(a) 4×2 Mesh of Quad4 Elements, $\nu = 1/3$:



(b) 16×2 Mesh of Quad4 Elements, $\nu = 1/3$:

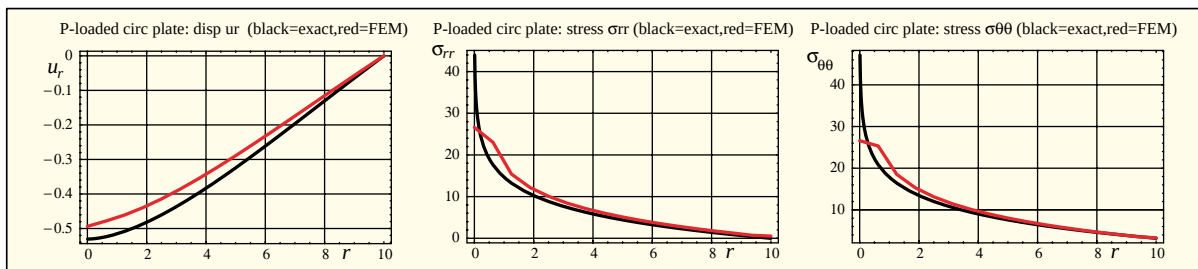


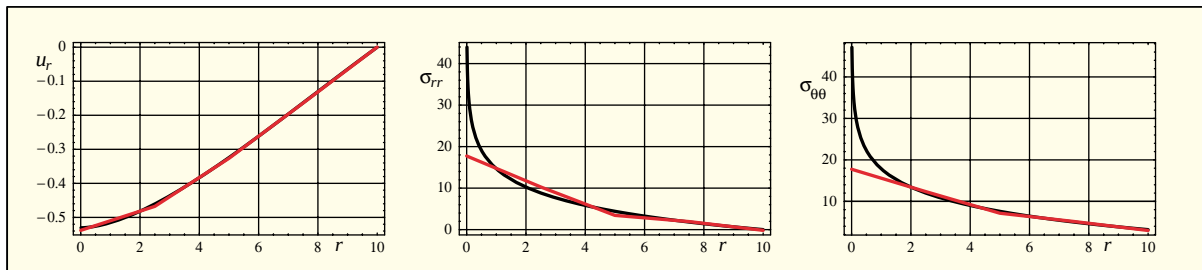
FIGURE 14.21. Point loaded circular plate benchmark: radial plots of u_r , σ_{rr} and $\sigma_{\theta\theta}$ for 4×2 and 16×2 element Quad4, with Poisson ratio $\nu = 1/3$. Plot is along the bottom plate surface $z = -h/2$.

Computed solution								
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz	r-force	z-force
1	0.0000	-0.5381	17.7424	0.3592	17.7424	1.9099	-1.3263	-0.2653
2	0.0000	-0.5409	0.0000	0.0000	0.0000	1.9099	0.0000	-1.0610
3	0.0000	-0.5381	-17.7424	-0.3592	-17.7424	1.9099	1.3263	-0.2653
4	0.0223	-0.4676	10.2753	0.0000	12.3574	0.9549	0.0000	0.0000
5	-0.0223	-0.4676	-10.2753	0.0000	-12.3574	0.9549	0.0000	0.0000
6	0.0308	-0.3265	3.4321	-0.0939	7.1442	0.1469	0.0000	0.0000
7	0.0000	-0.3274	0.0000	0.0000	0.0000	0.1469	0.0000	0.0000
8	-0.0308	-0.3265	-3.4321	0.0939	-7.1442	0.1469	0.0000	0.0000
9	0.0330	-0.1641	1.9422	0.0626	5.1496	0.2204	0.0000	0.0000
10	-0.0330	-0.1641	-1.9422	-0.0626	-5.1496	0.2204	0.0000	0.0000
11	0.0318	0.0002	-0.1718	-0.0462	2.9832	0.1469	0.0000	0.0000
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.1469	0.0000	1.5915
13	-0.0318	0.0002	0.1718	0.0462	-2.9832	0.1469	0.0000	0.0000

Thin plate (Kirchhoff) solution						
node	r-disp	z-disp	sigma-rr	sigma-zz	sigma- $\theta\theta$	sigma-rz
1	0.0000	-0.5305	43.9761	0.0000	47.1592	0.0000
2	0.0000	-0.5305	0.0000	0.0000	0.0000	0.0000
3	0.0000	-0.5305	-43.9761	0.0000	-47.1592	0.0000
4	0.0227	-0.4606	8.8254	0.0000	12.0085	0.0000
5	-0.0227	-0.4606	-8.8254	0.0000	-12.0085	0.0000
6	0.0306	-0.3243	4.4127	0.0000	7.5958	0.0000
7	0.0000	-0.3243	0.0000	0.0000	0.0000	0.0000
8	-0.0306	-0.3243	-4.4127	0.0000	-7.5958	0.0000
9	0.0330	-0.1634	1.8314	0.0000	5.0145	0.0000
10	-0.0330	-0.1634	-1.8314	0.0000	-5.0145	0.0000
11	0.0318	0.0000	0.0000	0.0000	3.1831	0.0000
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
13	-0.0318	0.0000	0.0000	0.0000	-3.1831	0.0000

FIGURE 14.22. Point loaded circular plate benchmark: computed results and exact solution for 2×1 Quad8 mesh, with Poisson ratio $\nu = 1/3$.

(a) 2×1 Mesh of Quad8 Elements, $\nu = 1/3$:



(b) 8×1 Mesh of Quad8 Elements, $\nu = 1/3$:

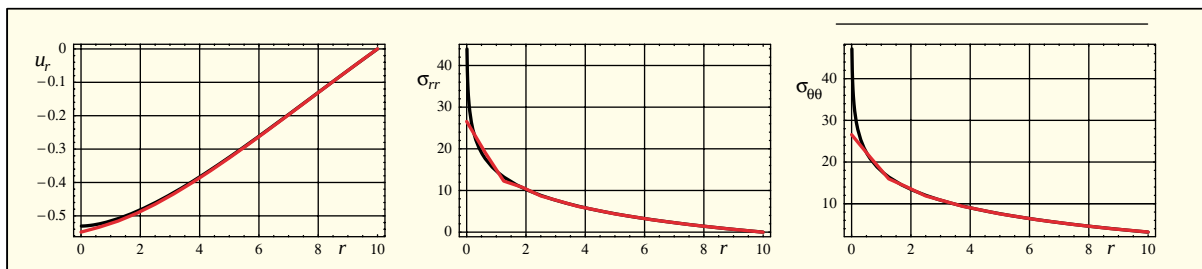


FIGURE 14.23. Point loaded circular plate benchmark: radial plots of u_r , σ_{rr} and $\sigma_{\theta\theta}$ for 2×1 and 8×1 Quad8 meshes, with Poisson ratio $\nu = 1/3$. Plot is along the bottom plate surface $z = -h/2$.

Homework Exercises for Chapter 14

Axisymmetric Solid Benchmark Problems

EXERCISE 14.1 [C:20] This exercise deals with the thick-tube benchmark example discussed in §13.2–5. The script for this exercise is in Cell 12 of the Notebook Quad4S0R.nb supplied with this Chapter.

- (a) Repeat the 4-element analysis using $\nu = 0.45$ and $\nu = 0.499$. Describe what happens to the accuracy of displacements and stresses when compared with the exact solution.
- (b) Can a more refined mesh fix the problems noted in (a)? To check, run 16 elements along the radial direction and report if things have improved.

EXERCISE 14.2 [C:20] A concrete pile embedded in a soil half-space, as defined in Figure E14.1. This problem is solved using a very coarse mesh in Cell 17 of the Notebook Quad4S0R.nb supplied with this Chapter. Repeat the analysis using a more refined mesh, like that suggested in the figure. Module `GenerateGradedRingNodeCoordinates` may be used to generate a graded regular mesh.

The total force applied to pile is $P = 500000$ lbs (this may be assumed to be uniformly distributed on the top pile surface, or placed as a point load). Other data: pile modulus $E_P = 300,000$ psi, soil modulus $E_S = E_P/50$, Poisson's ratio for pile $\nu_P = 0.1$, Poisson's ratio for soil $\nu_S = 0.40$, pile diameter $d = 10$ in, pile length $L = 250$ in, and mesh z-length $H = 1.2L$. Truncate the mesh at $R = 80$ in. Neglect all body forces.

Nodes on the “soil truncation boundary” should be fixed. Points on the z axis $r = 0$ should be on vertical rollers except for 1.

The most interesting numerical results are: (i) the top and bottom vertical displacement of the pile, (ii) the reaction forces at the bottom of the pile, and (iii) the normal stress σ_{zz} in the pile.

EXERCISE 14.3 [C:20] The spinning Mother Earth. Model one quadrant of the planet cross section with an axisymmetric finite element mesh as sketched in Figure E19.3 (note that all elements are 4-node quadrilaterals). The problem is solved in Cell 18 of the Notebook Quad4S0R.nb supplied with this Chapter, using a very coarse mesh of only 4 elements.

Use the Kg-force/m/sec unit system for this problem. The Earth spins with angular velocity $\omega = 2\pi \text{ rad}/24\text{hrs} = (2\pi/86400) \text{ sec}^{-1}$ about the z axis. The planet radius is $R = 6370 \text{ Km} = 6.37 \cdot 10^6 \text{ m}$. For E take 1/3 of the rigidity of steel, or $E = 7 \cdot 10^5 \text{ Kg/cm}^2 = 7 \cdot 10^9 \text{ Kg/m}^2$ as Love (Theory of Elasticity) recommends; Poisson's ratio $\nu = 0.3$ (this is my own guess), and mass density $\rho = 5.52$ times the water density. [Watch out for units: the centrifugal body force $\rho\omega^2 r$ should come up in Kg/m^3 .] All gravitational field effects (self weight) are ignored.

- (a) Get the equatorial “bulge” and the polar “flattening” in Km, and the maximum stress in MPa.
- (b) Where do the maximum normal stresses occur?

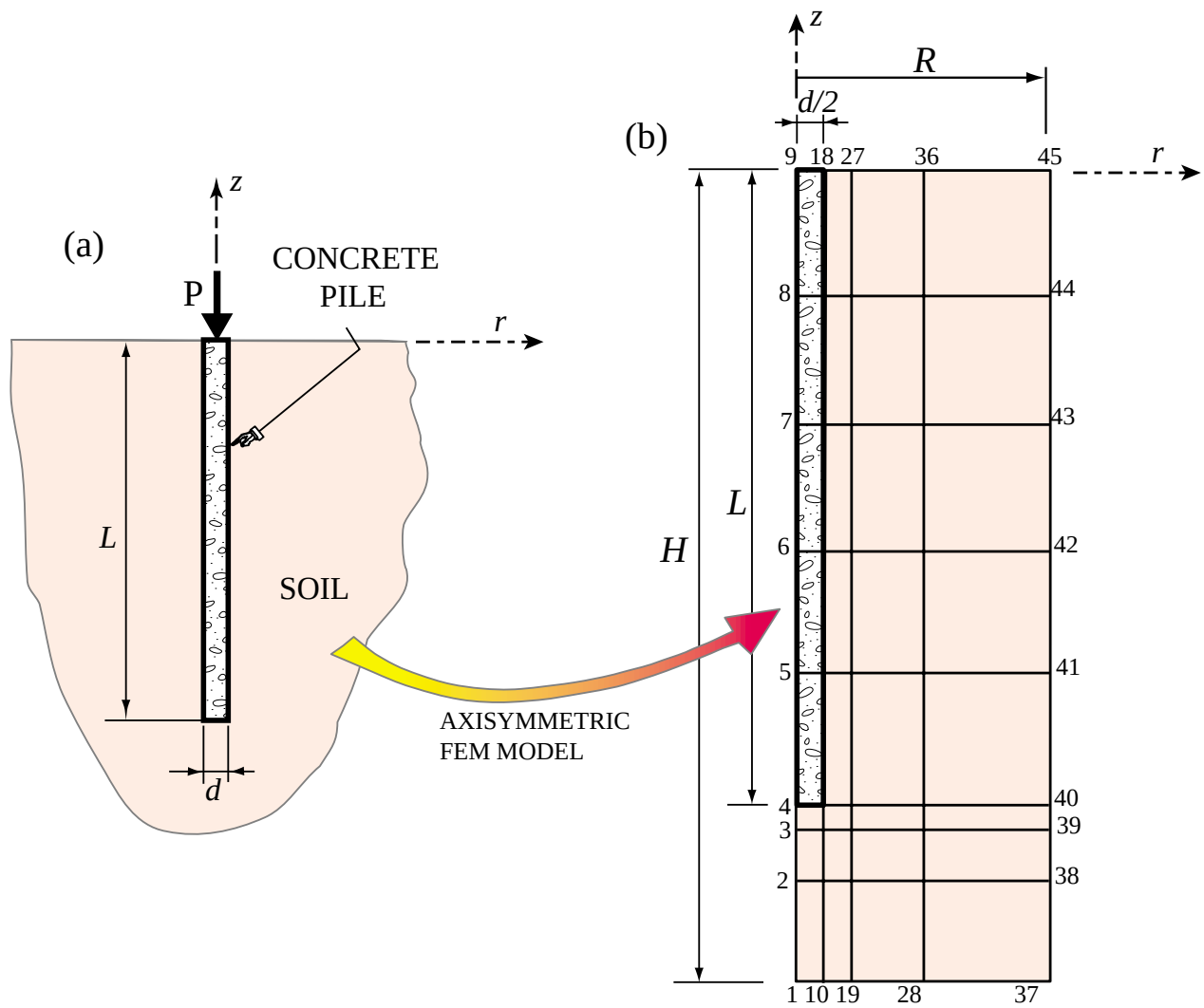


Figure E14.1. Pile embedded in soft soil.

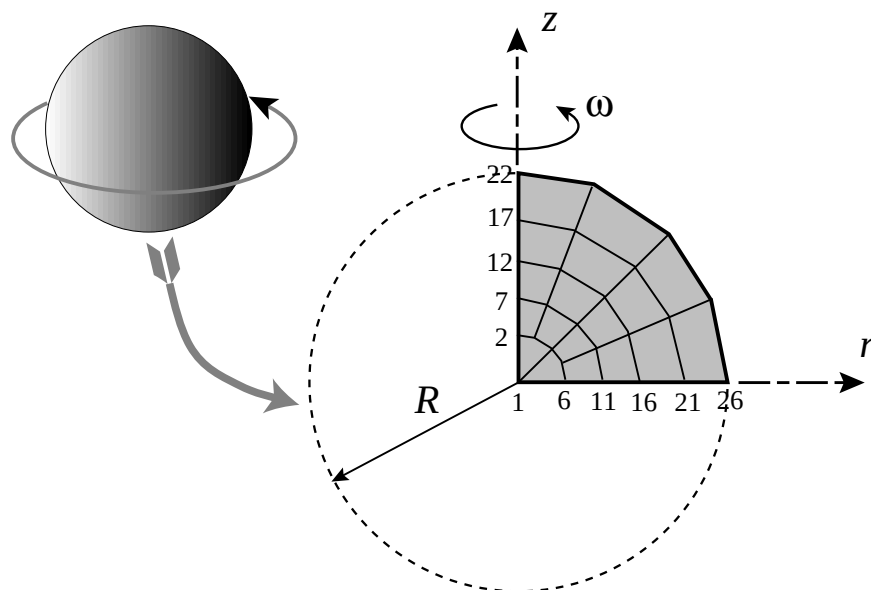


Figure E14.2. Our spinning planet.

15

Solid Elements: Overview

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§15.1. Introduction

Solid elements are three-dimensional finite elements that can model solid bodies and structures without any *a priori* geometric simplification.

Finite element models of this type have the advantage of *directness*. Geometric, constitutive and loading assumptions required to effect dimensionality reduction, for example to planar or axisymmetric behavior, are avoided. Boundary conditions on both forces and displacements can be more realistically treated. Another attractive feature is that the finite element mesh visually looks like the physical system. See Figure 15.1.

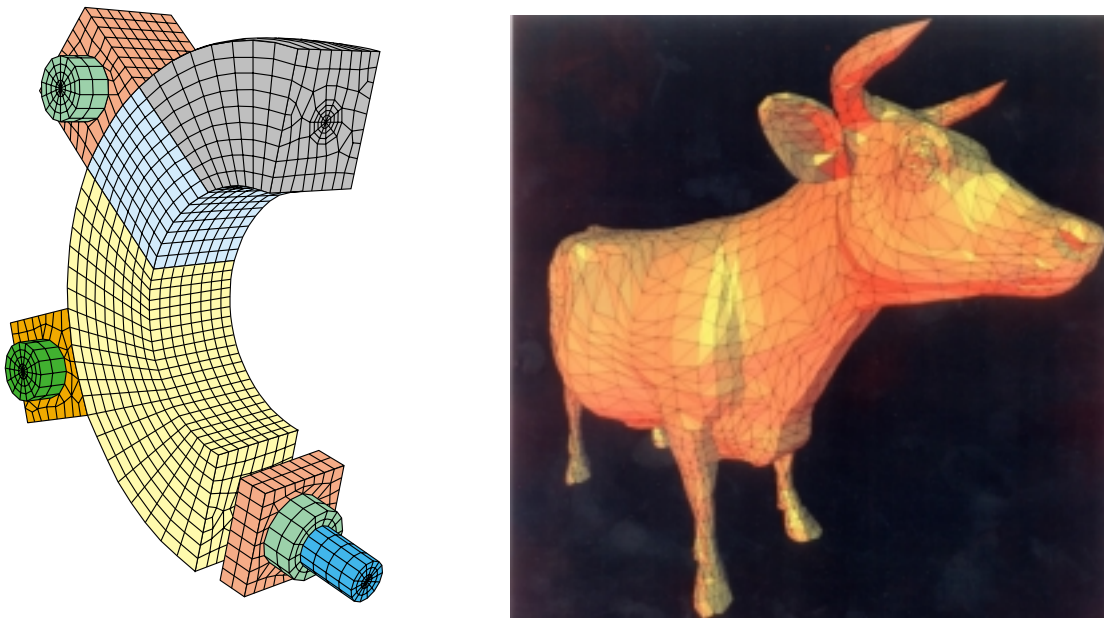


FIGURE 15.1. Use of solid elements simplifies visualization of the physical object being modeled.

This directness does not come for free. It is paid in terms of modeling, mesh preparation, computing and postprocessing effort. The rapid increase in computer time as the mesh is refined should be noted, as discussed in the “Curse of Dimensionality” section of a later Chapter. To keep these within reasonable limits it may be necessary to use coarser meshes than with two dimensional models. This in turn can degrade accuracy, negating the advantage of directness. Thus finite element users should not automatically look upon solid elements as snake oil. Several tradeoffs are at work.

In summary: use of solid elements should be restricted to problem and analysis stages, such as verification, where the generality and flexibility of full 3D models is warranted. They should be avoided during design stages. Furthermore they should also be avoided in thin-wall structures such as aerospace shells, since solid elements tend to perform poorly because of locking problems.

The present Chapter describes general attributes of solid elements for linear elastostatic problems.

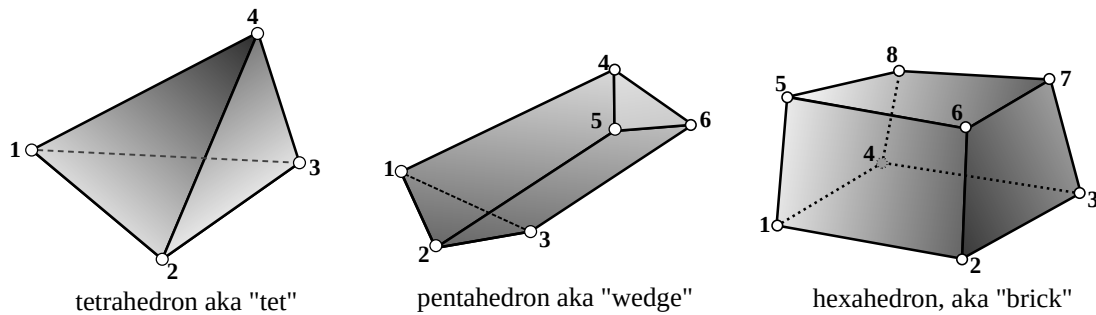


FIGURE 15.2. The three standard solid element geometries: tetrahedron (left), wedge (center) and brick (right). Only elements with corner nodes are shown.

§15.2. Geometrical Configurations

Two dimensional (2D) finite elements have two standard geometries: quadrilateral and triangle. All other geometric configurations, such as polygons with five or more sides, are classified as nonstandard or special.

Three dimensional (3D) finite elements offer more variety. There are three *standard geometries*: the tetrahedron, the wedge, and the hexahedron or “brick”. These have 4, 6 and 8 corners, respectively, with *three* faces meeting at each corner. See Figure 15.2. These elements can be used to build topologically regular meshes as illustrated in Figure 15.3.

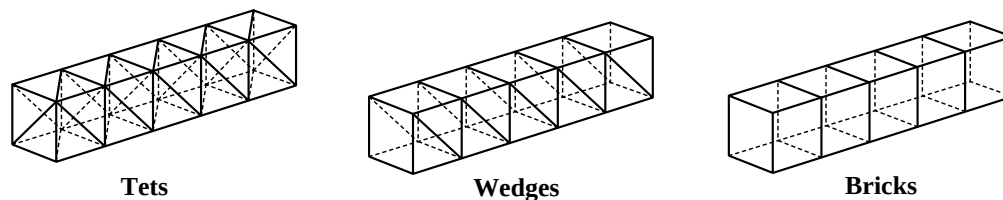


FIGURE 15.3. Regular 3D meshes can be built with cube-like repeating mesh units built with bricks, wedges or tetrahedra.

There are two *nonstandard geometries* that deserve consideration as they are occasionally useful to complete generated 3D meshes: the pyramid and the wrick. (The latter term is a contraction of “wedge” and “brick”) These have 5 and 7 corners, respectively. See Figure 15.4. One of the corners (the last numbered one in the elements pictured in that figure) is special in that *four* faces meet, which leads to a singular metric there. This singularity disqualifies those elements for use in stress analysis in highly stressed regions. However they may be acceptable away from such regions, and in vibration analysis.

Both standard and nonstandard elements can be refined with additional nodes. For example, Figure 15.4 shows the elements of Figures 15.1 and 15.3 equipped with midside nodes. These refined elements are of interest for more accurate stress analysis. Of course, the midside nodes may be moved away from the midpoints to fit curved geometries better.

Tables 15.1 and 15.2 summarize various properties of solid elements. These are discussed at length in this and ensuing chapters.

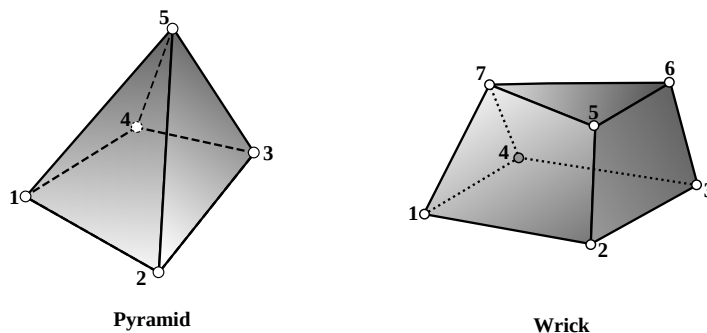


FIGURE 15.4. Two nonstandard solid element geometries: pyramid and wrick [w(edge)+(b)rick]. Four faces meet at corners 5 and 7, leading to a singular metric.

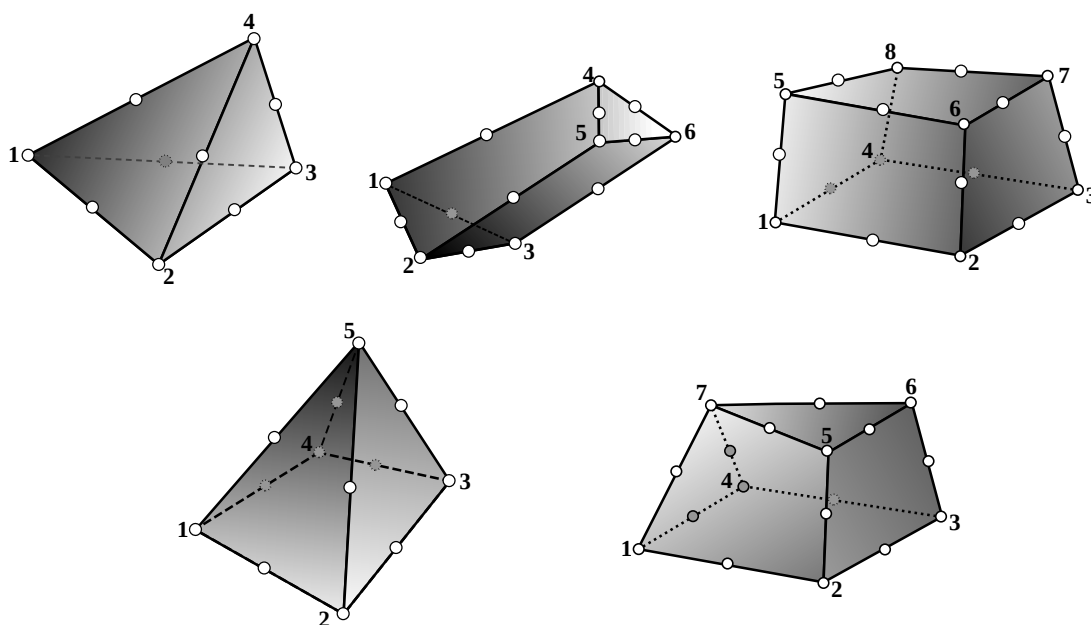


FIGURE 15.5. Solid elements refined with midside nodes. Although shown at midpoints for simplicity, in practice they can be placed away from such locations to fit curved geometries better.

§15.3. Isoparametric Definition

In this Chapter we will restrict consideration here to *isoparametric* solid elements with three translational degrees of freedom (DOF) per node. Much of the development of such elements can be carried out assuming an *arbitrary number of nodes* n . In fact a general “template module” can be written to form the element stiffness matrix and mass matrix.

Nodal quantities will be identified by the node subscript. Thus $\{x_i, y_i, z_i\}$ denote the node coordinates of the i^{th} node, while $\{u_{xi}, u_{yi}, u_{zi}\}$ are the nodal displacement DOFs. The shape function for the i^{th} node is denoted by N_i . These are expressed in term of natural coordinates which vary according to the type of element, as listed in table 15.2. Throughout this Chapter, however, the shape functions are left in generic form.

Table 15.1. Summary of Properties of Solid Elements

Region	Acronym *	C+E+F [†]	Global coords	Restrictions
Tetrahedron	Tetr	4+6+4	x, y, z	None
Wedge	Wedge	6+9+5	x, y, z	None
Pyramid	Pyra	5+8+5	x, y, z	None
Wrick	Wric	7+11+6	x, y, z	None
Hexahedron	Hexa	8+12+6	x, y, z	None

* Acronym may be followed by a node count, e.g. Tetr10 means a 10 node tetrahedron.
[†] C: corners, E: edges, F: faces.

Table 15.2. Natural Coordinates and Isoparametric Geometry Definition

Region	Natural coordinates*	Range of natural coordinates	Iso-P geometry definition in terms of n geometric nodes and shape functions N_i
Tetrahedron	$\zeta_1, \zeta_2, \zeta_3, \zeta_4$	$[0, 1]$	$\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ z_1 & z_2 & \dots & z_n \end{bmatrix} \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix}$
Wedge	$\zeta_1, \zeta_2, \zeta_3, \xi$	$\zeta_i: [0, 1], \xi: [-1, 1]$	same as tetrahedron
Pyramid	ξ, η, μ	$[-1, 1]$	same as hexahedron
Wrick	ξ, η, μ	$[-1, 1]$	same as hexahedron
Hexahedron	ξ, η, μ	$[-1, 1]$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ z_1 & z_2 & \dots & z_n \end{bmatrix} \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix}$

*NC constraints: $\zeta_1 + \zeta_2 + \zeta_3 = 1$ for wedges, $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 1$ for tetrahedra.

§15.3.1. Geometry Description

Following the isoparametric notation introduced by Felippa and Clough¹ the element geometry is described by

$$\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ z_1 & z_2 & \dots & z_n \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}. \quad (15.1)$$

¹ C. A. Felippa and R. W. Clough, The Finite Element Method in Solid Mechanics, in *Numerical Solution of Field Problems in Continuum Physics*, ed. by G. Birkhoff and R. S. Varga, SIAM–AMS Proceedings II, American Mathematical Society, Providence, R.I., 1969, pp. 210–252

The four rows of this matrix relation express the *geometric* conditions

$$\boxed{1 = \sum_{i=1}^n N_i, \quad x = \sum_{i=1}^n x_i N_i, \quad y = \sum_{i=1}^n y_i N_i, \quad z = \sum_{i=1}^n z_i N_i,} \quad (15.2)$$

which the shape functions must identically satisfy. The first one: sum of shape functions must be unity, is an essential part of the verification of *completeness*.

§15.3.2. Displacement Interpolation

The displacement interpolation is

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & \cdots & u_{xn} \\ u_{y1} & u_{y2} & \cdots & u_{yn} \\ u_{z1} & u_{z2} & \cdots & u_{zn} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}. \quad (15.3)$$

The three rows of this matrix relation express the *interpolation* conditions

$$\boxed{u_x = \sum_{i=1}^n u_{xi} N_i, \quad u_y = \sum_{i=1}^n u_{yi} N_i, \quad u_z = \sum_{i=1}^n u_{zi} N_i.} \quad (15.4)$$

The identical structure of the geometry definition (15.1) and displacement interpolation (15.3) characterizes an isoparametric element (iso=same).

For future development the element node displacements of (15.3) are collected in the $3n$ column vector $\mathbf{u}$ which is configured with the 3 components grouped node by node:

$$\mathbf{u}^T = [u_{x1} \quad u_{y1} \quad u_{z1} \quad u_{x2} \quad \cdots \quad u_{xn} \quad u_{yn} \quad u_{zn}] \quad (15.5)$$

§15.4. The Stiffness Matrix

For convenience in the stiffness matrix formulation the six components of the strains and stresses are grouped as follows to form into 6-component vectors:

$$\mathbf{e} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}. \quad (15.6)$$

§15.4.1. The Strain-Displacement Equations

The strains are related to the element node displacements by the $6 \times 3n$ strain-displacement matrix:

$$\mathbf{e} = \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_z}{\partial z} \\ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \\ \vdots \end{bmatrix} = \begin{bmatrix} N_{x1} & 0 & 0 & \dots & N_{xn} & 0 & 0 \\ 0 & N_{y1} & 0 & \dots & 0 & N_{yn} & 0 \\ 0 & 0 & N_{z1} & \dots & 0 & 0 & N_{zn} \\ N_{y1} & N_{x1} & 0 & \dots & N_{yn} & N_{xn} & 0 \\ 0 & N_{z1} & N_{y1} & \dots & 0 & N_{yn} & N_{xn} \\ N_{z1} & 0 & N_{x1} & \dots & N_{zn} & 0 & N_{xn} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{z1} \\ \vdots \\ u_{xn} \\ u_{yn} \\ u_{zn} \end{bmatrix} = \mathbf{B}\mathbf{u}, \quad (15.7)$$

in which N_{xi} , N_{yi} and N_{zi} denote the derivatives of shape function N_i with respect to x , y and z , respectively.

§15.4.2. The Constitutive Equations

We restrict attention to linear elastostatic analysis without initial stresses. For a general anisotropic material the stress-strain equations can be presented as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ \text{symm} & & & & & E_{66} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix} = \mathbf{E}\mathbf{e}. \quad (15.8)$$

The stress-strain matrix $\mathbf{E}$ will be assumed to be constant over the element. This matrix can be significantly simplified in the case of isotropic or orthotropic material. However for the actual element implementation those simplifications are not considered here.

§15.4.3. The Stiffness Matrix

The element stiffness matrix is formally given by the volume integral

$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^T \mathbf{E} \mathbf{B} dV \quad (15.9)$$

where the integral is taken over the element volume.

The usual treatment of (15.9) consists of evaluating the integral by numerical Gauss quadrature with n_G points:

$$\mathbf{K}^e = \sum_{k=1}^{n_G} w_k J_k \mathbf{B}_k^T \mathbf{E} \mathbf{B}_k \quad (15.10)$$

Here k is the integration point index; w_k is the integration weight, $\mathbf{B}_k$ is the stress-displacement matrix and J_k is the Jacobian determinant introduced later, respectively, evaluated at the integration point. Three dimensional integration rules are discussed in a later chapter.

Remark 15.1. The fast computation of the matrix product $\mathbf{B}^T \mathbf{E} \mathbf{B}$ is important in the efficient implementation of elements, because this evaluation is repeated at each integration point. For each node point pair i, j define

$$\mathbf{B}_i = \begin{bmatrix} N_{xi} & 0 & 0 \\ 0 & N_{yi} & 0 \\ 0 & 0 & N_{zi} \\ N_{yi} & N_{xi} & 0 \\ 0 & N_{zi} & N_{yi} \\ N_{zi} & 0 & N_{xi} \end{bmatrix}, \quad \mathbf{B}_j = \begin{bmatrix} N_{xj} & 0 & 0 \\ 0 & N_{yj} & 0 \\ 0 & 0 & N_{zj} \\ N_{yj} & N_{xj} & 0 \\ 0 & N_{zj} & N_{yi} \\ N_{zj} & 0 & N_{xj} \end{bmatrix}, \quad i, j = 1, \dots, n. \quad (15.11)$$

Then the 3×3 block $\mathbf{Q}_{ji} = \mathbf{B}_j^T \mathbf{E} \mathbf{B}_i = \mathbf{Q}_{ij}^T$ is efficiently computed as

$$\mathbf{C} = \begin{bmatrix} B_{xi} E_{11} + B_{yi} E_{14} + B_{zi} E_{16} & B_{yi} E_{12} + B_{xi} E_{14} + B_{zi} E_{15} & B_{zi} E_{13} + B_{yi} E_{15} + B_{xi} E_{16} \\ B_{xi} E_{12} + B_{yi} E_{24} + B_{zi} E_{26} & B_{yi} E_{22} + B_{xi} E_{24} + B_{zi} E_{25} & B_{zi} E_{23} + B_{yi} E_{25} + B_{xi} E_{26} \\ B_{xi} E_{13} + B_{yi} E_{34} + B_{zi} E_{36} & B_{yi} E_{23} + B_{xi} E_{34} + B_{zi} E_{35} & B_{zi} E_{33} + B_{yi} E_{35} + B_{xi} E_{36} \\ B_{xi} E_{14} + B_{yi} E_{44} + B_{zi} E_{46} & B_{yi} E_{24} + B_{xi} E_{44} + B_{zi} E_{45} & B_{zi} E_{34} + B_{yi} E_{45} + B_{xi} E_{46} \\ B_{xi} E_{15} + B_{yi} E_{45} + B_{zi} E_{56} & B_{yi} E_{25} + B_{xi} E_{45} + B_{zi} E_{55} & B_{zi} E_{35} + B_{yi} E_{55} + B_{xi} E_{56} \\ B_{xi} E_{16} + B_{yi} E_{46} + B_{zi} E_{66} & B_{yi} E_{26} + B_{xi} E_{46} + B_{zi} E_{56} & B_{zi} E_{36} + B_{yi} E_{56} + B_{xi} E_{66} \end{bmatrix} \quad (15.12)$$

$$\mathbf{Q}_{ij} = \begin{bmatrix} B_{xj} C_{11} + B_{yj} C_{41} + B_{zj} C_{61} & B_{xj} C_{12} + B_{yj} C_{42} + B_{zj} C_{62} & B_{xj} C_{13} + B_{yj} C_{43} + B_{zj} C_{63} \\ B_{yj} C_{21} + B_{xi} C_{41} + B_{zj} C_{51} & B_{yj} C_{22} + B_{xi} C_{42} + B_{zj} C_{52} & B_{yj} C_{23} + B_{xi} C_{43} + B_{zj} C_{53} \\ B_{zj} C_{31} + B_{yj} C_{51} + B_{xj} C_{61} & B_{zj} C_{32} + B_{yj} C_{52} + B_{xj} C_{62} & B_{zj} C_{33} + B_{yj} C_{53} + B_{xj} C_{63} \end{bmatrix}$$

The computation of each $\mathbf{Q}_{ij}$ block requires $54 + 27 = 81$ multiplications. (The slight savings for $i = j$ are not worth the coding complications.) For an element with n nodes and n_G Gauss integration points, $n_G n(n+1)/2$ multiplications are required since $\mathbf{Q}_{ji} = \mathbf{Q}_{ij}^T$, and the total effort associated with $\mathbf{B}^T \mathbf{E} \mathbf{B}$ is approximately $40n_G n^2$ multiplications. This computation, plus associated indexing to access the $\mathbf{K}$ entries, dominates the total effort. As an example: an 8-node brick integrated with a $2 \times 2 \times 2$ rule would require roughly 20,000 multiplications whereas a 20-node brick integrated by a $3 \times 3 \times 3$ rule would consume 432,000; thus the formation time ratio would be roughly 22:1.

For an isotropic material two thirds of the stress-strain coefficients in $\mathbf{E}$ are zero. Explicit recognition of that fact would cut the computation of $\mathbf{Q}_{ij}$ to $15 + 21 = 36$ multiplications, roughly a 2:1 speedup. This would complicate the program logic, however, because branching to this special case would be needed, and it is not clear whether the complication is likely to be worth the special effort. In any event, the “unrolled” evaluation of the $\mathbf{Q}_{ij}$ should be preferred to general matrix multiplication.

§15.5. The Mass Matrix

For deriving the consistent mass matrix the iso-P displacement interpolation (15.4) is rearranged as

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & \dots & N_n & 0 & 0 \\ 0 & N_1 & 0 & \dots & 0 & N_n & 0 \\ 0 & 0 & N_1 & \dots & 0 & 0 & N_n \end{bmatrix} \mathbf{u} = \mathbf{N} \mathbf{u}. \quad (15.13)$$

Using either the Lagrange dynamical equations or Hamilton’s variational principle, the consistent mass matrix is given by

$$\mathbf{M}_C^e = \int_V \rho \mathbf{N} \mathbf{N}^T dV. \quad (15.14)$$

where ρ is the mass density of the material. If ρ is constant over the element it may be taken out of the foregoing volume integral.

As in the case of the stiffness matrix, the expression (15.14) is usually evaluated by a Gauss quadrature rule. The sparsity of the shape function matrix $\mathbf{N}$ is easily accounted for when forming the 3×3 i, j blocks of $\mathbf{M}_C^e$.

Transformation of $\mathbf{M}_C^e$ to a lumped mass matrix $\mathbf{M}_L^e$ can be done through a variety of techniques. A good coverage is given in the book by Cook et al.².

² R. D. Cook, D. S. Malkus and M. E. Plesha, *Concepts and Application of Finite Element Methods*, 3rd ed., Wiley, New York, 1989

Homework Exercises for Chapter 15**Solid Elements: Overview**

EXERCISE 15.1 [A/C:10] Tabulate the rough number of multiplications needed to form $\mathbf{K}^e$ for elements with 4, 6, 8, 10, 15, 20 and 27 and 64 nodes, for integration rules containing 1, 4, 6, 8, 15, 27 and 64 Gauss points. (Use the formula found in Remark 15.1.) If the computer can do 10^9 multiplications per second, tabulate how many elements can be formed in one second of CPU time.

EXERCISE 15.2 [A/C:20] Find the approximate cost in multiplications to form the consistent mass matrix (15.14) of a solid element with n nodes if n_G Gauss points are used. Compare to the approximate count $40n_G n(n+1)$ to form the stiffness matrix $\mathbf{K}^e$ if the number of Gauss points n_G is the same.

16

The Linear Tetrahedron

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§16.1. Introduction

In this Chapter we study the construction of shape functions for three-dimensional solid elements, beginning with the 4-node tetrahedron. We start with this particular element for two reasons: the geometry is the simplest one, and no numerical integration is needed.

§16.2. The Linear Tetrahedron

The linear tetrahedron, shown in Figure 16.1(a), is not used often for stress analysis because of its poor performance.¹ Its main value in structural and solid mechanics is educational: it serves as a vehicle to introduce the basic steps of formulation of 3D solid elements, particularly as regards use of natural coordinate systems and node numbering conventions. It should be noted that 3D visualization is notoriously more difficult than 2D, so we need to proceed somewhat slowly here.

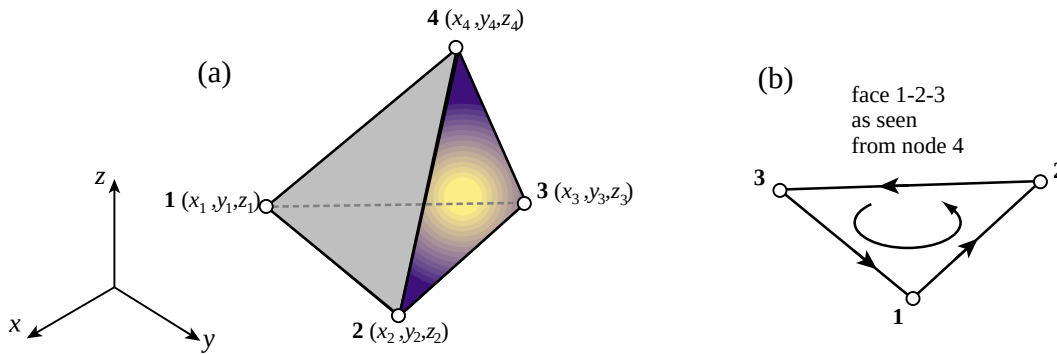


FIGURE 16.1. (a) The linear tetrahedron element: also called the 4-node tetrahedron; (b) Node numbering convention.

§16.2.1. Tetrahedron Geometry

Figure 16.1 shows a typical 4-node tetrahedron. Its geometry is fully defined by giving the location of the four corner nodes with respect to the global RCC system (x, y, z) :

$$x_i, y_i, z_i \quad (i = 1, 2, 3, 4). \quad (16.1)$$

The volume measure of the tetrahedron is denoted² by $\mathcal{V}$ and is given by the following determinant:

$$\mathcal{V} = \frac{1}{6} \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}. \quad (16.2)$$

¹ Derivative of shape functions are constant over the element volume. Strains and stresses recovered in this manner can be highly inaccurate. This makes the element dangerous for stress analysis. On the other hand, when the objective is merely to get values of primary variables, as in thermal analysis and computational gas dynamics, the linear tetrahedron is acceptable.

² This symbol (Upsilon) is used to avoid confusion with V , which denotes the volume of a generic body.

This volume is a *signed* quantity. It is positive if the corners are numbered in such a way that the volume is positive. A numbering rule that achieves this goal is as follows:

- (I) Pick a corner as initial one. In Figure 16.1(a) this is numbered 1.
- (II) Pick a face that will contain the first three corners. The excluded corner will be the last one.
- (III) Number these three corners in a *counterclockwise* sense when looking at the face from the excluded corner. See Figure 16.1(b).

In what follows we shall always assume that the numbering has been done in that manner so that $\mathcal{V} > 0$.³

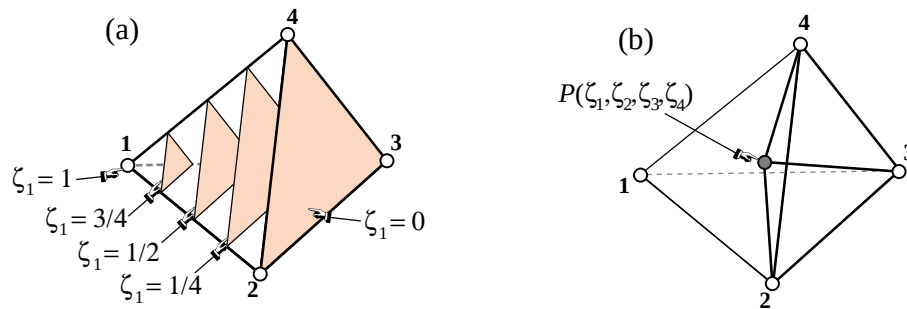


FIGURE 16.2. Tetrahedron natural coordinates: $\zeta_1, \zeta_2, \zeta_3, \zeta_4$.

§16.2.2. Tetrahedral Coordinates

The set of tetrahedral coordinates $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ is the three-dimensional analog of the triangular coordinate set discussed in Chapter 15 of IFEM. The value of ζ_i is one at corner i , zero at the other 3 corners (*i.e.* on the opposite face) and varies linearly as one traverses the distance from the corner to the face. The sum of the four coordinates is identically one:

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 1. \quad (16.3)$$

Any function *linear* in x, y, z , say $F(x, y, z)$, that takes the values F_i ($i = 1, 2, 3, 4$) at the corners may be interpolated in terms of the tetrahedron coordinates as

$$F(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = F_1\zeta_1 + F_2\zeta_2 + F_3\zeta_3 + F_4\zeta_4 = F_i\zeta_i. \quad (16.4)$$

Example 16.1. Suppose that $F(x, y, z) = 4x + 9y - 8z + 3$ and that the coordinates of corners 1,2,3,4 are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. The values of F at the corners are $F_1 = 3$, $F_2 = 7$, $F_3 = 12$ and $F_4 = -5$. Consequently $F(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = 3\zeta_1 + 7\zeta_2 + 12\zeta_3 - 5\zeta_4$.

³ The tetrahedron volume can be zero only if the four corners are coplanar. This case will be excluded.

§16.2.3. Coordinate Transformations

The geometric definition of the element in terms of these coordinates is obtained by applying the geometry definition (16.4) to x , y and z , and appending the sum-of-coordinates constraint (16.3):

$$\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix}. \quad (16.5)$$

Inverting this relation gives

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix} = \frac{1}{6\mathcal{V}} \begin{bmatrix} 6\mathcal{V}_1 & a_1 & b_1 & c_1 \\ 6\mathcal{V}_2 & a_2 & b_2 & c_2 \\ 6\mathcal{V}_3 & a_3 & b_3 & c_3 \\ 6\mathcal{V}_4 & a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}, \quad (16.6)$$

where the coefficients of this matrix can be calculated by forming the adjoints of the matrix in (16.5).

Remark 16.1. The values of a_i , b_i and c_i obtained by explicit inversion are

$$\begin{aligned} a_1 &= y_2z_{43} - y_3z_{42} + y_4z_{32}, & b_1 &= -x_2z_{43} + x_3z_{42} - x_4z_{32}, & c_1 &= x_2y_{43} - x_3y_{42} + x_4y_{32}, \\ a_2 &= -y_1z_{43} + y_3z_{41} - y_4z_{31}, & b_2 &= x_1z_{43} - x_3z_{41} + x_4z_{31}, & c_2 &= -x_1y_{43} + x_3y_{41} - x_4y_{31}, \\ a_3 &= y_1z_{42} - y_2z_{41} + y_4z_{21}, & b_3 &= -x_1z_{42} + x_2z_{41} - x_4z_{21}, & c_3 &= x_1y_{42} - x_2y_{41} + x_4y_{21}, \\ a_4 &= -y_1z_{32} + y_2z_{31} - y_3z_{21}, & b_4 &= x_1z_{32} - x_2z_{31} + x_3z_{21}, & c_4 &= -x_1y_{32} + x_2y_{31} - x_3y_{21}. \end{aligned} \quad (16.7)$$

in which the abbreviations $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$ and $z_{ij} = z_i - z_j$ are used. The volume is given explicitly by

$$6\mathcal{V} = x_{21}(y_{31}z_{41} - y_{41}z_{31}) + y_{21}(x_{41}z_{31} - x_{31}z_{41}) + z_{21}(x_{31}y_{41} - x_{41}y_{31}). \quad (16.8)$$

The values of $\mathcal{V}_i$ are of no interest in what follows.

§16.2.4. *Geometric Interpretation

Figure 16.2 illustrates two geometric interpretation of coordinate ζ_1 . In Figure 16.2(a), $\zeta_1 = C$, where C is a number between 0 and 1, is the equation of a plane parallel to the face 234. The plane coincides with that face if $\zeta_1 = 0$, it passes through corner node 1 if $\zeta_1 = 1$, and is interpolated linearly in between.

Figure 16.2(b) illustrates another interpretation that appears in many FEM books. Consider a point P of coordinates $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ inside the tetrahedron. Joining P to the corners we obtain four sub-tetrahedra 234 P , 341 P , 412 P and 123 P , whose volumes are $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$, respectively. Then ζ_i is the ratio $\mathcal{V}_i/\mathcal{V}$. Figure 16.2(b) pictures the sub-tetrahedron 234 P of volume $\mathcal{V}_1$. On account of this relation, tetrahedral coordinates are also called volume coordinates.

Remark 16.2. The interpretation as volume coordinates only holds for the tetrahedron defined by 4 corner nodes. It fails for higher order tetrahedra defined by additional nodes (e.g., midpoints). For this reason, the second interpretation, as well as the name “volume coordinates,” will not be used here.

§16.2.5. Partial Derivatives

From equations (16.5) and (16.6) we can easily find the following relations for the partial derivatives of Cartesian and tetrahedral coordinates

$$\frac{\partial x}{\partial \zeta_i} = x_i, \quad \frac{\partial y}{\partial \zeta_i} = y_i, \quad \frac{\partial z}{\partial \zeta_i} = z_i. \quad (16.9)$$

$$6V \frac{\partial \zeta_i}{\partial x} = a_i, \quad 6V \frac{\partial \zeta_i}{\partial y} = b_i, \quad 6V \frac{\partial \zeta_i}{\partial z} = c_i. \quad (16.10)$$

The derivatives of a function $F(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ with respect to the Cartesian coordinates follows from (16.10) and the chain rule:

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial \zeta_i} \frac{\partial \zeta_i}{\partial x} = \frac{1}{6V} \left(\frac{\partial F}{\partial \zeta_1} a_1 + \frac{\partial F}{\partial \zeta_2} a_2 + \frac{\partial F}{\partial \zeta_3} a_3 + \frac{\partial F}{\partial \zeta_4} a_4 \right) = \frac{1}{6V} \frac{\partial F}{\partial \zeta_i} a_i. \\ \frac{\partial F}{\partial y} &= \frac{\partial F}{\partial \zeta_i} \frac{\partial \zeta_i}{\partial y} = \frac{1}{6V} \left(\frac{\partial F}{\partial \zeta_1} b_1 + \frac{\partial F}{\partial \zeta_2} b_2 + \frac{\partial F}{\partial \zeta_3} b_3 + \frac{\partial F}{\partial \zeta_4} b_4 \right) = \frac{1}{6V} \frac{\partial F}{\partial \zeta_i} b_i. \\ \frac{\partial F}{\partial z} &= \frac{\partial F}{\partial \zeta_i} \frac{\partial \zeta_i}{\partial z} = \frac{1}{6V} \left(\frac{\partial F}{\partial \zeta_1} c_1 + \frac{\partial F}{\partial \zeta_2} c_2 + \frac{\partial F}{\partial \zeta_3} c_3 + \frac{\partial F}{\partial \zeta_4} c_4 \right) = \frac{1}{6V} \frac{\partial F}{\partial \zeta_i} c_i. \end{aligned} \quad (16.11)$$

§16.3. The Linear Tetrahedron

The simplest tetrahedron finite element for problems of variational order $m = 1$ is the four-node tetrahedron with *linear shape functions*. The shape functions are simply the tetrahedral coordinates: $N_i = \zeta_i, i = 1, 2, 3, 4$. This finite element is derived now for the elasticity problem, using the Total Potential Energy principle as source variational form.

§16.3.1. Displacement Interpolation

The displacement field over the tetrahedron is defined by the three components u_x, u_y and u_z . These are linearly interpolated over the element from their nodal values

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix}. \quad (16.12)$$

Putting this together with the geometric definition (16.4) we have the isoparametric definition of the 4-node tetrahedron as an elasticity element:

$$\begin{bmatrix} 1 \\ x \\ y \\ z \\ u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix}. \quad (16.13)$$

§16.3.2. The Strain Field

The strain field within the element is strongly connected to the displacement by the strain-displacement equations, which in indicial notation read

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \quad (16.14)$$

We transliterate this to matrix notation as follows. First, the six independent components of the stress tensor are arranged into a 6-component strain vector as follows:

$$\begin{aligned} \mathbf{e} &= [e_{11} \quad e_{22} \quad e_{33} \quad 2e_{12} \quad 2e_{23} \quad 2e_{31}]^T \\ &= [e_{xx} \quad e_{yy} \quad e_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zy}]^T. \end{aligned} \quad (16.15)$$

The second expression shows the engineering notation for the shear strains. Second, displacement components u_1 , u_2 and u_3 are rewritten as u_x , u_y and u_z , collected into a vector and linked to the displacement field by (16.14):

$$\mathbf{e} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{D} \vec{\mathbf{u}}. \quad (16.16)$$

Combining this with (16.12) and using the differentiation rules (16.11) we obtain the matrix relation between strains and nodal displacements:

$$\mathbf{e} = \mathbf{B} \mathbf{u}^e. \quad (16.17)$$

If the element nodal displacement vector is arranged component-wise:

$$\mathbf{u}^e = [u_{x1} \quad u_{x2} \quad u_{x3} \quad u_{x4} \quad u_{y1} \quad u_{y2} \quad \cdots \quad u_{z4}]^T, \quad (16.18)$$

the matrix $\mathbf{B}$ has the following configuration

$$\mathbf{B} = \frac{1}{6V} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & c_2 & c_3 & c_4 & b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 & 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 \end{bmatrix}. \quad (16.19)$$

If the node displacements are arranged node-wise:

$$\mathbf{u}^e = [u_{x1} \quad u_{y1} \quad u_{z1} \quad u_{x2} \quad u_{y2} \quad u_{z2} \quad \cdots \quad u_{z4}]^T, \quad (16.20)$$

the columns of $\mathbf{B}$ must be re-shuffled to yield

$$\mathbf{B} = \frac{1}{6V} \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 \\ 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 \\ 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 \\ b_1 & a_1 & 0 & b_2 & a_2 & 0 & b_3 & a_3 & 0 & b_4 & a_4 & 0 \\ 0 & c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 \\ c_1 & 0 & a_1 & c_2 & 0 & a_2 & c_3 & 0 & a_3 & c_4 & 0 & a_4 \end{bmatrix}. \quad (16.21)$$

The node-wise arrangement (16.20) of $\mathbf{u}^e$ is more common in practice because it facilitates the assembly process.

Note that both matrices (16.19) and (16.21) are constant over the element.

§16.3.3. The Stress Field

The stress field is related to the stress field by the strong connection

$$\sigma_{ij} = E_{ijkl} e_{kl} \quad (16.22)$$

To convert this to matrix notation we rearrange the 6 independent stress components to correspond to the strains (16.12) and link them by a 6×6 matrix of elastic moduli:

$$\begin{aligned} \boldsymbol{\sigma} &= [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{31}]^T = \\ &= [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx}]^T \end{aligned} \quad (16.23)$$

If the material is linearly elastic and no initial strains are considered, the constitutive equation may be compactly expressed as

$$\boldsymbol{\sigma} = \mathbf{E} \mathbf{e}. \quad (16.24)$$

where the elasticity matrix $\mathbf{E}$ is symmetric. For a general anisotropic material the expanded form of (16.24) is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ \text{symm} & & & & & E_{66} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix}, \quad (16.25)$$

in which E_{ij} are constitutive moduli. For an isotropic material of elastic modulus E and Poisson's ratio ν the foregoing relation simplifies to

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu \end{bmatrix}. \quad (16.26)$$

§16.4. The Element Stiffness Matrix

Introducing $\mathbf{e} = \mathbf{B}\mathbf{u}$ and $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$ into the strain energy functional restricted to the element volume and rendering the resulting algebraic form stationary with respect to the node displacements $\mathbf{u}^e$ we get the usual expression for the element stiffness matrix

$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^T \mathbf{E} \mathbf{B} dV. \quad (16.27)$$

Assuming that the elastic moduli are constant inside the element, the foregoing integrand is constant because matrix $\mathbf{B}$ is constant — cf. (16.19) or (16.21). Consequently

$$\mathbf{K}^e = \mathcal{V} \mathbf{B}^T \mathbf{E} \mathbf{B}. \quad (16.28)$$

This stiffness matrix is 12×12 . It can be directly evaluated in closed form using the above expression or, equivalently, by a one-point (centroid) integration rule.

§16.5. The Consistent Node Force Vector

A tetrahedral mesh may be subjected to given body forces in the volume and/or specified boundary tractions. Both have to be converted to node forces through an energy-based lumping procedure.

§16.5.1. Body Forces

Consider a body force field over the element, such as gravity or centrifugal forces, defined by its components

$$\mathbf{b} = [b_x \quad b_y \quad b_z]^T. \quad (16.29)$$

Inserting this into the TPE principle, the body force contribution gives

$$\mathbf{f}^e = \int_{V^e} \mathbf{N}^T \mathbf{b} dV. \quad (16.30)$$

Here $\mathbf{N}$ is the 3×12 matrix of shape functions that relates element field displacements to node displacements:

$$\vec{\mathbf{u}} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{N} \mathbf{u}^e. \quad (16.31)$$

For the component-wise node displacement ordering (16.18),

$$\mathbf{N} = \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{bmatrix} \quad (16.32)$$

For the node-wise displacement ordering (16.20),

$$\mathbf{N} = \begin{bmatrix} \zeta_1 & 0 & 0 & \zeta_2 & 0 & 0 & \zeta_3 & 0 & 0 & \zeta_4 & 0 & 0 \\ 0 & \zeta_1 & 0 & 0 & \zeta_2 & 0 & 0 & \zeta_3 & 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & \zeta_1 & 0 & 0 & \zeta_2 & 0 & 0 & \zeta_3 & 0 & 0 & \zeta_4 \end{bmatrix} \quad (16.33)$$

Even if the body forces are constant the integral is not constant over the element. Some useful formulae for such calculations are

$$\int_{V^e} \zeta_i dV = \frac{1}{4} \mathcal{V}, \quad (16.34)$$

and

$$\int_{V^e} \zeta_i \zeta_j dV = \begin{cases} \frac{1}{10} \mathcal{V} & \text{if } i = j, \\ \frac{1}{20} \mathcal{V} & \text{if } i \neq j. \end{cases} \quad (16.35)$$

The general rule for such integrals, which can be derived from the Beta function, is

$$\int_{V^e} \zeta_1^i \zeta_2^j \zeta_3^k \zeta_4^\ell dV = \frac{i! j! k! \ell!}{(i + j + k + \ell + 3)!} 6\mathcal{V}. \quad (16.36)$$

in which i, j, k and ℓ are nonnegative integers. This formula is only valid for tetrahedra with planar faces.

§16.5.2. Surface Traction

The most practically important case is that of surface tractions normal to an element face. This models the effect of pressure loads. The calculation of node forces for the case of a constant pressure acting on a tetrahedron face is the matter of one exercise.

§16.5.3. Element Implementation

The implementation of the linear tetrahedron in any programming language is very simple. An implementation in the form of a *Mathematica* module is shown in Figure 16.3. The module is invoked as

$$\text{Ke} = \text{Trig3IsoPMembraneStiffness}[\text{ncoor}, \text{Emat}, \{ \}, \text{options}]; \quad (16.37)$$

The arguments are

<code>ncoor</code>	Element node coordinates, arranged as a list: $\{\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}, \{x_3, y_3, z_3\}, \{x_4, y_4, z_4\}\}.$
<code>Emat</code>	A two-dimensional list storing the 6×6 matrix of elastic moduli as $\{\{E_{11}, E_{12}, E_{13}, E_{14}, E_{15}, E_{16}\}, \dots, \{E_{61}, E_{62}, E_{63}, E_{64}, E_{65}, E_{66}\}\}.$
<code>options</code>	A list of formation options. For this element it is simply $\{\text{numer}\}$, where <i>numer</i> is a logical flag. Flag is <i>True</i> to request floating point numeric work, <i>False</i> to request exact calculations.

The third argument is a placeholder and should be set to the empty list $\{ \}$.

The stiffness module calls module `IsoTetr4ShapeFunCarDer`, which is listed in Figure 16.4, to get the shape function Cartesian partial derivatives. These return in the 12×1 arrays *Bx*, *By* and *Bz*. The module also returns the jacobian determinant *Jdet*, which is six times the element volume.

Module `IsoTetr4ShapeFunCarDer` is written in a more complicated style than needed for this particular element. For example *J11* is simply *x1*, etc. It is actually configured to serve as a shape function derivative “template” for more refined tetrahedron elements, as described in the next Chapter. *r*

```

IsoTetr4Stiffness[ncoor_, Emat_, {}, options_] := Module[{i, n=4, nf=12,
  k, c, w, Jdet, zetalist, xyzlist, numer, Bx, By, Bz, Be, Ke},
  If [Length[options]>0, numer=options[[1]]];
  xyzlist={Table[ncoor[[i,1]],{i,n}],Table[ncoor[[i,2]],{i,n}],
    Table[ncoor[[i,3]],{i,n}]]; Ke=Table[0,{nf},{nf}];
  {Bx,By,Bz,Jdet}=IsoTetr4ShapeFunCarDer[xyzlist,{},numer];
  Be={Flatten[Table[{Bx[[i]],0,0},{i,n}]],
    Flatten[Table[{0,By[[i]],0},{i,n}]],
    Flatten[Table[{0,0,Bz[[i]]},{i,n}]],
    Flatten[Table[{By[[i]],Bx[[i]],0},{i,n}]],
    Flatten[Table[{0,Bz[[i]],By[[i]]},{i,n}]],
    Flatten[Table[{Bz[[i]],0,Bx[[i]]},{i,n}]]];
  Ke=(Jdet/6)*Transpose[Be].(Emat.Be);
  If [!numer,Ke=Simplify[Ke]]; Return[Ke]
];

```

FIGURE 16.3. Module to form the stiffness matrix of a linear tetrahedron (Tetr4) and outputs.

```

IsoTetr4ShapeFunCarDer[{xn_, yn_, zn_}, zetalist_, numer_] :=
Module[{dNz1, dNz2, dNz3, dNz4, Jmat, J11, J12, J13, J14,
  J21, J22, J23, J24, J31, J32, J33, J34, Jinv, Jdet, Bx, By, Bz},
  {dNz1, dNz2, dNz3, dNz4}={{{1,0,0,0},{0,1,0,0},{0,0,1,0},{0,0,0,1}}};
  J11=dNz1.xn; J12=dNz2.xn; J13=dNz3.xn; J14=dNz4.xn;
  J21=dNz1.yn; J22=dNz2.yn; J23=dNz3.yn; J24=dNz4.yn;
  J31=dNz1.zn; J32=dNz2.zn; J33=dNz3.zn; J34=dNz4.zn;
  Jmat={{1,1,1,1},{J11,J12,J13,J14},
    {J21,J22,J23,J24},{J31,J32,J33,J34}};
  Jdet=(J13*J22-J12*J23+J14*J23-J14*J22+J12*J24-J13*J24)*J31-
    (J13*J21-J11*J23+J14*J23-J14*J21+J11*J24-J13*J24)*J32+
    (J12*J21-J11*J22+J14*J22-J14*J21+J11*J24-J12*J24)*J33-
    (J12*J21-J11*J22+J13*J22-J13*J21+J11*J23-J12*J23)*J34;
  Jinv={{J22*(J34-J33)-J23*(J34-J32)+J24*(J33-J32),
    -J12*(J34-J33)+J13*(J34-J32)-J14*(J33-J32),
    J12*(J24-J23)-J13*(J24-J22)+J14*(J23-J22)},
    {-J21*(J34-J33)+J23*(J34-J31)-J24*(J33-J31),
    J11*(J34-J33)-J13*(J34-J31)+J14*(J33-J31),
    -J11*(J24-J23)+J13*(J24-J21)-J14*(J23-J21)},
    {J21*(J34-J32)-J22*(J34-J31)+J24*(J32-J31),
    -J11*(J34-J32)+J12*(J34-J31)-J14*(J32-J31),
    J11*(J24-J22)-J12*(J24-J21)+J14*(J22-J21)},
    {-J21*(J33-J32)+J22*(J33-J31)-J23*(J32-J31),
    J11*(J33-J32)-J12*(J33-J31)+J13*(J32-J31),
    -J11*(J23-J22)+J12*(J23-J21)-J13*(J22-J21)}};
  {Bx,By,Bz}=Transpose[Jinv].{dNz1,dNz2,dNz3,dNz4}/Jdet;
  Return[{Bx,By,Bz,Jdet}]
];

```

FIGURE 16.4. Module to compute shape function partial derivatives for linear tetrahedron (Tetr4). As noted in the text, it is deliberately written in a more general fashion than needed for this particular element.

```

ClearAll[Em,v]; Em=96; v=1/3;
Emat=Em/((1+v)*(1-2*v))*{{1-v,v,v,0,0,0},
  {v,1-v,v,0,0,0},{v,v,1-v,0,0,0},{0,0,0,1/2-v,0,0},
  {0,0,0,1/2-v,0},{0,0,0,0,0,1/2-v}};
Print["Emat=",Emat//MatrixForm];
ncoor={{2,3,4},{6,3,2},{2,5,1},{4,3,6}};
Ke=IsoTetr4Stiffness[ncoor,Emat,{},{False}];
Print["Ke=",Ke//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Ke]]]];

```

FIGURE 16.5. Test statements to exercise the module of Figure 16.3.

$$\text{Emat} = \begin{pmatrix} 144 & 72 & 72 & 0 & 0 & 0 \\ 72 & 144 & 72 & 0 & 0 & 0 \\ 72 & 72 & 144 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\text{Ke} = \begin{pmatrix} 149 & 108 & 24 & -1 & 6 & 12 & -54 & -48 & 0 & -94 & -66 & -36 \\ 108 & 344 & 54 & -24 & 104 & 42 & -24 & -216 & -12 & -60 & -232 & -84 \\ 24 & 54 & 113 & 0 & 30 & 35 & 0 & -24 & -54 & -24 & -60 & -94 \\ -1 & -24 & 0 & 29 & -18 & -12 & -18 & 24 & 0 & -10 & 18 & 12 \\ 6 & 104 & 30 & -18 & 44 & 18 & 12 & -72 & -12 & 0 & -76 & -36 \\ 12 & 42 & 35 & -12 & 18 & 29 & 0 & -24 & -18 & 0 & -36 & -46 \\ -54 & -24 & 0 & -18 & 12 & 0 & 36 & 0 & 0 & 36 & 12 & 0 \\ -48 & -216 & -24 & 24 & -72 & -24 & 0 & 144 & 0 & 24 & 144 & 48 \\ 0 & -12 & -54 & 0 & -12 & -18 & 0 & 0 & 36 & 0 & 24 & 36 \\ -94 & -60 & -24 & -10 & 0 & 0 & 36 & 24 & 0 & 68 & 36 & 24 \\ -66 & -232 & -60 & 18 & -76 & -36 & 12 & 144 & 24 & 36 & 164 & 72 \\ -36 & -84 & -94 & 12 & -36 & -46 & 0 & 48 & 36 & 24 & 72 & 104 \end{pmatrix}$$

eigs of Ke = {777.175, 201.363, 197.273, 42.9431, 21.3643, 19.8821, 0, 0, 0, 0, 0, 0}

FIGURE 16.6. Results from running test of Figure 16.5.

The stiffness module is exercised by the statements listed in Figure 16.5, which forms a tetrahedron with corner coordinates $\{x_1, y_1, z_1\} = \{2, 3, 4\}$, $\{x_2, y_2, z_2\} = \{6, 3, 2\}$, $\{x_3, y_3, z_3\} = \{2, 5, 1\}$ and $\{x_4, y_4, z_4\} = \{4, 3, 6\}$. Its volume is +24. The material is isotropic with elastic modulus $E = 96$ and Poisson's ratio $\nu = 1/3$. The results are shown in Figure 16.6. The computation of stiffness matrix eigenvalues is always a good programming test, since 6 eigenvalues (associated with rigid body modes) must be exactly zero and the other 6 real and positive. This is verified by the results.

Homework Exercises for Chapter 16

The Linear Tetrahedron

EXERCISE 16.1 [A:5] The tetrahedron element does not have fabrication properties, such as the thickness in the case of a plane stress element. Why?

EXERCISE 16.2 [A:15] Work out the formulas for a_i , b_i , c_i in terms of the corner coordinates x_i , y_i and z_i ($i = 1, 2, 3, 4$). Then write a compact formula for the volume $\mathcal{V}$. Hint: use the following script:

```
J={1,1,1,1},{x1,x2,x3,x4},{y1,y2,y3,y4},{z1,z2,z3,z4};
V6=Det[J]; Jinv=Simplify[Inverse[J]*V6];
{{R1,a1,b1,c1},{R2,a2,b2,c2},{R3,a3,b3,c3},{R4,a4,b4,c4}}=Jinv;
Print["{a1,a2,a3,a4}*V6=",{a1,a2,a3,a4}];
Print["{b1,b2,b3,b4}*V6=",{b1,b2,b3,b4}];
Print["{c1,c2,c3,c4}*V6=",{c1,c2,c3,c4}];
Print["V6=",V6];
```

EXERCISE 16.3 [A:20] Work out by hand the consistent node force vector $\mathbf{f}^e$ for the body force system $b_x = b_{x1}\zeta_1 + b_{x2}\zeta_2 + b_{x3}\zeta_3 + b_{x4}\zeta_4$, $b_y = 0$, $b_z = 0$, in which b_{xi} are given values at the nodes. Hint: use the integration rule (16.36). Specialize the result to $b_{x1} = b_{x2} = b_{x3} = b_{x4} = b_x$.

EXERCISE 16.4 [A:25] Face 1-2-3 of the 4-node tetrahedron is under pressure p acting normal to the face (positive pressure: $+p$ means it points into the body). Compute $\mathbf{f}^e$. [Hint: find the direction cosines n_j (needed to get the prescribed surface tractions $\hat{t}_i$) of the unit normal by developing the normal equation of plane 1-2-3].

17

The Quadratic Tetrahedron

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§17.1. Introduction

The technique used in Chapter 15 for the derivation of the linear triangle is now extended to the 10-node tetrahedron, also called the quadratic tetrahedron. This element can have both curved faces and edges. The extension is based on the isoparametric technique and the partial-derivative construction procedure followed the IFEM course for the 6-node triangle.

§17.2. The Quadratic Tetrahedron

The ten node tetrahedron shown in Figure 17.1 is the next complete-polynomial member of the isoparametric tetrahedron family.

The element has four corners with *local* numbers 1 through 4, which must be traversed following the same convention as the four node tetrahedron. It has six side nodes, with local numbers 5 through 10. Nodes 5,6,7 are located on sides 1-2, 2-3 and 3-1, whereas nodes 8,9,10 are located on sides 1-4, 2-4, and 3-4. The side nodes may be located arbitrarily subjected to positive-Jacobian-determinant constraints. Each element face is defined by six nodes. These do not necessarily lie on a plane, but they should not deviate too much from it.

The tetrahedral natural coordinates ζ_1 through ζ_4 are introduced in a fashion similar to that described for the linear tetrahedron in the previous Chapter. The main difference is that $\zeta_i = \text{constant}$ is not necessarily the equation of a plane.

The isoparametric element definition is

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & x_4 & \dots & x_{10} \\ y_1 & y_2 & y_3 & y_4 & \dots & y_{10} \\ z_1 & z_2 & z_3 & z_4 & \dots & z_{10} \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & \dots & u_{x10} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & \dots & u_{y10} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} & \dots & u_{z10} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ \vdots \\ N_{10}^e \end{bmatrix}. \quad (17.1)$$

The conventional (non-hierarchical) shape functions are given by

$$\begin{aligned} N_1^e &= \zeta_1(2\zeta_1 - 1), & N_2^e &= \zeta_2(2\zeta_2 - 1) \\ N_3^e &= \zeta_3(2\zeta_3 - 1), & N_4^e &= \zeta_4(2\zeta_4 - 1) \\ N_5^e &= 4\zeta_1\zeta_2, & N_6^e &= 4\zeta_2\zeta_3 \\ N_7^e &= 4\zeta_3\zeta_1, & N_8^e &= 4\zeta_1\zeta_4 \\ N_9^e &= 4\zeta_2\zeta_4, & N_{10}^e &= 4\zeta_3\zeta_4 \end{aligned}. \quad (17.2)$$

These shape functions are similar in form to those of the six node quadratic triangle discussed in IFEM. If the element is curved (that is, the six nodes that define each face are not on a plane), the tetrahedron coordinates no longer fall on planes, but form a curvilinear system.

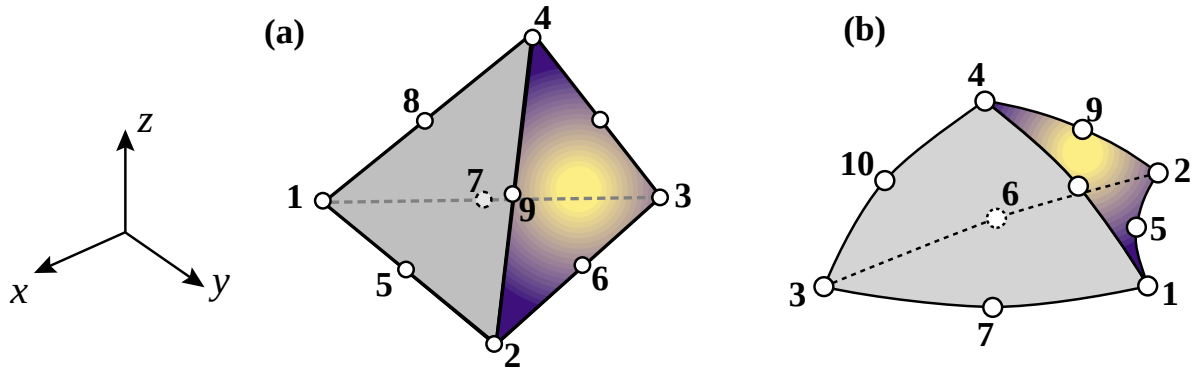


FIGURE 17.1. The ten-node (quadratic) tetrahedron. (a): element with planar faces, (b): element with curved faces.

§17.3. Partial Derivative Calculations

The main task involved in writing the shape function subroutine for the 10-node tetrahedron is the computation of the partial derivatives of the functions in (17.1) with respect to x , y and z at any point in the element. For this purpose consider a generic scalar function, $w(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, that is quadratically interpolated over the ten-node tetrahedron with the shape functions (17.2):

$$w = [w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad \dots \quad w_{10}] \begin{bmatrix} \zeta_1(2\zeta_1 - 1) \\ \zeta_2(2\zeta_2 - 1) \\ \zeta_3(2\zeta_3 - 1) \\ \zeta_4(2\zeta_4 - 1) \\ 4\zeta_1\zeta_2 \\ 4\zeta_2\zeta_3 \\ \vdots \\ 4\zeta_3\zeta_4 \end{bmatrix}. \quad (17.3)$$

Symbol w may represent x , y , z , u_x , u_y or u_z in the isoparametric representation (7.1), or other element-varying quantities such as body forces or temperatures.

Taking partials with respect to x , y and z , and applying the chain rule twice we get

$$\begin{aligned} \frac{\partial w}{\partial x} &= \sum w_i \frac{\partial N_i}{\partial x} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial x} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial x} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial x} + \frac{\partial N_i}{\partial \zeta_4} \frac{\partial \zeta_4}{\partial x} \right), \\ \frac{\partial w}{\partial y} &= \sum w_i \frac{\partial N_i}{\partial y} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial y} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial y} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial y} + \frac{\partial N_i}{\partial \zeta_4} \frac{\partial \zeta_4}{\partial y} \right), \\ \frac{\partial w}{\partial z} &= \sum w_i \frac{\partial N_i}{\partial z} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial z} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial z} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial z} + \frac{\partial N_i}{\partial \zeta_4} \frac{\partial \zeta_4}{\partial z} \right), \end{aligned} \quad (17.4)$$

where all sums are understood to run from $i = 1$ through 10, and element superscripts on the shape

functions have been suppressed for clarity. In matrix form:

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_4}{\partial x} \\ \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_4}{\partial y} \\ \frac{\partial \zeta_1}{\partial z} & \frac{\partial \zeta_2}{\partial z} & \frac{\partial \zeta_3}{\partial z} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} \begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_2} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_3} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \quad (17.5)$$

Transposing both sides of (17.5) while switching the left and right hand sides, yields a form exploited below:

$$\begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} & \sum w_i \frac{\partial N_i}{\partial \zeta_2} & \sum w_i \frac{\partial N_i}{\partial \zeta_3} & \sum w_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}. \quad (17.6)$$

Now make $w \equiv x, y, z$ and stack the results row-wise:

$$\begin{bmatrix} \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} & \sum x_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} & \sum y_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum z_i \frac{\partial N_i}{\partial \zeta_1} & \sum z_i \frac{\partial N_i}{\partial \zeta_2} & \sum z_i \frac{\partial N_i}{\partial \zeta_3} & \sum z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix}. \quad (17.7)$$

This is a linear system with the required unknowns in the second matrix, but its coefficient matrix is not square. To achieve that, differentiate both sides of the identity $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 1$ with respect to x, y and z , and insert as first row:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} & \sum x_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} & \sum y_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum z_i \frac{\partial N_i}{\partial \zeta_1} & \sum z_i \frac{\partial N_i}{\partial \zeta_2} & \sum z_i \frac{\partial N_i}{\partial \zeta_3} & \sum z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial 1}{\partial x} & \frac{\partial 1}{\partial y} & \frac{\partial 1}{\partial z} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix}. \quad (17.8)$$

But $\partial x/\partial x = \partial y/\partial y = \partial z/\partial z = 1$ and $\partial 1/\partial x = \partial 1/\partial y = \partial 1/\partial z = \partial x/\partial y = \partial x/\partial z = \partial y/\partial x = \partial y/\partial z = \partial z/\partial x = \partial z/\partial y = 0$ because x, y and z are independent coordinates. Consequently we arrive at a system of linear equations of order 4 with three right-hand sides. Using the summation

convention to get rid of sum symbols the system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_i \frac{\partial N_i}{\partial \zeta_1} & x_i \frac{\partial N_i}{\partial \zeta_2} & x_i \frac{\partial N_i}{\partial \zeta_3} & x_i \frac{\partial N_i}{\partial \zeta_4} \\ y_i \frac{\partial N_i}{\partial \zeta_1} & y_i \frac{\partial N_i}{\partial \zeta_2} & y_i \frac{\partial N_i}{\partial \zeta_3} & y_i \frac{\partial N_i}{\partial \zeta_4} \\ z_i \frac{\partial N_i}{\partial \zeta_1} & z_i \frac{\partial N_i}{\partial \zeta_2} & z_i \frac{\partial N_i}{\partial \zeta_3} & z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (17.9)$$

In compact matrix notation

$$\mathbf{J} \mathbf{P} = \mathbf{I}_{aug}. \quad (17.10)$$

where

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ J_{x1} & J_{x2} & J_{x3} & J_{x4} \\ J_{y1} & J_{y2} & J_{y3} & J_{y4} \\ J_{z1} & J_{z2} & J_{z3} & J_{z4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_i \frac{\partial N_i}{\partial \zeta_1} & x_i \frac{\partial N_i}{\partial \zeta_2} & x_i \frac{\partial N_i}{\partial \zeta_3} & x_i \frac{\partial N_i}{\partial \zeta_4} \\ y_i \frac{\partial N_i}{\partial \zeta_1} & y_i \frac{\partial N_i}{\partial \zeta_2} & y_i \frac{\partial N_i}{\partial \zeta_3} & y_i \frac{\partial N_i}{\partial \zeta_4} \\ z_i \frac{\partial N_i}{\partial \zeta_1} & z_i \frac{\partial N_i}{\partial \zeta_2} & z_i \frac{\partial N_i}{\partial \zeta_3} & z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix}. \quad (17.11)$$

and $\mathbf{I}_{aug}$ is the 3×3 identity matrix augmented with a zero first row. Taking the partials of the shape functions (17.2) with respect to the tetrahedron coordinates and substituting into the above yields

$$\begin{aligned} J_{x1} &= x_1(4\zeta_1 - 1) + 4x_5\zeta_2 + 4x_7\zeta_3 + 4x_8\zeta_4, \\ J_{x2} &= x_2(4\zeta_2 - 1) + 4x_6\zeta_3 + 4x_5\zeta_1 + 4x_9\zeta_4, \\ J_{x3} &= x_3(4\zeta_3 - 1) + 4x_7\zeta_1 + 4x_6\zeta_2 + 4x_{10}\zeta_4, \\ J_{x4} &= x_4(4\zeta_4 - 1) + 4x_8\zeta_1 + 4x_9\zeta_2 + 4x_{10}\zeta_3. \end{aligned} \quad (17.12)$$

For J_{yi} and J_{zi} replace x_i by y_i and z_i , respectively, into the above formulas.

Summarizing, to compute the x-y-z partials for a function w interpolated as per (17.3) the recipe is: form the linear system (17.10) from the geometric data, solve for the 12 tetrahedron coordinates partials, and substitute these into (17.5).

Remark 17.1. By analogy with the isoparametric brick elements, matrix $\mathbf{J}$ of (17.10) may be called a Jacobian matrix. However, the factor J that appears in the element-of-volume transformation

$$dV^e = J d\zeta_1 d\zeta_2 d\zeta_3 d\zeta_4, \quad (17.13)$$

is not $\det \mathbf{J} = |\mathbf{J}|$, but $J = \frac{1}{6} |\mathbf{J}|$. If the element has straight sides with side nodes at the midpoints, J is constant and equal to the volume $\mathcal{V}$ of the tetrahedron, which is given by the usual formula

$$J = \mathcal{V} = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \quad (17.14)$$

as in Chapter 6. In this case J is constant over the element.

§17.3.1. Implementation Considerations

To speed up computations in the shape function subroutine it is important to observe that the Jacobian matrix has the special structure (17.11). We can take advantage of this by subtracting the first column of $\mathbf{J}$ from the last three columns. This manipulation reduces (17.10) to the solution of a 3×3 linear system:

$$\begin{bmatrix} J_{x2} - J_{x1} & J_{x3} - J_{x1} & J_{x4} - J_{x1} \\ J_{y2} - J_{y1} & J_{y3} - J_{y1} & J_{y4} - J_{y1} \\ J_{z2} - J_{z1} & J_{z3} - J_{z1} & J_{z4} - J_{z1} \end{bmatrix} \begin{bmatrix} \frac{\partial(\zeta_2 - \zeta_1)}{\partial x} & \frac{\partial(\zeta_2 - \zeta_1)}{\partial y} & \frac{\partial(\zeta_2 - \zeta_1)}{\partial z} \\ \frac{\partial(\zeta_3 - \zeta_1)}{\partial x} & \frac{\partial(\zeta_3 - \zeta_1)}{\partial y} & \frac{\partial(\zeta_3 - \zeta_1)}{\partial z} \\ \frac{\partial(\zeta_4 - \zeta_1)}{\partial x} & \frac{\partial(\zeta_4 - \zeta_1)}{\partial y} & \frac{\partial(\zeta_4 - \zeta_1)}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (17.15)$$

or

$$\bar{\mathbf{J}} \bar{\mathbf{P}} = \mathbf{I} \quad (17.16)$$

where $\mathbf{I}$ is the 3×3 identity matrix, and the modified Jacobian matrix is shown on the left hand side. Therefore $\bar{\mathbf{P}}$ is just the inverse of $\bar{\mathbf{J}}$, which can be readily calculated by Cramer's rule. Finally, the first row of $\mathbf{P}$ is recovered from the constraints

$$\frac{\partial \zeta_1}{\partial x} + \frac{\partial \zeta_2}{\partial x} + \frac{\partial \zeta_3}{\partial x} + \frac{\partial \zeta_4}{\partial x} = 0, \quad (17.17)$$

and similarly for the y and z partials. This is done as follows: take the column sum s_x of the computed first column of $\bar{\mathbf{P}}$; then

$$\frac{\partial \zeta_1}{\partial x} = -\frac{1}{4}s_x, \quad (17.18)$$

from which the other x can be recovered. This operation is repeated over the other columns.

To tie up with the notation used in Chapter 6 for the four-node tetrahedron, we may denote the partials as

$$\begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \quad (17.19)$$

For the 4-node tetrahedron, $J = \frac{1}{6}\mathcal{V}$ was constant over the element, but now it generally will vary with position unless the tetrahedron has planar faces.

§17.4. Gauss Rules over Tetrahedra

We mention the first two numerical integration rules, which find applications in the evaluation of element stiffness matrix and consistent force vector.

One point rule (exact for constant and linear polynomials over plane-face tetrahedra):

$$\frac{1}{V} \int_{V^e} F(\zeta_1, \zeta_2, \zeta_3, \zeta_4) dV^e \approx F\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right). \quad (17.20)$$

Four-point rule (exact for constant through quadratic polynomials over plane-face tetrahedra):

$$\frac{1}{V} \int_{V^e} F(\zeta_1, \zeta_2, \zeta_3, \zeta_4) dV^e \approx \frac{1}{4}F(\alpha, \beta, \beta, \beta) + \frac{1}{4}F(\beta, \alpha, \beta, \beta) + \frac{1}{4}F(\beta, \beta, \alpha, \beta) + \frac{1}{4}F(\beta, \beta, \beta, \alpha), \quad (17.21)$$

in which $\alpha = (5 + 3\sqrt{5})/20 = 0.58541020$, $\beta = (5 - \sqrt{5})/20 = 0.13819660$. More details on this and other integration rules are posted in Chapter 17: A Compendium of Gauss Integration Rules for FEM.

§17.5. The Element Stiffness Matrix

In three dimensional elasticity this element has $10 \times 3 = 30$ degrees of freedom. For the following derivations we assume that they are arranged as

$$\mathbf{u}^e = [u_{x1} \ u_{y1} \ u_{z1} \ u_{x2} \ u_{y2} \ u_{z2} \ \dots \ u_{x10} \ u_{y10} \ u_{z10}]^T. \quad (17.22)$$

The 6×30 strain-displacement matrix for the 10-node tetrahedron is

$$\mathbf{B} = \begin{bmatrix} q_{x1} & q_{x2} & \dots & q_{x1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & q_{y1} & q_{y2} & \dots & q_{y10} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & q_{z1} & q_{z2} & \dots & q_{z10} \\ q_{y1} & q_{y2} & \dots & q_{y10} & q_{x1} & q_{x2} & \dots & q_{x10} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & q_{z1} & q_{z2} & \dots & q_{z10} & q_{y1} & q_{y2} & \dots & q_{y10} \\ q_{x1} & q_{x2} & \dots & q_{x10} & 0 & 0 & \dots & 0 & q_{z1} & q_{z2} & \dots & q_{z10} \end{bmatrix}. \quad (17.23)$$

where (summation convention implied on $j = 1, 2, 3, 4$):

$$\begin{aligned} q_{xi} &= \frac{\partial N_i}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial x} = J^{-1} \frac{\partial N_i}{\partial \zeta_j} a_j, \\ q_{yi} &= \frac{\partial N_i}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial y} = J^{-1} \frac{\partial N_i}{\partial \zeta_j} b_j, \\ q_{zi} &= \frac{\partial N_i}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial z} = J^{-1} \frac{\partial N_i}{\partial \zeta_j} c_j, \end{aligned} \quad (17.24)$$

The stiffness matrix $\mathbf{K}^e$ is evaluated by numerical integration

$$\mathbf{K}^e = \sum_{k=1}^p w_k \mathbf{B}^T(\zeta_{ik}) \mathbf{E} \mathbf{B}(\zeta_{ik}) J(\zeta_{ik}) \quad (17.25)$$

where p is the number of Gauss points, (ζ_{ik}) denotes the coordinate quartet $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ at the k^{th} integration point, and w_k are the corresponding weights. The stress-strain matrix $\mathbf{E}$ is the same as in Chapter 6. The four-point rule integrates this element with the correct rank.

§17.6. The Consistent Node Force Vector

Consider a body force field over the element defined by its components

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}. \quad (17.26)$$

The consistent node force vector is given by

$$\mathbf{f}^e = \int_{V^e} \mathbf{N}^T \mathbf{b} dV^e, \quad (17.27)$$

where $\mathbf{N}$ is the 3×30 matrix of shape functions that relates element displacements to node displacements:

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{N} \mathbf{u}. \quad (17.28)$$

For this element and the node displacement ordering (17.22)

$$\mathbf{N} = \begin{bmatrix} N_1 & \dots & N_{10} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & N_1 & \dots & N_{10} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & N_1 & \dots & N_{10} \end{bmatrix}, \quad (17.29)$$

where the shape functions are given by (17.2).

Homework Exercises for Chapter 17**The Quadratic Tetrahedron**

EXERCISE 17.1 [A:15] The 10-node tetrahedron element is converted into an 11-point tetrahedron by adding node point 11 located at the centroid $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 1/4$. What is the shape function N_{11}^e ? (You do not need to write the full element definition).

EXERCISE 17.2 [A:15] The next full-polynomial, isoparametric member of the tetrahedron family is the cubic tetrahedron, which has 21 node points. Where do you think the nodes are located?

EXERCISE 17.3 [A/C:25] Derive the shape functions for the 21-node tetrahedron.

EXERCISE 17.4 [A:15] Justify the rule (17.18).

EXERCISE 17.5 [A/C:20] Compute $\mathbf{f}^e$ for a straight-face 10-node tetrahedron if the body forces $b_x = b_y = 0$ and b_z is constant, using the 4-point rule (17.21) to evaluate the integral in (17.27). (You need to give only the z force components).

18

Hexahedron Elements

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§18.1. Introduction

The generalization of a quadrilateral three-dimensions is a *hexahedron*, also known in the finite element literature as *brick*. A hexahedron is topologically equivalent to a cube. It has eight corners, twelve edges or sides, and six faces.

Finite elements with this geometry are extensively used in modeling three-dimensional solids. Hexahedra also have been the motivating factor for the development of “Ahmad-Pawsey” shell elements through the use of the “degenerated solid” concept.

The construction of hexahedra shape functions and the computation of the stiffness matrix was greatly facilitated by three advances in finite element technology: natural coordinates, isoparametric description and numerical integration. Together these revolutionized the finite element field in the mid-1960’s.

§18.1.1. Natural Coordinates

Before presenting examples of hexahedron elements, we have to introduce the appropriate *natural coordinate system* for that geometry. The natural coordinates for this geometry are called ξ , η and μ , and are called *isoparametric hexahedral coordinates* or simply *natural coordinates*.

These coordinates are illustrated in Figure 18.1. As can be seen they are very similar to the quadrilateral coordinates ξ and η used in IFEM. They vary from -1 on one face to +1 on the opposite face, taking the value zero on the “median” face. As in the quadrilateral, this particular choice of limits was made to facilitate the use of the standard Gauss integration formulas.

§18.1.2. Corner Numbering Rules

The eight corners of a hexahedron element are locally numbered 1, 2 . . . 8. The corner numbering rule is similar to that given for the 4-node tetrahedron in Chapter 14. Again the purpose is to guarantee a positive volume (or, more precisely, a positive Jacobian determinant at every point). The transcription of those rules to the hexahedron element is as follows:

1. Chose one starting corner, which is given number 1, and one initial face pertaining to that corner (given a starting corner, there are three possible faces meeting at that corner that may be selected).
2. Number the other 3 corners as 2,3,4 traversing the initial face counterclockwise¹ while one looks at the initial face from the opposite one.
3. Number the corners of the opposite face directly opposite 1,2,3,4 as 5,6,7,8, respectively.

¹ “Anticlockwise” in British.

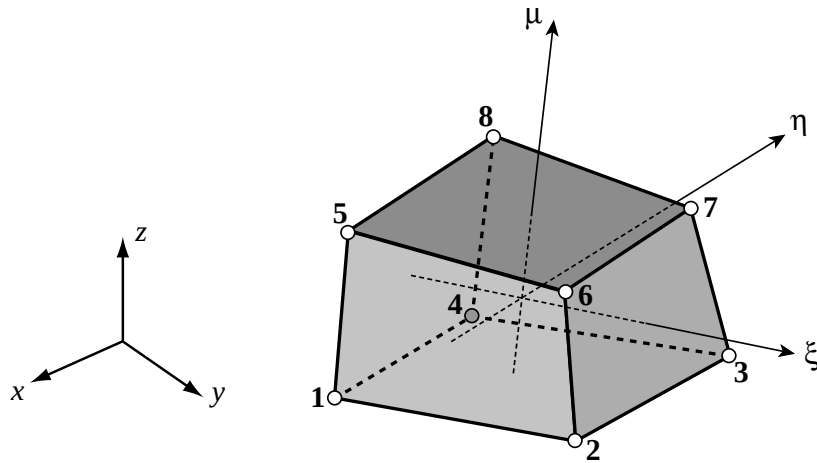


Figure 18.1. The 8-node hexahedron and the natural coordinates ξ , η , μ .
The definition of the latter is the same for higher order models.

The definition of ξ , η and μ can be now be made more precise:

ξ goes from -1 from (center of) face 1485 to $+1$ on face 2376

η goes from -1 from (center of) on face 1265 to $+1$ on face 3487

μ goes from -1 from (center of) on face 1234 to $+1$ on face 5678

The center of a face is the intersection of the two medians.

§18.2. The Eight Node (Trilinear) Hexahedron

The eight-node hexahedron shown in Figure 18.2 is the simplest member of the hexahedron family. It is defined by

$$\begin{bmatrix} 1 \\ x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\ v_{x1} & v_{x2} & v_{x3} & v_{x4} & v_{x5} & v_{x6} & v_{x7} & v_{x8} \\ v_{y1} & v_{y2} & v_{y3} & v_{y4} & v_{y5} & v_{y6} & v_{y7} & v_{y8} \\ v_{z1} & v_{z2} & v_{z3} & v_{z4} & v_{z5} & v_{z6} & v_{z7} & v_{z8} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_8^{(e)} \end{bmatrix} \quad (18.1)$$

The hexahedron coordinates of the corners are (see Figure 18.1)

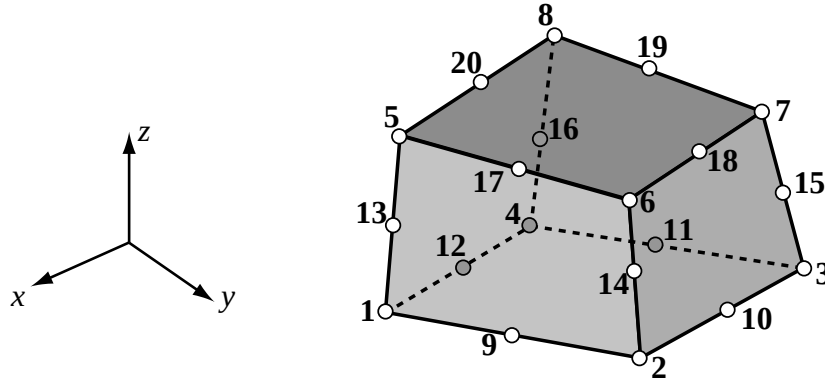


Figure 18.2. The 20-node hexahedron element — note node numbering conventions.

node	ξ	η	μ
1	-1	-1	-1
2	+1	-1	-1
3	+1	+1	-1
4	-1	+1	-1
5	-1	-1	+1
6	+1	-1	+1
7	+1	+1	+1
8	-1	+1	+1

The shape functions are

$$\begin{aligned}
 N_1^{(e)} &= \frac{1}{8}(1 - \xi)(1 - \eta)(1 - \mu), & N_2^{(e)} &= \frac{1}{8}(1 + \xi)(1 - \eta)(1 - \mu) \\
 N_3^{(e)} &= \frac{1}{8}(1 + \xi)(1 + \eta)(1 - \mu), & N_4^{(e)} &= \frac{1}{8}(1 - \xi)(1 + \eta)(1 - \mu) \\
 N_5^{(e)} &= \frac{1}{8}(1 - \xi)(1 - \eta)(1 + \mu), & N_6^{(e)} &= \frac{1}{8}(1 + \xi)(1 - \eta)(1 + \mu) \\
 N_7^{(e)} &= \frac{1}{8}(1 + \xi)(1 + \eta)(1 + \mu), & N_8^{(e)} &= \frac{1}{8}(1 - \xi)(1 + \eta)(1 + \mu)
 \end{aligned} \tag{18.2}$$

These eight formulas can be summarized in a single expression:

$$N_1^{(e)} = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \mu\mu_i) \tag{18.3}$$

where ξ_i , η_i and μ_i denote the coordinates of the i^{th} node.

§18.3. The 20-node (Serendipity) Hexahedron

The 20-node hexahedron is the analog of the 8-node “serendipity” quadrilateral. The 8 corner nodes are augmented with 12 side nodes which are usually located at the midpoints of the sides.

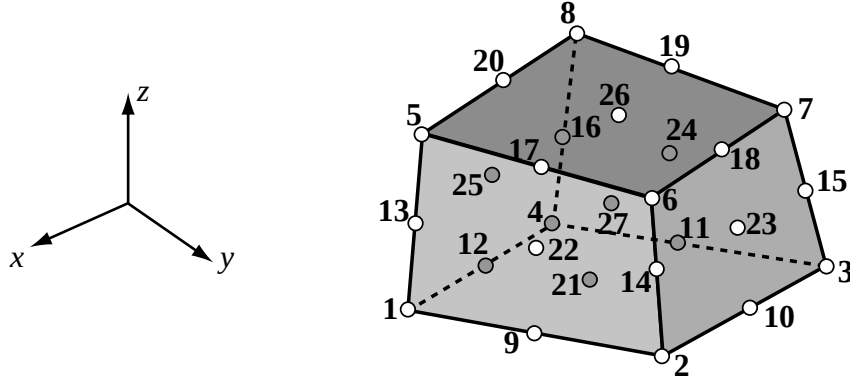


Figure 18.3. The 27-node hexahedron element — note node numbering conventions.

The numbering scheme is illustrated in Figure 18.2. For elasticity applications this element have $20 \times 3 = 60$ degrees of freedom.

The 8-node quadrilateral studied in IFEM cannot represent a complete biquadratic expansion in the quadrilateral coordinates ξ and η , that is, the nine terms $1, \xi, \eta, \xi^2, \dots, \xi^2\eta^2$. One has to go to the 9-node (biquadratic) quadrilateral to achieve that.

Likewise, the 20 node hexahedron is incapable of accomodating a full triquadratic expansion in ξ, η and μ ; that is $1, \xi, \eta, \mu, \eta^2, \dots, \xi^2\eta^2\mu^2$. A 27-node hexahedron is required for that. That element is described in the next section.

The shape functions of the 20-node hexahedron can be grouped as follows. For the corner nodes $i = 1, 2, \dots, 8$:

$$N_i^{(e)} = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \mu\mu_i)(\xi\xi_i + \eta\eta_i + \mu\mu_i - 2). \quad (18.4)$$

For the midside nodes $i = 9, 11, 17, 19$:

$$N_i^{(e)} = \frac{1}{4}(1 - \xi^2)(1 + \eta\eta_i)(1 + \mu\mu_i). \quad (18.5)$$

For the midside nodes $i = 10, 12, 18, 20$:

$$N_i^{(e)} = \frac{1}{4}(1 - \eta^2)(1 + \xi\xi_i)(1 + \mu\mu_i). \quad (18.6)$$

For the midside nodes $i = 13, 14, 15, 16$:

$$N_i^{(e)} = \frac{1}{4}(1 - \mu^2)(1 + \xi\xi_i)(1 + \eta\eta_i). \quad (18.7)$$

§18.4. The 27-Node Hexahedron

A 27-node hexahedron can indeed be constructed by adding 7 more nodes: 6 on each face center, and 1 interior node at the hexahedron center. See Figure 18.3. In elasticity application such an element has $27 \times 3 = 81$ degrees of freedom.

(To be completed).

§18.5. Partial Derivatives

The calculation of partial derivatives of hexahedron shape functions with respect to Cartesian coordinates follows techniques similar to that discussed for two-dimensional quadrilateral elements in IFEM. Only the size of the matrices changes because of the appearance of the third dimension.

§18.5.1. The Jacobian

The derivatives of the shape functions are given by the usual chain rule formulas:

$$\begin{aligned}\frac{\partial N_i^{(e)}}{\partial x} &= \frac{\partial N_i^{(e)}}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_i^{(e)}}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial N_i^{(e)}}{\partial \mu} \frac{\partial \mu}{\partial x}, \\ \frac{\partial N_i^{(e)}}{\partial y} &= \frac{\partial N_i^{(e)}}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_i^{(e)}}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial N_i^{(e)}}{\partial \mu} \frac{\partial \mu}{\partial y}, \\ \frac{\partial N_i^{(e)}}{\partial z} &= \frac{\partial N_i^{(e)}}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial N_i^{(e)}}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial N_i^{(e)}}{\partial \mu} \frac{\partial \mu}{\partial z}.\end{aligned}\tag{18.8}$$

In matrix form

$$\begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial x} \\ \frac{\partial N_i^{(e)}}{\partial y} \\ \frac{\partial N_i^{(e)}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \mu}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \mu}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \mu}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial \xi} \\ \frac{\partial N_i^{(e)}}{\partial \eta} \\ \frac{\partial N_i^{(e)}}{\partial \mu} \end{bmatrix}.\tag{18.9}$$

The 3×3 matrix that appears in (18.9) is $\mathbf{J}^{-1}$, the inverse of:

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \mu)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \mu} & \frac{\partial y}{\partial \mu} & \frac{\partial z}{\partial \mu} \end{bmatrix}.\tag{18.10}$$

Matrix $\mathbf{J}$ is called the *Jacobian matrix* of (x, y, z) with respect to (ξ, η, μ) . In the finite element literature, matrices $\mathbf{J}$ and $\mathbf{J}^{-1}$ are called simply the *Jacobian* and *inverse Jacobian*, respectively, although such a short name is sometimes ambiguous. The notation

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \mu)}, \quad \mathbf{J}^{-1} = \frac{\partial(\xi, \eta, \mu)}{\partial(x, y, z)}.\tag{18.11}$$

is standard in multivariable calculus and suggests that the Jacobian may be viewed as a generalization of the ordinary derivative, to which it reduces for a scalar function $\mathbf{x} = x(\xi)$.

§18.5.2. Computing the Jacobian Matrix

The isoparametric definition of hexahedron element geometry is

$$x = x_i N_i^{(e)}, \quad y = y_i N_i^{(e)}, \quad z = z_i N_i^{(e)},\tag{18.12}$$

where the summation convention is understood to apply over $i = 1, 2, \dots, n$, in which n denotes the number of element nodes.

Differentiating these relations with respect to the hexahedron coordinates we construct the matrix $\mathbf{J}$ as follows:

$$\mathbf{J} = \begin{bmatrix} x_i \frac{\partial N_i^{(e)}}{\partial \xi} & y_i \frac{\partial N_i^{(e)}}{\partial \xi} & z_i \frac{\partial N_i^{(e)}}{\partial \xi} \\ x_i \frac{\partial N_i^{(e)}}{\partial \eta} & y_i \frac{\partial N_i^{(e)}}{\partial \eta} & z_i \frac{\partial N_i^{(e)}}{\partial \eta} \\ x_i \frac{\partial N_i^{(e)}}{\partial \mu} & y_i \frac{\partial N_i^{(e)}}{\partial \mu} & z_i \frac{\partial N_i^{(e)}}{\partial \mu} \end{bmatrix}. \quad (18.13)$$

Given a point of hexahedron coordinates (ξ, η, μ) the Jacobian $\mathbf{J}$ can be easily formed using the above formula, and numerically inverted to form $\mathbf{J}^{-1}$.

Remark 18.1. The inversion formula for a matrix of order 3 is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad (18.14)$$

where

$$\begin{aligned} A_{11} &= a_{22}a_{33} - a_{23}a_{32}, & A_{22} &= a_{33}a_{11} - a_{31}a_{13}, \\ A_{33} &= a_{11}a_{22} - a_{12}a_{21}, & A_{12} &= a_{23}a_{31} - a_{21}a_{33}, \\ A_{23} &= a_{31}a_{12} - a_{32}a_{11}, & A_{31} &= a_{12}a_{23} - a_{13}a_{22}, \\ A_{21} &= a_{32}a_{13} - a_{12}a_{33}, & A_{32} &= a_{13}a_{21} - a_{23}a_{11}, \\ A_{13} &= a_{21}a_{22} - a_{31}a_{22}, & |\mathbf{A}| &= a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}. \end{aligned} \quad (18.15)$$

(The determinant can in fact be computed in 9 different ways.)

§18.6. The Strain Displacement Matrix

Having obtained the shape function derivatives, the matrix $\mathbf{B}$ for a hexahedron element displays the usual structure for 3D elements:

$$\mathbf{B} = \mathbf{D}\Phi = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{bmatrix} \mathbf{q} & 0 & 0 \\ 0 & \mathbf{q} & 0 \\ 0 & 0 & \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_x & 0 & 0 \\ 0 & \mathbf{q}_y & 0 \\ 0 & 0 & \mathbf{q}_z \\ \mathbf{q}_y & \mathbf{q}_x & 0 \\ 0 & \mathbf{q}_z & \mathbf{q}_y \\ \mathbf{q}_z & 0 & \mathbf{q}_x \end{bmatrix} \quad (18.16)$$

where

$$\begin{aligned}\mathbf{q} &= [N_1^{(e)} \quad \dots \quad N_n^{(e)}] \\ \mathbf{q}_x &= \left[\frac{\partial N_1^{(e)}}{\partial x} \quad \dots \quad \frac{\partial N_n^{(e)}}{\partial x} \right] \\ \mathbf{q}_y &= \left[\frac{\partial N_1^{(e)}}{\partial y} \quad \dots \quad \frac{\partial N_n^{(e)}}{\partial y} \right] \\ \mathbf{q}_z &= \left[\frac{\partial N_1^{(e)}}{\partial z} \quad \dots \quad \frac{\partial N_n^{(e)}}{\partial z} \right]\end{aligned}$$

are row vectors of length n , n being the number of nodes in the element.

§18.7. Stiffness Matrix Evaluation

The element stiffness matrix is given by

$$\mathbf{K}^{(e)} = \int_{V^{(e)}} \mathbf{B}^T \mathbf{E} \mathbf{B} dV^{(e)}. \quad (18.17)$$

As in the two-dimensional case, this is replaced by a numerical integration formula which now involves a triple loop over conventional Gauss quadrature rules. Assuming that the stress-strain matrix $\mathbf{E}$ is constant over the element,

$$\mathbf{K}^{(e)} = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \sum_{k=1}^{p_3} w_i w_j w_k \mathbf{B}_{ijk}^T \mathbf{E} \mathbf{B}_{ijk} J_{ik\ell}. \quad (18.18)$$

Here p_1 , p_2 and p_3 are the number of Gauss points in the ξ , η and μ direction, respectively, while $\mathbf{B}_{ijk}$ and $\mathbf{J}_{ik\ell}$ are abbreviations for

$$\mathbf{B}_{ijk} \equiv \mathbf{B}(\xi_i, \eta_j, \mu_k), \quad J_{ik\ell} \equiv \det \mathbf{J}(\xi_i, \eta_j, \mu_k). \quad (18.19)$$

Usually the number of integration points is taken the same in all directions: $p = p_1 = p_2 = p_3$. The total number of Gauss points is thus p^3 . Each point adds at most 6 to the stiffness matrix rank. The minimum rank-sufficient rules for the 8-node and 20-node hexahedra are $p = 2$ and $p = 3$, respectively.

Remark 18.2. The computation of consistent node forces corresponding to body forces is straightforward. The treatment of prescribed surface tractions such as pressure, presents, however, some computational difficulties because hexahedron faces are not generally plane.

§18.7.1. Selecting the Integration Rule

Usually the number of integration points is taken the same in all directions: $p = p_1 = p_2 = p_3$. The total number of Gauss points is thus p^3 . Each point adds at most 6 to the stiffness matrix rank. For the 8-node hexahedron this rule gives $p = 2$ because $2^3 \times 6 = 48 > 24 - 6$. For other configurations see Exercise 18.3.

Homework Exercises for Chapter 18

Hexahedron Elements

EXERCISE 18.1 [A:20] Find the shape functions associated with the 16-node hexahedron depicted in Figure E18.1(a) for node points 1 and 9. (This kind of element is historically important as a pit stop on the way the “degenerated solid” thick-shell elements developed in the late 1960s.)

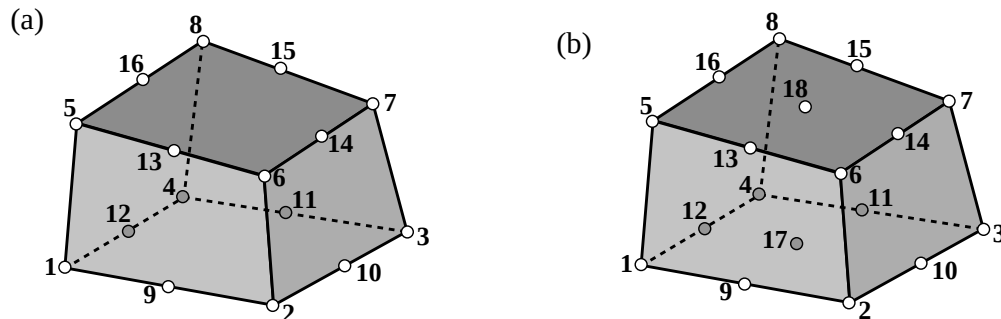


Figure E18.1. (a): 16-node hexahedron for Exercise 18.1;
(b): 18-node hexahedron for Exercise 18.2.

EXERCISE 18.2 [A:20] Find the shape functions associated with the 18-node hexahedron depicted in Figure E18.1(b) for node points 1, 9 and 17. (This node configuration is used for some solid-shell elements discussed in the last part of the course.)

EXERCISE 18.3 [A:15] Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for (a) the 20-node hexahedron, (b) the 27-node hexahedron, (c) the the 16-node hexahedron of Exercise 18.1 and (d) the 18-node hexahedron of Exercise 18.2. For the last two, would a formula containing less Gauss sample points in the μ direction (for example: $3 \times 3 \times 1$, work, at least on paper?

19

Pyramid Solid Elements

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§19.1. Introduction

Pyramid solid elements are useful for transition between bricks and tetrahedra in automated 3D mesh generation. Figure 19.1 depicts the 3 elements developed in this Chapter. Elements with 5, 13 and 14 nodes are identified as Pyra5, Pyra13 and Pyra14, respectively, in the sequel. A brief description follows.

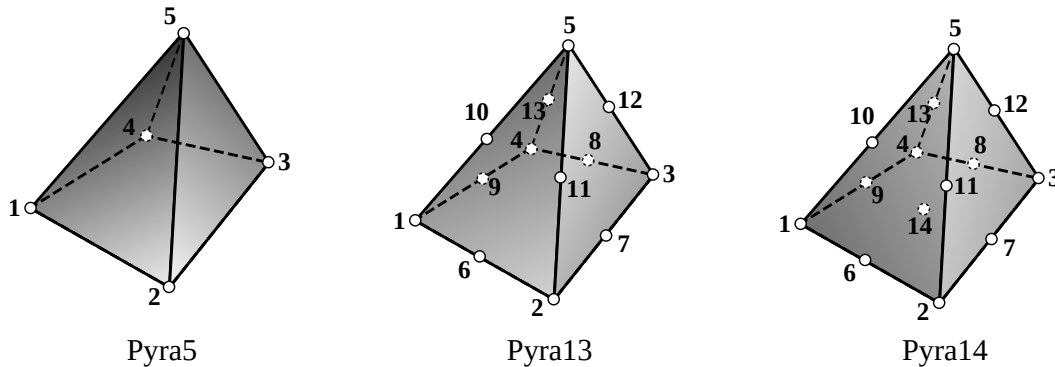


FIGURE 19.1. The three pyramid elements considered in this Chapter.

Pyra5. A isoparametric 5-node pyramid element that is useful as transition between 8-node bricks and 4-node tetrahedra. Although this element was available in several solid codes developed since 1972 (and presently in several general purpose FEM codes, notably ANSYS), the rederivation here serves to illustrate some aspects of the derivation and implementation of the more complicated elements.

Pyra13. A isoparametric 13-node pyramid element obtained from Pyra5 by adding 8 midside nodes. It is useful as transition between 20-node (serendipity) bricks and 10-node tetrahedra.

Pyra14. A isoparametric 14-node pyramid element derived from Pyra13 by injecting a node at the center of the quadrilateral face. It is useful as transition between 27-node (Lagrangian) bricks and 10-node tetrahedra.

The shape function set for Pyra5 is unique and has been known since the early 1970s. The sets for Pyra13 and Pyra14 are not unique. Some instances were implemented in mid-1970 proprietary codes used by the nuclear reactor industry of the time. The shape functions presented here are instances of a parametric family obtainable through the template approach. It was felt that a comprehensive but lengthy derivation of all possible sets was not worthwhile because these elements fulfill a limited function.

§19.2. Geometric Description

The major ingredients of the pyramid geometry are illustrated in Figure 19.2. A pyramid has 5 corners, 8 edges and 5 faces. One of the faces is a quadrilateral, called the *base*, which may be warped. The corner opposite to the base is the *apex*. Four triangular faces, called *apex faces*, meet at the apex. The apex faces are planar in Pyra5 but may be warped in Pyra13 and Pyra14.

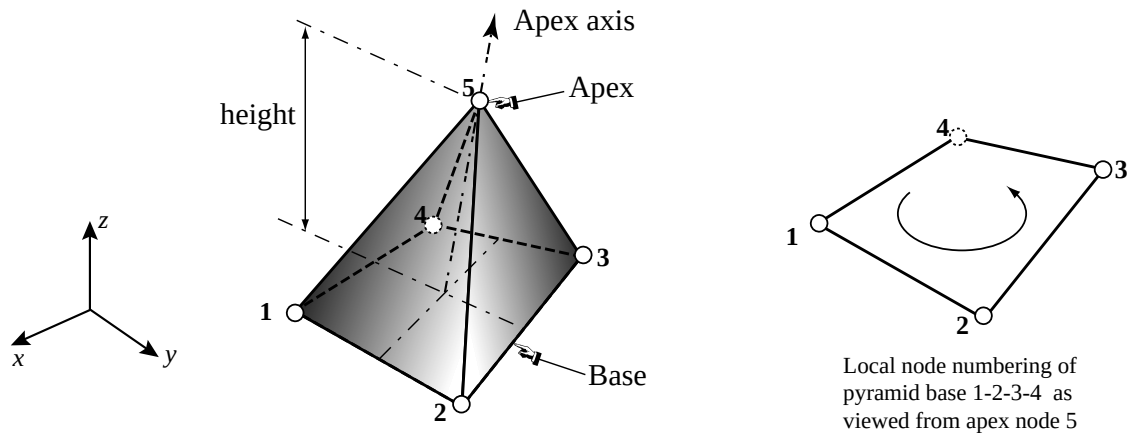


FIGURE 19.2. Nomenclature for pyramid geometry.

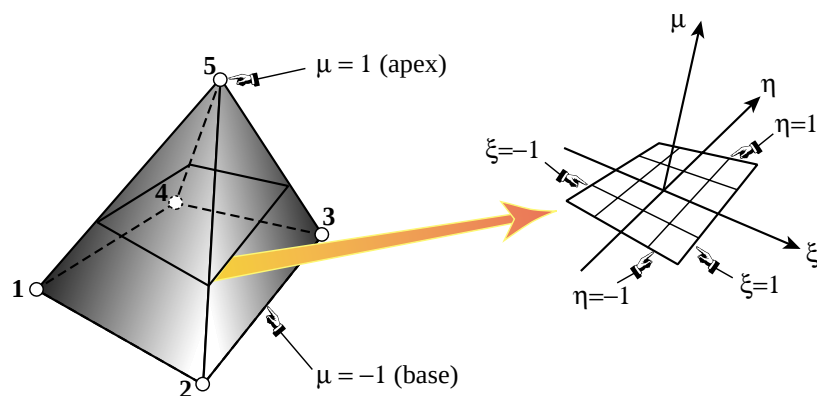
By analogy to pyramidal monuments, the normal distance from the quadrilateral base to the apex is called the *height*. The line joining the base center with the apex is the *apex axis*. This *apex direction* is not generally normal to the base.

§19.3. The Isoparametric Definition

This section summarizes the formulation of element stiffness and mass matrices of arbitrary isoparametric (iso-P) solid elements. The following formulation steps are well known; this material serves primarily to introduce notation.

An iso-P solid element has n nodes of global Cartesian coordinates $\{x_i, y_i, z_i\}$, $i = 1, 2, \dots, n$, which define the geometry of the element. The node displacements are $\{u_{xi}, u_{yi}, u_{zi}\}$, which are stacked node-wise to form the $3n \times 1$ node displacement vector $\mathbf{u}^{(e)}$.

The global coordinates of a generic point P in the element are $\{x, y, z\}$ and the displacement components at P are $\{u_x, u_y, u_z\}$ its displacement components.

FIGURE 19.3. The natural coordinates (ξ, η, μ) for the pyramid element.

All elements considered here employ the natural coordinates $\{\xi, \eta, \mu\}$ of brick and brick-like elements, which range from -1 to $+1$. For the special case of the pyramid geometry those coordinates

are illustrated in Figure 19.3. Coordinate μ is special in that it identifies the apex direction.

The shape functions are denoted by $N_i = N_i(\xi, \eta, \mu)$. The isoparametric definition of a 3D element with n nodes is

$$\begin{bmatrix} 1 \\ x \\ y \\ z \\ u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & x_4 & \dots & x_n \\ y_1 & y_2 & y_3 & y_4 & \dots & y_n \\ z_1 & z_2 & z_3 & z_4 & \dots & z_n \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & \dots & u_{xn} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & \dots & u_{yn} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} & \dots & u_{zn} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ \vdots \\ N_n \end{bmatrix}. \quad (19.1)$$

For later use in the expression of the mass matrix, define

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots & N_n \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{z1} \\ \vdots \\ u_{zn} \end{bmatrix} = \mathbf{N}\mathbf{u}^{(e)}. \quad (19.2)$$

The entries of the 3×3 Jacobian $\mathbf{J}$ of $\{x, y, z\}$ with respect to $\{\xi, \eta, \mu\}$ are given by

$$\begin{aligned} \mathbf{J} &= \frac{\partial(x, y, z)}{\partial(\xi, \eta, \mu)} = \begin{bmatrix} \partial x / \partial \xi & \partial y / \partial \xi & \partial z / \partial \xi \\ \partial x / \partial \eta & \partial y / \partial \eta & \partial z / \partial \eta \\ \partial x / \partial \mu & \partial y / \partial \mu & \partial z / \partial \mu \end{bmatrix} = \begin{bmatrix} J_{x\xi} & J_{y\xi} & J_{z\xi} \\ J_{x\eta} & J_{y\eta} & J_{z\eta} \\ J_{x\mu} & J_{y\mu} & J_{z\mu} \end{bmatrix}, \\ J_{x\xi} &= \sum_{i=1}^n \frac{\partial N_i}{\partial \xi} x_i, \quad J_{x\eta} = \sum_{i=1}^n \frac{\partial N_i}{\partial \eta} x_i, \quad J_{x\mu} = \sum_{i=1}^n \frac{\partial N_i}{\partial \mu} x_i, \\ J_{y\xi} &= \sum_{i=1}^n \frac{\partial N_i}{\partial \xi} y_i, \quad J_{y\eta} = \sum_{i=1}^n \frac{\partial N_i}{\partial \eta} y_i, \quad J_{y\mu} = \sum_{i=1}^n \frac{\partial N_i}{\partial \mu} y_i, \\ J_{z\xi} &= \sum_{i=1}^n \frac{\partial N_i}{\partial \xi} z_i, \quad J_{z\eta} = \sum_{i=1}^n \frac{\partial N_i}{\partial \eta} z_i, \quad J_{z\mu} = \sum_{i=1}^n \frac{\partial N_i}{\partial \mu} z_i, \end{aligned} \quad (19.3)$$

The inverse Jacobian is

$$\begin{aligned} \mathbf{J}^{-1} &= \frac{\partial(\xi, \eta, \mu)}{\partial(x, y, z)} = \begin{bmatrix} \partial \xi / \partial x & \partial \eta / \partial x & \partial \mu / \partial x \\ \partial \xi / \partial y & \partial \eta / \partial y & \partial \mu / \partial y \\ \partial \xi / \partial z & \partial \eta / \partial z & \partial \mu / \partial z \end{bmatrix} = \begin{bmatrix} J_{\xi x} & J_{\eta x} & J_{\mu x} \\ J_{\xi y} & J_{\eta y} & J_{\mu y} \\ J_{\xi z} & J_{\eta z} & J_{\mu z} \end{bmatrix} \\ &= \frac{1}{J} \begin{bmatrix} J_{y\eta} J_{z\mu} - J_{y\mu} J_{z\eta} & J_{y\mu} J_{z\xi} - J_{y\xi} J_{z\mu} & J_{y\xi} J_{z\eta} - J_{y\eta} J_{z\xi} \\ J_{x\mu} J_{z\eta} - J_{x\eta} J_{z\mu} & J_{x\xi} J_{z\mu} - J_{x\mu} J_{z\xi} & J_{x\eta} J_{z\xi} - J_{x\xi} J_{z\eta} \\ J_{x\eta} J_{y\mu} - J_{x\mu} J_{y\eta} & J_{x\mu} J_{y\xi} - J_{x\xi} J_{y\mu} & J_{x\xi} J_{y\eta} - J_{x\eta} J_{y\xi} \end{bmatrix}, \end{aligned}$$

$$\text{where } J = \det(\mathbf{J}) = J_{x\xi} J_{y\eta} J_{z\mu} + J_{x\eta} J_{y\mu} J_{z\xi} + J_{x\mu} J_{y\xi} J_{z\eta} - J_{x\mu} J_{y\eta} J_{z\xi} - J_{x\xi} J_{y\mu} J_{z\eta} - J_{x\eta} J_{y\xi} J_{z\mu}. \quad (19.4)$$

The Cartesian derivatives of the shape functions follow as

$$\begin{aligned} N_{xi} &= \frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} J_{\xi x} + \frac{\partial N_i}{\partial \eta} J_{\eta x} + \frac{\partial N_i}{\partial \mu} J_{\mu x}, \\ N_{yi} &= \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} J_{\xi y} + \frac{\partial N_i}{\partial \eta} J_{\eta y} + \frac{\partial N_i}{\partial \mu} J_{\mu y}, \\ N_{zi} &= \frac{\partial N_i}{\partial z} = \frac{\partial N_i}{\partial \xi} J_{\xi z} + \frac{\partial N_i}{\partial \eta} J_{\eta z} + \frac{\partial N_i}{\partial \mu} J_{\mu z}, \end{aligned} \quad (19.5)$$

or in matrix form

$$\begin{bmatrix} N_{xi} \\ N_{yi} \\ N_{zi} \end{bmatrix} = \begin{bmatrix} \partial N_i / \partial x \\ \partial N_i / \partial y \\ \partial N_i / \partial z \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \partial N_i / \partial \xi \\ \partial N_i / \partial \eta \\ \partial N_i / \partial \mu \end{bmatrix}, \quad i = 1, 2, \dots, n. \quad (19.6)$$

For code checking, it is useful to have the following expressions for a regular pyramid of flat rectangular $a \times b$ base, planar apex faces, apex axis normal to base, and height h :

$$\mathbf{J} = \begin{bmatrix} \frac{1}{4}a(1-\mu) & 0 & 0 \\ 0 & \frac{1}{4}b(1-\mu) & 0 \\ -\frac{1}{4}a\xi & -\frac{1}{4}b\eta & \frac{1}{2}h \end{bmatrix}, \quad J = \det \mathbf{J} = \frac{1}{32}abh(1-\mu)^2. \quad (19.7)$$

Note that at the apex $\mu = 1$, $J = 0$. Thus displacement gradients and strains are undefined at the apex.

The 6×3 strain displacement submatrix $\mathbf{B}_i$ associated with the contribution of node i is configured as

$$\mathbf{B}_i = \begin{bmatrix} N_{xi} & 0 & 0 \\ 0 & N_{yi} & 0 \\ 0 & 0 & N_{zi} \\ N_{yi} & N_{xi} & 0 \\ 0 & N_{zi} & N_{xi} \\ N_{zi} & 0 & N_{xi} \end{bmatrix}, \quad i = 1, 2, \dots, n. \quad (19.8)$$

Stacking the n submatrices $\mathbf{B}_i$ row-wise one gets the complete $6 \times 3n$ strain-displacement matrix $\mathbf{B}$ that relate strains to element node displacements:

$$\mathbf{e} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \dots \quad \mathbf{B}_n] \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{z1} \\ \vdots \\ u_{zn} \end{bmatrix} = \mathbf{B} \mathbf{u}^{(e)}. \quad (19.9)$$

The stresses are linked to the strains by the elastic constitutive equation

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ \text{symm} & & & & & E_{66} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}, \quad (19.10)$$

or

$$\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}. \quad (19.11)$$

The stiffness and (consistent) mass matrix of the element are given by the integrals over the element volume $V^{(e)}$:

$$\begin{aligned} \mathbf{K}^{(e)} &= \int_{V^{(e)}} \mathbf{B}^T \mathbf{E} \mathbf{B} dV = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{E} \mathbf{B} J d\xi d\eta d\mu, \\ \mathbf{M}^{(e)} &= \int_{V^{(e)}} \rho \mathbf{N}^T \mathbf{N} dV = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \rho \mathbf{N}^T \mathbf{N} J d\xi d\eta d\mu, \end{aligned} \quad (19.12)$$

in which ρ is the mass density. These matrices are evaluated by numerical quadrature at N_p Gauss points. Assuming $\mathbf{E}$ and ρ to be constant over the element, the numerically integrated stiffness and mass are given by

$$\mathbf{K}^{(e)} = \sum_{p=1}^{N_p} w_p J_p \mathbf{B}_p^T \mathbf{E} \mathbf{B}_p, \quad \mathbf{M}^{(e)} = \sum_{p=1}^{N_p} w_p J_p \rho \mathbf{N}_p^T \mathbf{N}_p. \quad (19.13)$$

where $\mathbf{N}_p$, $\mathbf{B}_p$ and J_p are evaluations of $\mathbf{N}$, $\mathbf{B}$ and $J = \det \mathbf{J}$, respectively, at Gauss points defined by abscissae $\{\xi_p, \eta_p, \mu_p\}$, and the w_p are quadrature weights.

§19.4. Shape Functions of 5-Node Pyramid

To derive the shape functions of the Pyra5 shown in Figure 19.4(b) we start with the isoparametric eight node brick. Denoting by $\{\xi_i, \eta_i, \mu_i\}$ the natural coordinates of the i^{th} node, the brick shape functions are

$$\tilde{N}_i = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \mu\mu_i), \quad i = 1, \dots, 8 \quad (19.14)$$

Nodes 5-6-7-8 are collapsed to apex 5, as illustrated in Figure 19.4. The shape functions of the 5-node pyramid are

$$\begin{aligned} N_i &= \tilde{N}_i, \quad i = 1, 2, 3, 4, \\ N_5 &= \tilde{N}_5 + \tilde{N}_6 + \tilde{N}_7 + \tilde{N}_8 = \frac{1}{2}(1 + \mu). \end{aligned} \quad (19.15)$$

It is easily checked that these functions satisfy compatibility across the base and apex faces when the pyramid is connected to tetrahedra elements (across apex faces) and brick elements (across the base). The sum of the shape functions (19.15) is identically one. Consequently the element is conforming and complete.

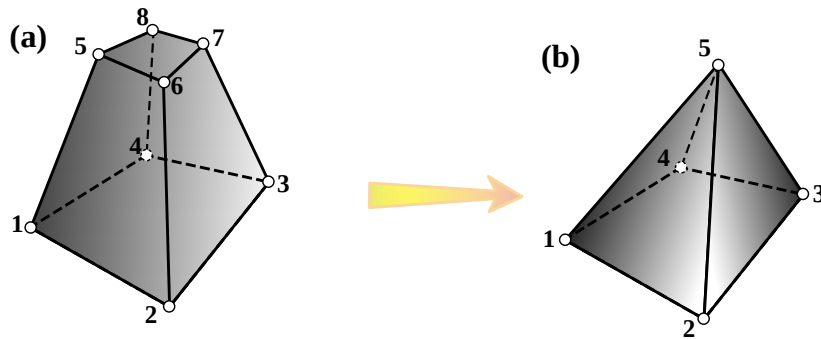


FIGURE 19.4. Morphing of 8-node brick to 5-node pyramid.

The partial derivatives with respect to $\{\xi, \eta, \mu\}$ are

$$\begin{aligned}\frac{\partial N_i}{\partial \xi} &= \{-\frac{1}{8}(1-\eta)(1-\mu), \quad \frac{1}{8}(1-\eta)(1-\mu), \quad \frac{1}{8}(1+\eta)(1-\mu), \quad -\frac{1}{8}(1+\eta)(1-\mu), \quad 0\}, \\ \frac{\partial N_i}{\partial \eta} &= \{-\frac{1}{8}(1-\xi)(1-\mu), \quad -\frac{1}{8}(1+\xi)(1-\mu), \quad \frac{1}{8}(1+\xi)(1-\mu), \quad \frac{1}{8}(1-\xi)(1-\mu), \quad 0\}, \\ \frac{\partial N_i}{\partial \mu} &= \{-\frac{1}{8}(1-\xi)(1-\eta), \quad -\frac{1}{8}(1+\xi)(1-\eta), \quad -\frac{1}{8}(1+\xi)(1+\eta), \quad -\frac{1}{8}(1-\xi)(1+\eta), \quad 1/2\}.\end{aligned}\quad (19.16)$$

The remaining computations follow as described in the previous section. It is recommended to evaluate this element by a 5-point rule specialized to the pyramid geometry, which is described in §7.1.

§19.5. The 13-Node Pyramid

§19.5.1. Shape Functions

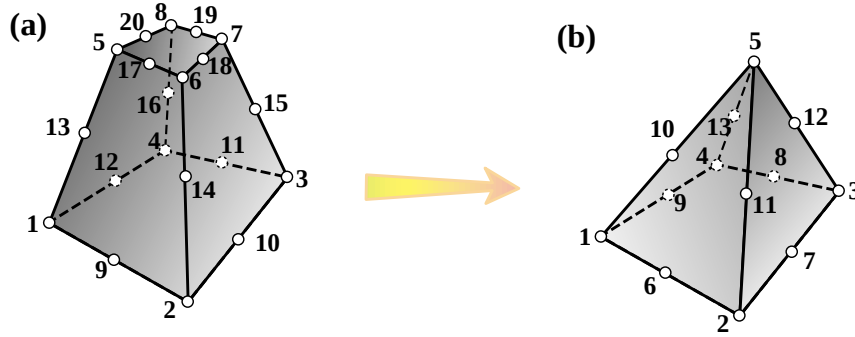


FIGURE 19.5. Morphing of 20-node brick to 13-node pyramid.

The shape functions of the 13-node pyramid shown in Figure 19.5(b) are derived in a two-stage process. In the first stage we start with the 20-node brick shown in Figure AFEMNineteen:fig:TwentyNodeBrickMorphingToPyramid(a). The well known shape functions of this element are:

$$\begin{aligned}\tilde{N}_1 &= -\frac{1}{8}(1-\xi)(1-\eta)(1-\mu)(2+\xi+\eta+\mu), & \tilde{N}_2 &= -\frac{1}{8}(1+\xi)(1-\eta)(1-\mu)(2-\xi+\eta+\mu), \\ \tilde{N}_3 &= -\frac{1}{8}(1+\xi)(1+\eta)(1-\mu)(2-\xi-\eta+\mu), & \tilde{N}_4 &= -\frac{1}{8}(1-\xi)(1+\eta)(1-\mu)(2+\xi-\eta+\mu), \\ \tilde{N}_5 &= -\frac{1}{8}(1-\xi)(1-\eta)(1+\mu)(2+\xi+\eta-\mu), & \tilde{N}_6 &= -\frac{1}{8}(1+\xi)(1-\eta)(1+\mu)(2-\xi+\eta-\mu), \\ \tilde{N}_7 &= -\frac{1}{8}(1+\xi)(1+\eta)(1+\mu)(2-\xi-\eta-\mu), & \tilde{N}_8 &= -\frac{1}{8}(1-\xi)(1+\eta)(1+\mu)(2+\xi-\eta-\mu), \\ \tilde{N}_9 &= \frac{1}{4}(1-\xi^2)(1-\eta)(1-\mu), & \tilde{N}_{10} &= \frac{1}{4}(1+\xi)(1-\eta^2)(1-\mu), \\ \tilde{N}_{11} &= \frac{1}{4}(1-\xi^2)(1+\eta)(1-\mu), & \tilde{N}_{12} &= \frac{1}{4}(1-\xi)(1-\eta^2)(1-\mu), \\ \tilde{N}_{13} &= \frac{1}{4}(1-\xi)(1-\eta)(1-\mu^2), & \tilde{N}_{14} &= \frac{1}{4}(1+\xi)(1-\eta)(1-\mu^2), \\ \tilde{N}_{15} &= \frac{1}{4}(1+\xi)(1+\eta)(1-\mu^2), & \tilde{N}_{16} &= \frac{1}{4}(1-\xi)(1+\eta)(1-\mu^2), \\ \tilde{N}_{17} &= \frac{1}{4}(1-\xi^2)(1-\eta)(1+\mu), & \tilde{N}_{18} &= \frac{1}{4}(1+\xi)(1-\eta^2)(1+\mu), \\ \tilde{N}_{19} &= \frac{1}{4}(1-\xi^2)(1+\eta)(1+\mu), & \tilde{N}_{20} &= \frac{1}{4}(1-\xi)(1-\eta^2)(1+\mu).\end{aligned}\quad (19.17)$$

Eight nodes of the brick face 5-6-7-8-17-18-19-20 are collapsed to node 5 at the pyramid apex. The resulting shape functions are

$$\begin{aligned}
 N_i^* &= \tilde{N}_i, \quad i = 1, 2, 3, 4, \\
 N_i^* &= \tilde{N}_{i+3}, \quad i = 6, 7, 8, 9, \\
 N_i &= \tilde{N}_{i+3}, \quad i = 10, 11, 12, 13, \\
 N_5 &= \tilde{N}_5 + \tilde{N}_6 + \tilde{N}_7 + \tilde{N}_8 + \tilde{N}_{17} + \tilde{N}_{18} + \tilde{N}_{19} + \tilde{N}_{20} = \frac{1}{2}\mu(1 + \mu).
 \end{aligned} \tag{19.18}$$

Note that shape functions except N_5 , N_{10} , N_{11} , N_{12} and N_{13} are marked with a star (*) in (19.18) because they will be corrected in the next stage. We check displacement compatibility over the apex faces 125, 235, 345 and 415, with the shape functions of an attached 10-node tetrahedron. Compatibility is found to be violated except on the edges. To remedy this problem we inject the following “face bubble functions”

$$\begin{aligned}
 N_{125} &= \frac{1}{16}(1 - \xi^2)(1 - \mu^2)(1 - \eta), \quad N_{345} = \frac{1}{16}(1 - \xi^2)(1 - \mu^2)(1 + \eta) \\
 N_{235} &= \frac{1}{16}(1 - \eta^2)(1 - \mu^2)(1 + \xi), \quad N_{415} = \frac{1}{16}(1 - \eta^2)(1 - \mu^2)(1 - \xi).
 \end{aligned} \tag{19.19}$$

Function N_{125} only “bubbles out” over apex face 1-2-5 ($\eta = -1$); likewise N_{235} , N_{345} and N_{415} bubble out over faces 2-3-5 ($\xi = 1$), 3-4-5 ($\eta = 1$) and 4-1-5 ($\xi = -1$), respectively. They vanish over the base 1-2-3-4 ($\mu = -1$) and thus do not impair conformity with a 20-brick element attached there.

The corrections necessary to establish conformity, found through *Mathematica*, are: $N_1 = N_1^* + N_{125} + N_{415}$, $N_2 = N_2^* + N_{235} + N_{125}$, $N_3 = N_3^* + N_{345} + N_{235}$, $N_4 = N_4^* + N_{415} + N_{345}$, $N_6 = N_6^* - 2N_{125}$, $N_7 = N_7^* - 2N_{235}$, $N_8 = N_8^* - 2N_{345}$, $N_9 = N_9^* - 2N_{415}$. Adding up it is obvious that $N_1 + N_2 + N_3 + N_4 + N_6 + N_7 + N_8 + N_9 = N_1^* + N_2^* + N_3^* + N_4^* + N_6^* + N_7^* + N_8^* + N_9^*$. Thus the sum of all shape functions remain unity, and completeness is preserved. The final set of shape functions is

$$\begin{aligned}
 N_1 &= -\frac{1}{16}(1 - \xi)(1 - \eta)(1 - \mu)(4 + 3\xi + 3\eta + 2\xi\eta + 2\mu + \xi\mu + \eta\mu + 2\xi\eta\mu), \\
 N_2 &= -\frac{1}{16}(1 + \xi)(1 - \eta)(1 - \mu)(4 - 3\xi + 3\eta - 2\xi\eta + 2\mu - \xi\mu + \eta\mu - 2\xi\eta\mu), \\
 N_3 &= -\frac{1}{16}(1 + \xi)(1 + \eta)(1 - \mu)(4 - 3\xi - 3\eta + 2\xi\eta + 2\mu - \xi\mu - \eta\mu + 2\xi\eta\mu), \\
 N_4 &= -\frac{1}{16}(1 - \xi)(1 + \eta)(1 - \mu)(4 + 3\xi - 3\eta - 2\xi\eta + 2\mu + \xi\mu - \eta\mu - 2\xi\eta\mu), \\
 N_5 &= \frac{1}{2}\mu(1 + \mu), \\
 N_6 &= \frac{1}{8}(1 - \xi^2)(1 - \eta)(1 - \mu)(2 + \eta + \eta\mu), \quad N_7 = \frac{1}{8}(1 + \xi)(1 - \eta^2)(1 - \mu)(2 - \xi - \xi\mu), \\
 N_8 &= \frac{1}{8}(1 - \xi^2)(1 + \eta)(1 - \mu)(2 - \eta - \eta\mu), \quad N_9 = \frac{1}{8}(1 - \xi)(1 - \eta^2)(1 - \mu)(2 + \xi + \xi\mu), \\
 N_{10} &= \frac{1}{4}(1 - \xi)(1 - \eta)(1 - \mu^2), \quad N_{11} = \frac{1}{4}(1 + \xi)(1 - \eta)(1 - \mu^2), \\
 N_{12} &= \frac{1}{4}(1 + \xi)(1 + \eta)(1 - \mu^2), \quad N_{13} = \frac{1}{4}(1 - \xi)(1 + \eta)(1 - \mu^2).
 \end{aligned} \tag{19.20}$$

§19.5.2. Compatibility Verification

The compatibility check with a quadratic tetrahedron over an apex face will be illustrated here for face 1-2-3 with midpoints 6-10-11. Evaluate (19.20) at $\eta = -1$, which is the equation of face 1-2-3. The nonzero shape functions are:

$$\begin{aligned} N_1 &= -\frac{1}{8}(1-\mu)(1-\xi)(1+\mu+\xi-\mu\xi), & N_2 &= -\frac{1}{8}(1-\mu)(1+\xi)(1+\mu-\xi+\mu\xi), \\ N_5 &= \frac{1}{2}\mu(1+\mu), & N_6 &= \frac{1}{4}(1-\mu)^2(1-\xi^2), \\ N_{10} &= \frac{1}{2}(1-\mu^2)(1-\xi), & N_{11} &= \frac{1}{2}(1-\mu^2)(1+\xi). \end{aligned} \quad (19.21)$$

That face is taken to match that of a 10-node tetrahedron, with local corner numbers 1,2,3 and local midpoints 5-6-7 (corner 4 of the tetrahedron is away from that face). The natural coordinates of the tetrahedron are ζ_i , $i = 1, 2, 3, 4$. The equation of face 1-2-3 is $\zeta_4 = 0$. Evaluating the shape functions at $\zeta_4 = 0$ one gets

$$\begin{aligned} N_1 &= \zeta_1(2\zeta_1 - 1), & N_2 &= \zeta_2(2\zeta_2 - 1), \\ N_3 &= \zeta_3(2\zeta_3 - 1), & N_5 &= 4\zeta_1\zeta_2, \\ N_6 &= 4\zeta_2\zeta_3, & N_7 &= 4\zeta_1\zeta_3. \end{aligned} \quad (19.22)$$

The relation between natural coordinates of the pyramid and tetrahedron over that face is $\zeta_1 = \frac{1}{4}(1-\xi)(1-\mu)$, $\zeta_2 = \frac{1}{4}(1+\xi)(1-\mu)$, and $\zeta_3 = 1 - \zeta_1 - \zeta_2 = \frac{1}{2}(1+\mu)$. Substituting these relations into (19.22) the shape functions become identical to those in (19.21). This proves conformity for an arbitrary face geometry.

Compatibility verification with a 20-node brick over the pyramid base is immediate since the shape functions of the pyramid are identical over that face, and the 4 face bubbles vanish over it.

§19.5.3. Natural Derivatives

The natural derivatives are

$$\begin{aligned} \partial N_1 / \partial \xi &= \frac{1}{16}(1-\eta)(1-\mu)(1+6\xi+\eta+4\xi\eta+\mu+2\xi\mu-\eta\mu+4\xi\eta\mu), \\ \partial N_2 / \partial \xi &= -\frac{1}{16}(1-\eta)(1-\mu)(1-6\xi+\eta-4\xi\eta+\mu-2\xi\mu-\eta\mu-4\xi\eta\mu), \\ \partial N_3 / \partial \xi &= -\frac{1}{16}(1+\eta)(1-\mu)(1-6\xi-\eta+4\xi\eta+\mu-2\xi\mu+\eta\mu+4\xi\eta\mu), \\ \partial N_4 / \partial \xi &= \frac{1}{16}(1+\eta)(1-\mu)(1+6\xi-\eta-4\xi\eta+\mu+2\xi\mu+\eta\mu-4\xi\eta\mu), \\ \partial N_5 / \partial \xi &= 0, \\ \partial N_6 / \partial \xi &= -\frac{1}{4}\xi(1-\eta)(1-\mu)(2+\eta+\eta\mu), & \partial N_7 / \partial \xi &= \frac{1}{8}(1-\eta^2)(1-\mu)(1-2\xi-\mu-2\xi\mu), \\ \partial N_8 / \partial \xi &= -\frac{1}{4}\xi(1+\eta)(1-\mu)(2-\eta-\eta\mu), & \partial N_9 / \partial \xi &= -\frac{1}{8}(1-\eta^2)(1-\mu)(1+2\xi-\mu+2\xi\mu), \\ \partial N_{10} / \partial \xi &= -\frac{1}{4}(1-\eta)(1-\mu^2), & \partial N_{11} / \partial \xi &= \frac{1}{4}(1-\eta)(1-\mu^2), \\ \partial N_{12} / \partial \xi &= \frac{1}{4}(1+\eta)(1-\mu^2), & \partial N_{13} / \partial \xi &= -\frac{1}{4}(1+\eta)(1-\mu^2), \end{aligned} \quad (19.23)$$

$$\begin{aligned}
\partial N_1/\partial \eta &= \frac{1}{16}(1-\xi)(1-\mu)(1+\xi+6\eta+4\xi\eta+\mu-\xi\mu+2\eta\mu+4\xi\eta\mu), \\
\partial N_2/\partial \eta &= \frac{1}{16}(1+\xi)(1-\mu)(1-\xi+6\eta-4\xi\eta+\mu+\xi\mu+2\eta\mu-4\xi\eta\mu), \\
\partial N_3/\partial \eta &= -\frac{1}{16}(1+\xi)(1-\mu)(1-\xi-6\eta+4\xi\eta+\mu+\xi\mu-2\eta\mu+4\xi\eta\mu), \\
\partial N_4/\partial \eta &= -\frac{1}{16}(1-\xi)(1-\mu)(1+\xi-6\eta-4\xi\eta+\mu-\xi\mu-2\eta\mu-4\xi\eta\mu), \\
\partial N_5/\partial \eta &= 0, \\
\partial N_6/\partial \eta &= -\frac{1}{8}(1-\xi^2)(1-\mu)(1+2\eta-\mu+2\eta\mu), \quad \partial N_7/\partial \eta = -\frac{1}{4}(1+\xi)\eta(1-\mu)(2-\xi-\xi\mu), \\
\partial N_8/\partial \eta &= \frac{1}{8}(1-\xi^2)(1-\mu)(1-2\eta-\mu-2\eta\mu), \quad \partial N_9/\partial \eta = -\frac{1}{4}(1-\xi)\eta(1-\mu)(2+\xi+\xi\mu), \\
\partial N_{10}/\partial \eta &= -\frac{1}{4}(1-\xi)(1-\mu^2), \quad \partial N_{11}/\partial \eta = -\frac{1}{4}(1+\xi)(1-\mu^2), \\
\partial N_{12}/\partial \eta &= \frac{1}{4}(1+\xi)(1-\mu^2), \quad \partial N_{13}/\partial \eta = \frac{1}{4}(1-\xi)(1-\mu^2),
\end{aligned} \tag{19.24}$$

$$\begin{aligned}
\partial N_1/\partial \mu &= \frac{1}{8}(1-\xi)(1-\eta)(1+\xi+\eta+2\mu+\xi\mu+\eta\mu+2\xi\eta\mu), \\
\partial N_2/\partial \mu &= \frac{1}{8}(1+\xi)(1-\eta)(1-\xi+\eta+2\mu-\xi\mu+\eta\mu-2\xi\eta\mu), \\
\partial N_3/\partial \mu &= \frac{1}{8}(1+\xi)(1+\eta)(1-\xi-\eta+2\mu-\xi\mu-\eta\mu+2\xi\eta\mu), \\
\partial N_4/\partial \mu &= \frac{1}{8}(1-\xi)(1+\eta)(1+\xi-\eta+2\mu+\xi\mu-\eta\mu-2\xi\eta\mu), \\
\partial N_5/\partial \mu &= 1/2 + \mu, \\
\partial N_6/\partial \mu &= -\frac{1}{4}(1-\xi^2)(1-\eta)(1+\eta\mu), \quad \partial N_7/\partial \mu = -\frac{1}{4}(1+\xi)(1-\eta^2)(1-\xi\mu), \\
\partial N_8/\partial \mu &= -\frac{1}{4}(1-\xi^2)(1+\eta)(1-\eta\mu), \quad \partial N_9/\partial \mu = -\frac{1}{4}(1-\xi)(1-\eta^2)(1+\xi\mu), \\
\partial N_{10}/\partial \mu &= -\frac{1}{2}(1-\xi)(1-\eta)\mu, \quad \partial N_{11}/\partial \mu = -\frac{1}{2}(1+\xi)(1-\eta)\mu, \\
\partial N_{12}/\partial \mu &= -\frac{1}{2}(1+\xi)(1+\eta)\mu, \quad \partial N_{13}/\partial \mu = -\frac{1}{2}(1-\xi)(1+\eta)\mu.
\end{aligned} \tag{19.25}$$

§19.6. The 14-Node Pyramid

§19.6.1. Shape Functions

The shape functions of the 14-node pyramid shown in Figure 19.6(a) are constructed by starting from the shape functions of the 13-node pyramid. A hierarchical shape function for node 14, located at the center of the base and with coordinates $\xi = \eta = 0, \mu = -1$, is

$$N_c = \frac{1}{2}(1-\xi^2)(1-\eta^2)(1-\mu), \tag{19.26}$$

This function vanishes at all nodes of the 13-node pyramid, is zero on all four apex faces, and takes the value one at the new node 14. The shape functions of nodes 1 through 4 and 6 through 9 have to be corrected by a multiple of (19.26) so they vanish at node 14 ($\xi = \eta = 0, \mu = -1$). The

complete shape function set is

$$\begin{aligned}
N_1 &= -\frac{1}{16}(1-\xi)(1-\eta)(1-\mu)(4+3\xi+3\eta+2\xi\eta+2\mu+\xi\mu+\eta\mu+2\xi\eta\mu)+\frac{1}{4}N_c, \\
N_2 &= -\frac{1}{16}(1+\xi)(1-\eta)(1-\mu)(4-3\xi+3\eta-2\xi\eta+2\mu-\xi\mu+\eta\mu-2\xi\eta\mu)+\frac{1}{4}N_c, \\
N_3 &= -\frac{1}{16}(1+\xi)(1+\eta)(1-\mu)(4-3\xi-3\eta+2\xi\eta+2\mu-\xi\mu-\eta\mu+2\xi\eta\mu)+\frac{1}{4}N_c, \\
N_4 &= -\frac{1}{16}(1-\xi)(1+\eta)(1-\mu)(4+3\xi-3\eta-2\xi\eta+2\mu+\xi\mu-\eta\mu-2\xi\eta\mu)+\frac{1}{4}N_c, \\
N_5 &= \frac{1}{2}\mu(1+\mu), \\
N_6 &= \frac{1}{8}(1-\xi^2)(1-\eta)(1-\mu)(2+\eta+\eta\mu)-\frac{1}{2}N_c, \\
N_7 &= \frac{1}{8}(1+\xi)(1-\eta^2)(1-\mu)(2-\xi-\xi\mu)-\frac{1}{2}N_c, \\
N_8 &= \frac{1}{8}(1-\xi^2)(1+\eta)(1-\mu)(2-\eta-\eta\mu)-\frac{1}{2}N_c, \\
N_9 &= \frac{1}{8}(1-\xi)(1-\eta^2)(1-\mu)(2+\xi+\xi\mu)-\frac{1}{2}N_c, \\
N_{10} &= \frac{1}{4}(1-\xi)(1-\eta)(1-\mu^2), \quad N_{11} = \frac{1}{4}(1+\xi)(1-\eta)(1-\mu^2), \\
N_{12} &= \frac{1}{4}(1+\xi)(1+\eta)(1-\mu^2), \quad N_{13} = \frac{1}{4}(1-\xi)(1+\eta)(1-\mu^2), \\
N_{14} &= N_c.
\end{aligned} \tag{19.27}$$

Because N_c vanishes on all apex faces, it is not necessary to recheck for compatibility with the 10-node tetrahedron.

§19.6.2. Natural Derivatives

The natural derivatives are obtained by adding the $\{\xi, \eta, \mu\}$ derivatives of N_c to the natural derivatives of Pyra13 given by (19.23) through (19.25). Defining for convenience

$$N_{c\xi} = \frac{\partial N_c}{\partial \xi} = -\xi(1-\eta^2)(1-\mu), \quad N_{c\eta} = \frac{\partial N_c}{\partial \eta} = -\eta(1-\xi^2)(1-\mu), \quad N_{c\mu} = \frac{\partial N_c}{\partial \mu} = -\frac{1}{2}(1-\xi^2)(1-\eta^2), \tag{19.28}$$

we get

$$\begin{aligned}
\partial N_1 / \partial \xi &= \frac{1}{16}(1-\eta)(1-\mu)(1+6\xi+\eta+4\xi\eta+\mu+2\xi\mu-\eta\mu+4\xi\eta\mu)+\frac{1}{4}N_{c\xi}, \\
\partial N_2 / \partial \xi &= -\frac{1}{16}(1-\eta)(1-\mu)(1-6\xi+\eta-4\xi\eta+\mu-2\xi\mu-\eta\mu-4\xi\eta\mu)+\frac{1}{4}N_{c\xi}, \\
\partial N_3 / \partial \xi &= -\frac{1}{16}(1+\eta)(1-\mu)(1-6\xi-\eta+4\xi\eta+\mu-2\xi\mu+\eta\mu+4\xi\eta\mu)+\frac{1}{4}N_{c\xi}, \\
\partial N_4 / \partial \xi &= \frac{1}{16}(1+\eta)(1-\mu)(1+6\xi-\eta-4\xi\eta+\mu+2\xi\mu+\eta\mu-4\xi\eta\mu)+\frac{1}{4}N_{c\xi}, \quad \partial N_5 / \partial \xi = 0, \\
\partial N_6 / \partial \xi &= -\frac{1}{4}\xi(1-\eta)(1-\mu)(2+\eta+\eta\mu)-\frac{1}{2}N_{c\xi}, \quad \partial N_7 / \partial \xi = \frac{1}{8}(1-\eta^2)(1-\mu)(1-2\xi-\mu-2\xi\mu)-\frac{1}{2}N_{c\xi}, \\
\partial N_8 / \partial \xi &= -\frac{1}{4}\xi(1+\eta)(1-\mu)(2-\eta-\eta\mu)-\frac{1}{2}N_{c\xi}, \quad \partial N_9 / \partial \xi = -\frac{1}{8}(1-\eta^2)(1-\mu)(1+2\xi-\mu+2\xi\mu)-\frac{1}{2}N_{c\xi}, \\
\partial N_{10} / \partial \xi &= -\frac{1}{4}(1-\eta)(1-\mu^2), \quad \partial N_{11} / \partial \xi = \frac{1}{4}(1-\eta)(1-\mu^2), \\
\partial N_{12} / \partial \xi &= \frac{1}{4}(1+\eta)(1-\mu^2), \quad \partial N_{13} / \partial \xi = -\frac{1}{4}(1+\eta)(1-\mu^2), \quad \partial N_{14} / \partial \xi = N_{c\xi}.
\end{aligned} \tag{19.29}$$

$$\begin{aligned}
\partial N_1/\partial \eta &= \frac{1}{16}(1-\xi)(1-\mu)(1+\xi+6\eta+4\xi\eta+\mu-\xi\mu+2\eta\mu+4\xi\eta\mu) + \frac{1}{4}N_{c\eta}, \\
\partial N_2/\partial \eta &= \frac{1}{16}(1+\xi)(1-\mu)(1-\xi+6\eta-4\xi\eta+\mu+\xi\mu+2\eta\mu-4\xi\eta\mu) + \frac{1}{4}N_{c\eta}, \\
\partial N_3/\partial \eta &= -\frac{1}{16}(1+\xi)(1-\mu)(1-\xi-6\eta+4\xi\eta+\mu+\xi\mu-2\eta\mu+4\xi\eta\mu) + \frac{1}{4}N_{c\eta}, \\
\partial N_4/\partial \eta &= -\frac{1}{16}(1-\xi)(1-\mu)(1+\xi-6\eta-4\xi\eta+\mu-\xi\mu-2\eta\mu-4\xi\eta\mu) + \frac{1}{4}N_{c\eta}, \quad \partial N_5/\partial \eta = 0, \\
\partial N_6/\partial \eta &= -\frac{1}{8}(1-\xi^2)(1-\mu)(1+2\eta-\mu+2\eta\mu) - \frac{1}{2}N_{c\eta}, \quad \partial N_7/\partial \eta = -\frac{1}{4}(1+\xi)\eta(1-\mu)(2-\xi-\xi\mu) - \frac{1}{2}N_{c\eta}, \\
\partial N_8/\partial \eta &= \frac{1}{8}(1-\xi^2)(1-\mu)(1-2\eta-\mu-2\eta\mu) - \frac{1}{2}N_{c\eta}, \quad \partial N_9/\partial \eta = -\frac{1}{4}(1-\xi)\eta(1-\mu)(2+\xi+\xi\mu) - \frac{1}{2}N_{c\eta}, \\
\partial N_{10}/\partial \eta &= -\frac{1}{4}(1-\xi)(1-\mu^2), \quad \partial N_{11}/\partial \eta = -\frac{1}{4}(1+\xi)(1-\mu^2), \\
\partial N_{12}/\partial \eta &= \frac{1}{4}(1+\xi)(1-\mu^2), \quad \partial N_{13}/\partial \eta = \frac{1}{4}(1-\xi)(1-\mu^2), \quad \partial N_{14}/\partial \eta = N_{c\eta}.
\end{aligned} \tag{19.30}$$

$$\begin{aligned}
\partial N_1/\partial \mu &= \frac{1}{8}(1-\xi)(1-\eta)(1+\xi+\eta+2\mu+\xi\mu+\eta\mu+2\xi\eta\mu) + \frac{1}{4}N_{c\mu}, \\
\partial N_2/\partial \mu &= \frac{1}{8}(1+\xi)(1-\eta)(1-\xi+\eta+2\mu-\xi\mu+\eta\mu-2\xi\eta\mu) + \frac{1}{4}N_{c\mu}, \\
\partial N_3/\partial \mu &= \frac{1}{8}(1+\xi)(1+\eta)(1-\xi-\eta+2\mu-\xi\mu-\eta\mu+2\xi\eta\mu) + \frac{1}{4}N_{c\mu}, \\
\partial N_4/\partial \mu &= \frac{1}{8}(1-\xi)(1+\eta)(1+\xi-\eta+2\mu+\xi\mu-\eta\mu-2\xi\eta\mu) + \frac{1}{4}N_{c\mu}, \quad \partial N_5/\partial \mu = 1/2 + \mu, \\
\partial N_6/\partial \mu &= -\frac{1}{4}(1-\xi^2)(1-\eta)(1+\eta\mu) - \frac{1}{2}N_{c\mu}, \quad \partial N_7/\partial \mu = -\frac{1}{4}(1+\xi)(1-\eta^2)(1-\xi\mu) - \frac{1}{2}N_{c\mu}, \\
\partial N_8/\partial \mu &= -\frac{1}{4}(1-\xi^2)(1+\eta)(1-\eta\mu) - \frac{1}{2}N_{c\mu}, \quad \partial N_9/\partial \mu = -\frac{1}{4}(1-\xi)(1-\eta^2)(1+\xi\mu) - \frac{1}{2}N_{c\mu}, \\
\partial N_{10}/\partial \mu &= -\frac{1}{2}(1-\xi)(1-\eta)\mu, \quad \partial N_{11}/\partial \mu = -\frac{1}{2}(1+\xi)(1-\eta)\mu, \\
\partial N_{12}/\partial \mu &= -\frac{1}{2}(1+\xi)(1+\eta)\mu, \quad \partial N_{13}/\partial \mu = -\frac{1}{2}(1-\xi)(1+\eta)\mu, \quad \partial N_{14}/\partial \mu = N_{c\mu}.
\end{aligned} \tag{19.31}$$

§19.6.3. *Local Node Mapping

The local numbering of the Pyra14 element has to be mapped as shown in Figure 19.6 for compatibility with Lagrangian solid elements of certain codes that follow an *IJK* scheme. This is easily done in the implementation of the shape functions by appropriate array re-indexing.

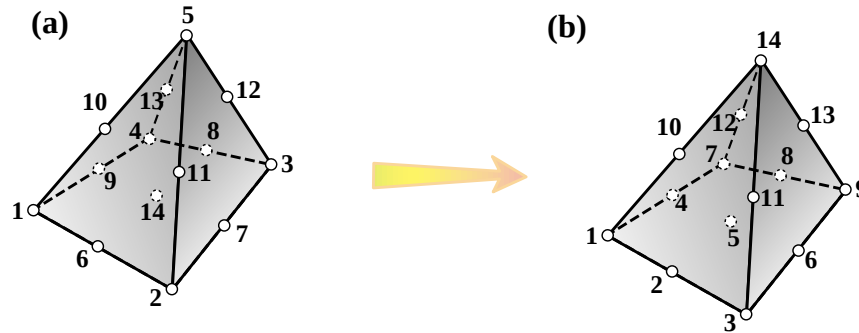


FIGURE 19.6. Mapping of local node numbering of 14-node pyramid.

§19.7. Numerical Quadrature

The minimum number of Gauss points to get a rank-sufficient stiffness for the Pyra5, Pyra13 and Pyra14 elements is 2, 6 and 6, respectively. The minimum number to get a full-rank mass matrix is 5, 13 and 14, respectively (that is, same as the number of nodes). These counts can be amply covered by the standard $2 \times 2 \times 2$ product rule for Pyra5 and the $3 \times 3 \times 3$ product rule for Pyra13

and Pyra14. However, it is possible to reduce the number of points by using rules customized to the pyramid geometry. New rules of this nature, developed with *Mathematica*, are presented here.

The new rules are “ $\{\xi, \mu\}$ symmetric” in the following sense: if $\{\xi_i = \alpha, \eta_i = \beta, \mu_i = \gamma\}$ is a Gauss point with weight w_i , the four locations $\{\pm\alpha, \pm\beta, \gamma\}$ are Gauss points with the same weight w_i . This grouping is called a 4-point star. It is appropriate for the pyramid geometry, which treats the height direction μ differently from $\{\xi, \eta\}$. A consequence is that Gauss points appear in multiples of 4 except if $\alpha = \beta = 0$, in which case they coalesce to one point on the apex axis.

Other constraints on the new rules are the usual ones in FEM work: all points must be inside the element, and all weights must be positive.

Table 19.1. The 1-point Pyramid Gauss Rule

Point i	Abscissa $\{\xi_i, \eta_i, \mu_i\}$	Weight w_i
1	$\{0, 0, -1/2\}$	128/27
Exact (reference pyramid): 1, μ , and any function odd in ξ or η , e.g. $\xi, \eta, \xi\eta, \xi\mu^2 \dots$		

§19.7.1. Rules with 1 and 5 Points

The simplest rule has one point at the centroid and is defined in Table 19.1. The exactness remark in Table 19.1 applies to the “reference pyramid” which has a flat rectangular base. The rule is appropriate for computation of volumes and first moments. As such it would be useful for consistent body loads and lumped mass of the low order element Pyra5.

The next useful rule has five points and is specified in Table 19.2. It is made up of a 4-point star plus one point located on the apex axis. It is recommended for the Pyra5 element in lieu of the standard $2 \times 2 \times 2$ product rule, since it gives rank-sufficient (and well conditioned) $\mathbf{K}$ and $\mathbf{M}$ with a 5:8 reduction in the number of integration points.

§19.7.2. Rules with 6 through 9 Points

Three more $\{\xi, \eta\}$ symmetric rules with 6, 8 and 9 points were constructed with the objective of using them for the stiffness matrix of Pyra13 and Pyra14. These are made up with one, two and two 4-point stars, respectively, complemented with two, zero and one apex-axis points, respectively. Although tests show that they provide a rank-sufficient $\mathbf{K}$, they are not recommended for the following reason.

When the 8-node serendipity iso-P quadrilateral stiffness is underintegrated by a 2×2 product Gauss rule, it is well known that the element develops a spurious zero-energy mode (ZEM) that has zero strains at the 4 Gauss points. That ZEM has two opposite sides “bulging parabolically” outward whereas the other two sides bulge inward.

The analogous pattern for the Pyra13 and Pyra14 elements is a in-out bulging base mode that propagates with a similar shape in the apex direction, vanishing at the apex. Under the 6-, 8-

Table 19.2. The 5-point Pyramid Gauss Rule

Point i	Abscissa $\{\xi_i, \eta_i, \mu_i\}$	Weight w_i
1	$\{-g_1, -g_1, g_2\}$	w_1
2	$\{g_1, -g_1, g_2\}$	w_1
3	$\{g_1, g_1, g_2\}$	w_1
4	$\{-g_1, g_1, g_2\}$	w_1
5	$\{0, 0, g_3\}$	w_2
$g_1 = (8/5)\sqrt{2/15} = 0.584237394672177188,$ $g_2 = -2/3, g_3 = 2/5, w_1 = 81/100, w_2 = 125/27.$		
Exact (reference pyramid): $1, \mu, \xi^2, \eta^2, \mu^2, \xi\eta, \xi\mu, \eta\mu, \xi^2\mu, \eta^2\mu,$ as well as any function odd in ξ or η , such as $\xi, \eta, \xi\eta, \xi\mu, \eta\mu \dots$		

Table 19.3. Pyra13-Pyra14 Deformational Stiffness Condition Numbers for Various Integration Rules

Quadrature rule	Pyra13 $C^d(\mathbf{K})$	Pyra14 $C^d(\mathbf{K})$
new: 6 points	83324.5	∞
new: 8 points	7784220	6569760
standard $2 \times 2 \times 2$	∞	∞
new: 9 points	31370.7	26643.3
new: 13 points	480.14	416.808
standard $3 \times 3 \times 3$	104.01	99.995
Elements have flat square bases 2×2 , height $h = 3$, isotropic material with $\nu = 0$.		

and 9-point integration rules this pattern becomes a low energy mode (LEM), characterized by an eigenvector $\mathbf{v}_{LEM}$. This is not actually a spurious mode but it has a very low associated eigenvalue $\lambda_{min} = \lambda_{LEM} = \mathbf{v}_{LEM}^T \mathbf{K} \mathbf{v}_{LEM} > 0$. Here the designation λ_{min} ignores the six rigid body modes (RBMs) which have zero associated eigenvalues. The ratio

$$C^d(\mathbf{K}) = \lambda_{max}/\lambda_{min} \quad (\text{with 6 RBMs excluded}) \quad (19.32)$$

is called the deformational stiffness condition number. Table 19.3 collects the $C^d(\mathbf{K})$ computed for the new rules as well as the standard $2 \times 2 \times 2$ and $3 \times 3 \times 3$ product rules, for an element of good aspect ratio. As expected, under the $2 \times 2 \times 2$ product rule the LEM becomes a ZEM (spurious mode) and $C^d = \infty$. Of the new pyramid-customized rules only the 13-point quadrature compares well with the more expensive $3 \times 3 \times 3$ product rule. Consequently rules with 6, 8 and 9 points are not recommended for the Pyra13 and Pyra14 elements.

Table 19.4. The 13-point Pyramid Gauss Rule

Point i	Abscissa $\{\xi_i, \eta_i, \mu_i\}$	Weight w_i
1	$\{-g_1, -g_1, g_4\}$	w_1
2	$\{g_1, -g_1, g_4\}$	w_1
3	$\{g_1, g_1, g_4\}$	w_1
4	$\{-g_1, g_1, g_4\}$	w_1
5	$\{-g_2, 0, g_5\}$	w_2
6	$\{g_2, 0, g_5\}$	w_2
7	$\{0, -g_2, g_5\}$	w_2
8	$\{0, g_2, g_5\}$	w_2
9	$\{0, 0, g_6\}$	w_3
10	$\{-g_3, -g_3, g_7\}$	w_4
11	$\{g_3, -g_3, g_7\}$	w_4
12	$\{g_3, g_3, g_7\}$	w_4
13	$\{-g_3, g_3, g_7\}$	w_4

$$g_1 = (7/8)\sqrt{35/59} = 0.673931986207731726,$$

$$g_2 = (224/37)\sqrt{336633710/33088740423} = 0.610639618865075532,$$

$$g_3 = (1/56)\sqrt{37043/35} = 0.580939660561084423,$$

$$g_4 = -1/7 = -0.1428571428571428571,$$

$$g_5 = -9/28 = -0.321428571428571429,$$

$$g_6 = 1490761/2842826 = 0.524394036075370072,$$

$$g_7 = -127/153 = -0.830065359477124183,$$

$$w_1 = 170569/331200 = 0.515003019323671498,$$

$$w_2 = 276710106577408/1075923777052725 = 0.2571837452420646589,$$

$$w_3 = 10663383340655070643544192/4310170528879365193704375$$

$$= 2.474004977113405936,$$

$$w_4 = 12827693806929/30577384040000 = 0.419515737191525950.$$

Exact (reference pyramid): $1, \mu, \xi^2, \eta^2, \mu^2, \xi^2\mu, \eta^2\mu, \mu^3, \xi^2\eta^2, \xi^2\mu^2, \eta^2\mu^2, \xi^2\mu^3, \eta^2\mu^3, \xi^2\eta^2\mu, \xi^2\eta^2\mu^2$, and any function odd in ξ or η .

§19.7.3. A Rule with 13 Points

The 13-point rule mentioned above is defined in Table 19.4. It is made up of three 4-point stars plus one point on the apex axis. As noted, it is recommended for the computation of the stiffness matrix of Pyra13 and Pyra14. For the mass matrix of Pyra14 it falls short of giving full rank since 14 (=42/3) points would be needed, but that deficiency is not usually important* in element assemblies.

One effect of the 13-point rule should be noted. Consistent node forces for face pressures or body loads for Pyra13 and Pyra14 are not exactly the same as those obtained if the $3 \times 3 \times 3$ rules

* Unless the mass matrix is inverted by itself, which rarely happens outside of explicit dynamics.

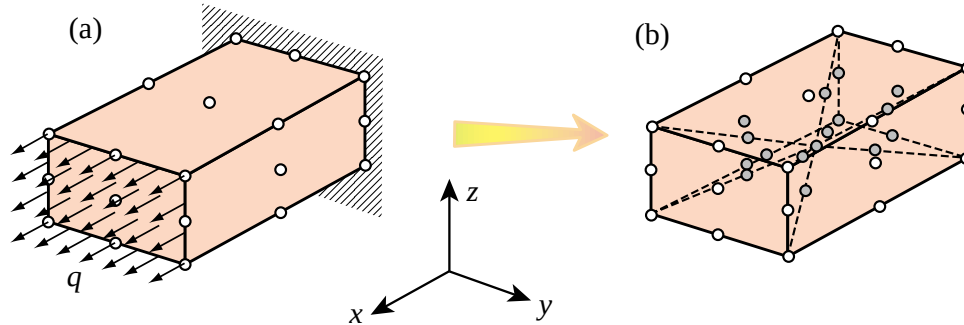


FIGURE 19.7. Patch test on cubic volume: (a) prescribed force patch test illustrated on a Pyra14 assembly; (b) division of volume into six Pyra14 elements.

is used. The reason is that some terms such as $\xi^2 \eta^2 \mu^4$, which appear when the variation of the Jacobian over faces or the volume is considered, are not exactly integrated by the 13-point rule.

The difference is not very significant: about 1 to 5% for uniform face pressures, but the discrepancy it should be taken into account when running tests with few elements. In particular, *patch tests involving Py13 and Pyra14 should be run using the $3 \times 3 \times 3$ rule for the pyramids*. This recommendation is discussed further in the next section.

§19.8. Patch Tests

A series of patch tests were run to verify the formulation of the new Pyra13 and Pyra14 elements as well as their mixability with 10-node tetrahedra. The tests were carried out as follows.

1. A cube-like prismatic volume was divided into six pyramidal volumes by injecting a center node, as depicted in Figure 19.7. The pyramid bases are the volume faces and the center node is the apex of all pyramids. Figure 19.8 illustrates the volume decomposition in more detail.
2. Each pyramid volume was treated as either a single pyramid element, or as a combination of two 10-node tetrahedra (Tet10) elements, as shown in Figure 19.8 for one of the pyramids. The two-tet configuration was implemented as a pyramid macroelement for convenience in mix-and-match tests. The Tet10 elements were integrated exactly using a 4-point Gauss rule. The replacement was done randomly.
3. The cube was subjected to prescribed displacement and prescribed force patch tests for uniform stress-strain states. Figure 19.7 pictures a force patch test in which one face of the cubic volume is subject to a uniform normal surface traction q while the opposite face is fixed (but allowed to contract or expand); the result should be a uniform $\sigma_{xx} = q$ over all elements.

Tests with pyramid-only configurations were successful for the Pyra5, Pyra13 and Pyra14 models. Mixtures of Tet10 elements with Pyra13 and Pyra14 elements initially failed the test by about 25%. This was traced back to two error sources:

- (i) The face-bubble corrective shape functions (19) were missing decay-away-from-face functions. This was responsible for about 22% error, which was fixed when the proper decay functions were incorporated into the shape function sets (19.20) and (19.27).
- (ii) The element integration rule effect discussed in §7.3 was responsible for the remaining error of about 2-3%. This was eliminated by using the $3 \times 3 \times 3$ rule and the correct consistent

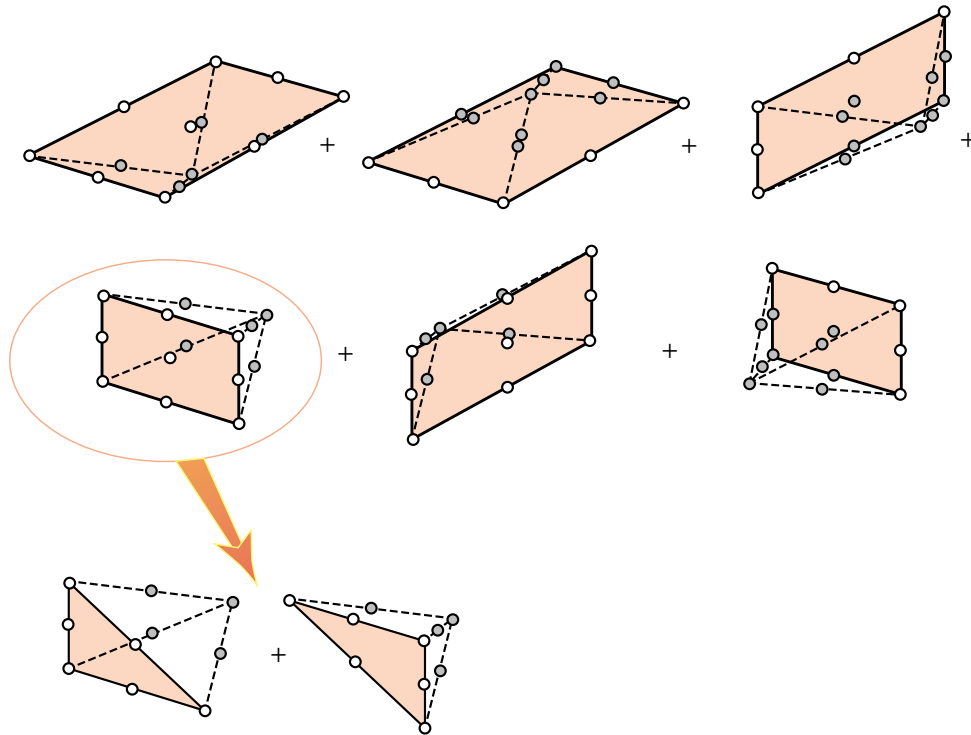


FIGURE 19.8. Pyra14 elements resulting from the patch test decomposition of Figure 19.7(b). One of them is shown replaced by a two-Tet10 macroelement to check Pyra14-Tet10 mixability. These replacements were done randomly.

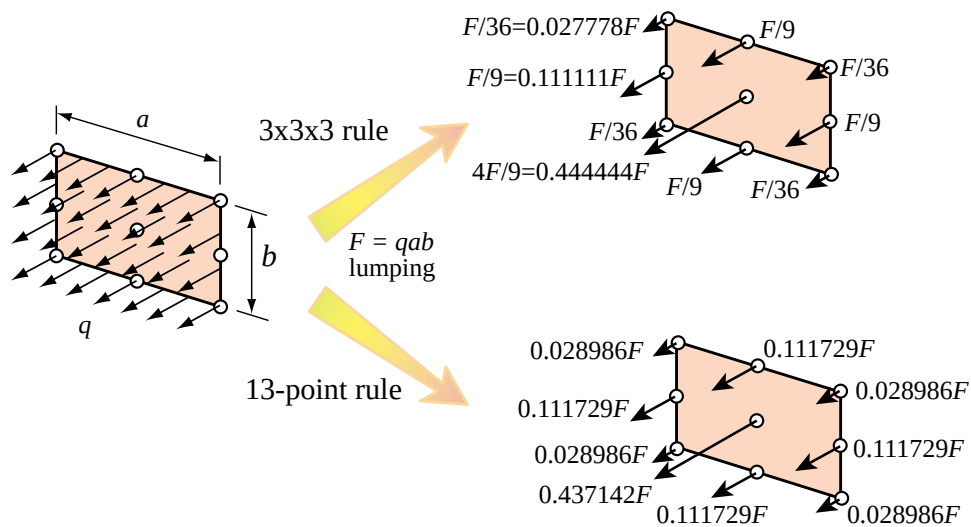


FIGURE 19.9. Consistent forces for pressure over a 9-node quadrilateral face.

node forces for face pressures.

The last effect is further illustrated in Figure 19.9. This shows the consistent node forces associated to a uniform normal surface traction q acting over a $a \times b$ 9-node rectangular face, which corresponds to a Pyra14 patch test of the type depicted in Figure AFEMNineteen:fig:PatchTestOnCubicVolume(a).

For the $3 \times 3 \times 3$ rule the consistent forces follows the well know proportion 16 : 4 : 1 for center:midpoints:corners. If the 13-point rule is used, the consistent forces change by 1-5%. Similar effects were observed in Pyra13 and apex faces.

21

Element Morphing

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§21.1. Introduction

The subject of this Chapter is a finite element fabrication, testing and optimization method called *morphing*. The word generically means “to undergo transformation” (Merriam-Webster dictionary; From Gr. morphos: shape.)

The term is nowadays used in image processing: mapping the image of an object into that of another one by means of computer software, a process sometimes endowed with animation. But the idea preceeds the computer as illustrated by well known woodcuts of M. C. Escher, e.g., Figure 21.1.

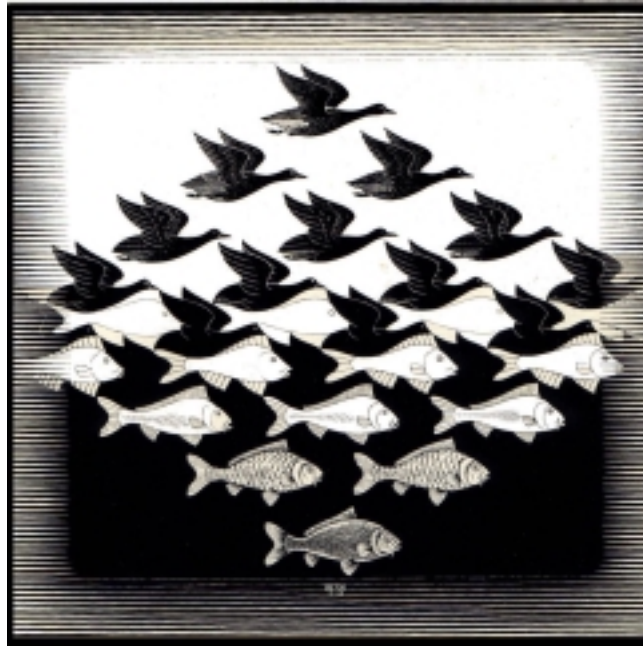


FIGURE 21.1. Escher’s Sky and Water: morphing as shape transformation.

In the finite element field, the author introduced the term *morphing* to identify the mapping of a *source element* or *macroelement* to a *target element* through kinematic constraints. (A macroelement is an assembly of few elements that may be view as a repeating mesh unit.) The target element has a smaller number of degrees of freedom and often a smaller dimensionality.

The process was originally called *retrofitting* of a parent element to a child element in []. This terminology has been changed because “child element” has been widely adopted since in adaptive mesh refinement processes.

The feedback process, which may be called “demorphing” fits appropriately under case 2 below.

What are the reasons for morphing? Improving element performance is the dominant motive. Two cases may be considered:

Improving the source. The source model contains adjustable parameters that survive in the target element. Improving the performance of the latter may yields values for those parameters, which are fed back.

Deriving special elements. The source element is fixed. Morphing is used to produced functionally specialized instances; for example a solid shell element from a general isoparametric brick. The

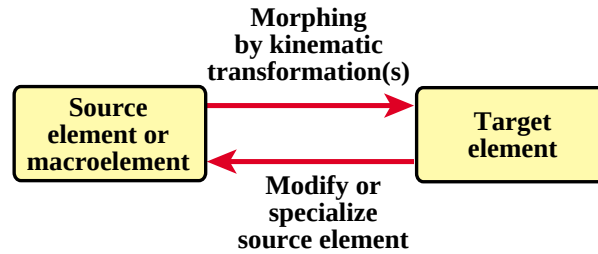


FIGURE 21.2. Element morphing and demorphing processes.

transformation may introduced free parameters in the target. These parameters are chosen to improve the performance of the target element. In turn this may pose constraints in the transformation. The process is best explained through specific examples that convey the scope of what can be accomplished.

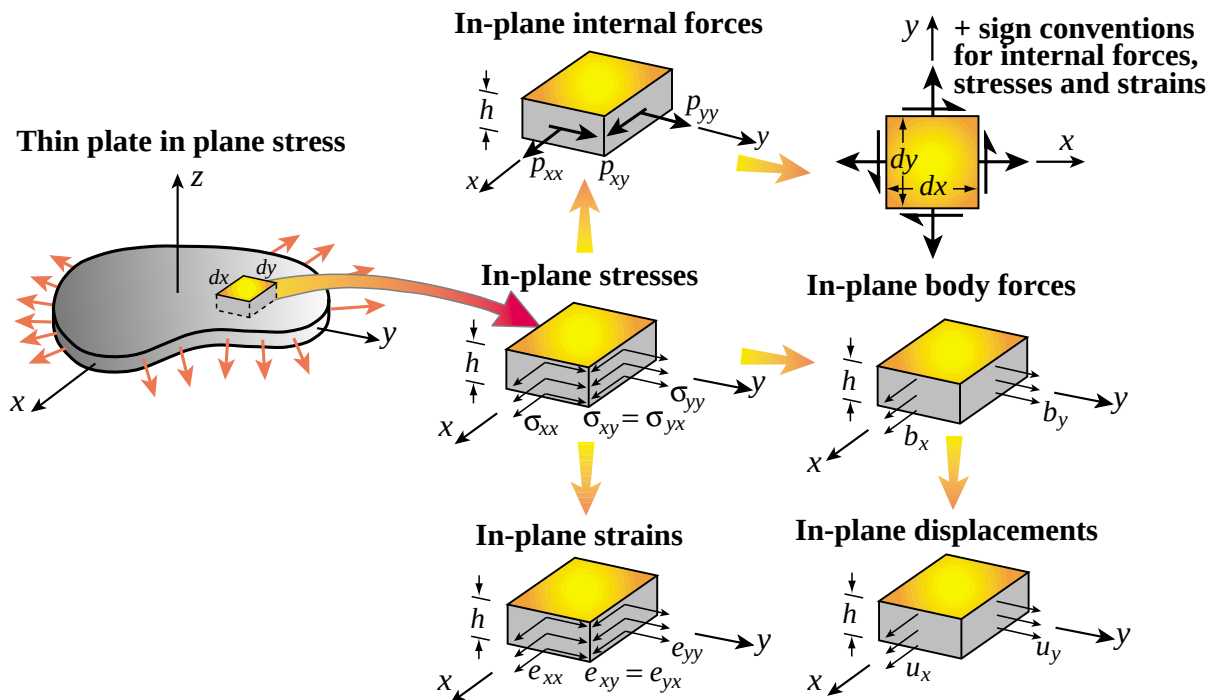


FIGURE 21.3. Notation for displacements, strains, stresses and forces in a thin plate in plane stress state.

§21.2. Plate in Plane Stress

Several of the following examples pertain to a thin plate in a plane stress (also known as membrane) state. Notation and sign conventions for displacements, strains, stresses and forces in this mathematical model are collected in Figure 21.3. Assumptions associated with plane stress are discussed in detail in Chapter 14 of the Introduction to Finite Element Notes. Here we simply recall

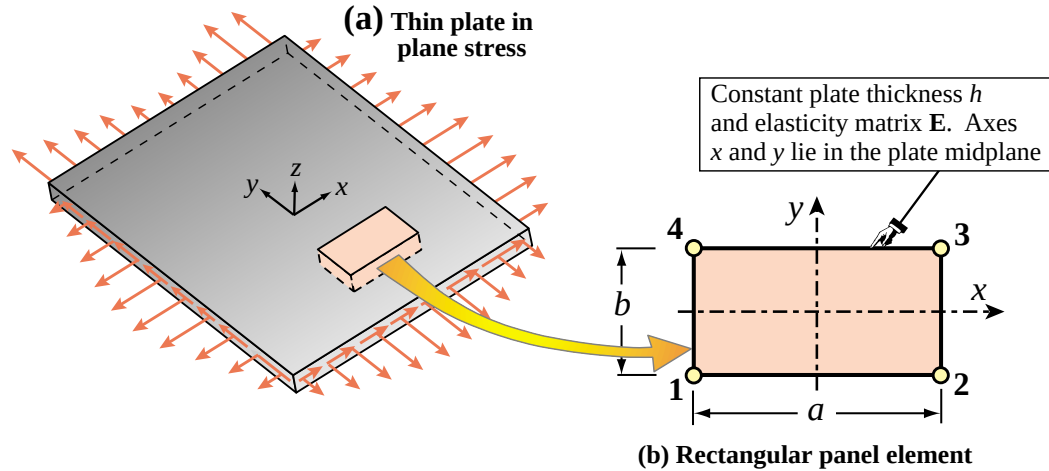


FIGURE 21.4. The four-node, 8DOF rectangular panel modeling a thin plate in plane stress.

the in-plane governing equations

$$\begin{aligned}
 \mathbf{e} &= \mathbf{D} \mathbf{u} \quad \text{or} \quad \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \\
 \boldsymbol{\sigma} &= \mathbf{E} \mathbf{e} \quad \text{or} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \\
 \mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} &= \mathbf{0} \quad \text{or} \quad \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
 \end{aligned} \tag{21.1}$$

As regards stresses in the z direction, one assumes $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. The strains $\{e_{zz}, e_{xz}, e_{yz}\}$ are determined from the three-dimensional strain-stress (compliance) constitutive equations. If the plate material is isotropic with elastic modulus E and Poisson's ratio ν , the matrix constitutive equation $\boldsymbol{\sigma} = \mathbf{E} \mathbf{e}$ and its inverse $\mathbf{e} = \mathbf{E}^{-1} \boldsymbol{\sigma}$ become

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \quad \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}. \tag{21.2}$$

For the isotropic case $e_{xz} = e_{yz} = 0$, whereas the thickness strain is

$$e_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}) = -\frac{\nu}{1-\nu} (e_{xx} + e_{yy}). \tag{21.3}$$

§21.3. Source Element 1: 4-node Rectangular Panel

The 4-node, 8-DOF rectangular element modeling a elastic thin plate in plate stress, sketched in Figure 21.4(a), is used as source element for the following examples. The element is called *rectangular panel* for brevity. As [] narrates, this is one of the two oldest continuum structural

elements, having been developed in the 1950s to model wing skin panels. This geometry is simple enough to be amenable to complete analytical development, but it is not trivial. Thus it provides a good illustration of the morphing process as well as of its strengths and pitfalls.

§21.3.1. Rectangular Panel Template

The rectangular panel geometry is defined in Figure 21.4(b). Axes x and y are aligned with the sides for convenience. The element has constant thickness h and elasticity matrix $\mathbf{E}$. The 8 displacement degrees of freedom and associated node forces are collected in the vectors

$$\begin{aligned}\mathbf{u}_{RP} &= [u_{x1} \ u_{y1} \ u_{x2} \ u_{y2} \ u_{x3} \ u_{y3} \ u_{x4} \ u_{y4}]^T, \\ \mathbf{f}_{RP} &= [f_{x1} \ f_{y1} \ f_{x2} \ f_{y2} \ f_{x3} \ f_{y3} \ f_{x4} \ f_{y4}]^T.\end{aligned}\quad (21.4)$$

The element stiffness matrix may be expressed in template form as [refs] the sum of a *basic stiffness* $\mathbf{K}_b$ and a *higher order stiffness* $\mathbf{K}_h$:

$$\begin{aligned}\mathbf{K} &= \mathbf{K}_b + \mathbf{K}_h = V \mathbf{H}_c^T \mathbf{E} \mathbf{H}_c + V \mathbf{H}_h^T \mathbf{W}^T \mathbf{R} \mathbf{W} \mathbf{H}_h, \\ \mathbf{H}_c &= \frac{1}{2ab} \begin{bmatrix} -b & 0 & b & 0 & b & 0 & -b & 0 \\ 0 & -a & 0 & -a & 0 & a & 0 & a \\ -a & -b & -a & b & a & b & a & -b \end{bmatrix}, \\ \mathbf{H}_h &= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}, \\ \mathbf{W} &= \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix}, \quad V = abh.\end{aligned}\quad (21.5)$$

Here R_{11} , R_{12} and R_{22} are three free parameters with dimension of elastic moduli. The list $\{R_{11}, R_{12}, R_{22}\}$ is called the *template signature*. All possible elements of this type that pass the Individual Patch Test (IET) of Bergan and Hanssen are instances of (21.5). Note that the basic stiffness $\mathbf{K}_b$ is the same for all possible elements whereas the higher order stiffness $\mathbf{K}_h$ depends on the signature $\{R_{11}, R_{12}, R_{22}\}$. The former specifies the element response to rigid body modes and constant strain states, whereas the latter characterizes the response to bending deformations. A particular template instance is defined by specifying its signature. Hence the only difference between elements of this type is their response to bending modes.

§21.3.2. Morphing to Bar

The morphing of the rectangular panel to a 2-node, prismatic, x -aligned bar element is diagrammed in Figure 21.5. The x and y direction are called the *axial* and *transverse* directions, respectively. The material is assumed *isotropic*, obeying the constitutive equations (21.2). The target is the 2-node bar element shown in Figure 21.5(d), with node displacements and forces

$$\mathbf{u}_{bar} = \begin{bmatrix} u_{xi} \\ u_{xj} \end{bmatrix}, \quad \mathbf{f}_{bar} = \begin{bmatrix} f_{xi} \\ f_{xj} \end{bmatrix}.\quad (21.6)$$

The morphing process is carried out in two steps for convenience. First, the panel is assumed to deform symmetrically with respect to axes x and y placed along the rectangle medians, as sketched

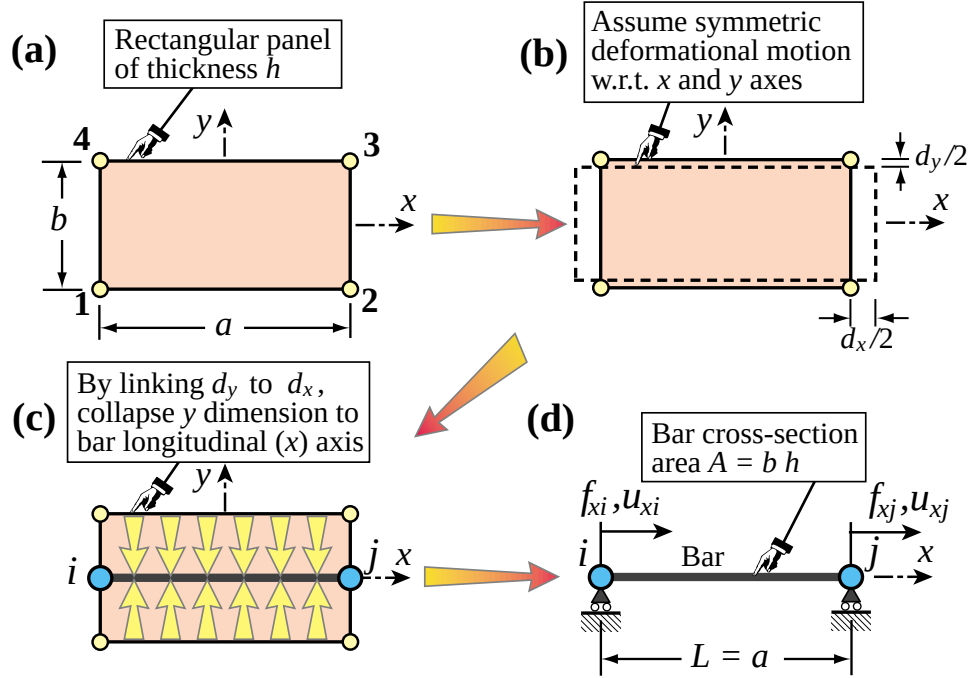


FIGURE 21.5. Morphing of the rectangular panel of Figure 21.4 (source element) to a 2-node prismatic bar element in the axial x direction.

in Figure 21.5(b). The axial and transverse elongations are called d_x and d_y , respectively, collected in array $\mathbf{d}_{RP}$. By inspection $u_{x1} = u_{x4} = -\frac{1}{2}d_x$, $u_{x2} = u_{x3} = \frac{1}{2}d_x$, $u_{y1} = u_{y2} = -\frac{1}{2}d_y$ and $u_{y3} = u_{y4} = \frac{1}{2}d_y$. Those relations are collected in the transformation

$$\mathbf{u}_{RP} = \mathbf{T}_d \mathbf{d}_{RP} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}^T \mathbf{d}_{RP} \quad (21.7)$$

Applying (21.7) to the panel gives the deformational stiffness equations

$$\frac{E b h}{1 - \nu^2} \begin{bmatrix} \frac{1}{a} & \frac{\nu}{b} \\ \frac{\nu}{b} & \frac{a}{b^2} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} f_{x2} + f_{x3} - f_{x1} - f_{x4} \\ f_{y3} + f_{y4} - f_{y1} - f_{y2} \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}. \quad (21.8)$$

Note that the higher order stiffness parameters $\{R_{11}, R_{12}, R_{22}\}$ are gone from (21.8) because $\mathbf{H} \mathbf{T}_d = \mathbf{0}$. Next, to get rid of the y dimension it is necessary to link d_y and d_x , bringing the 2D nature of the problem into play. If the y motion is unrestrained, Poisson's effect says that $d_y = -\nu b d_x / a$, which may also be obtained by solving the second of (21.8) for $f_y = 0$. On the other hand, $d_y = 0$ if that motion is precluded. The degree of transverse restraint can be parametrized by taking $d_y = -(1 - \alpha) \nu b d_x / a$, where α varies from 0 for unconstrained to 1 for fully constrained. Grouping that relation with $d_x = u_{xj} - u_{xi}$ gives the transformation to the bar freedoms:

$$\mathbf{d}_{RP} = \mathbf{T}_b \mathbf{u}_{bar} = \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \frac{(1 - \alpha) \nu b}{a} & -\frac{(1 - \alpha) \nu b}{a} \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{xj} \end{bmatrix}. \quad (21.9)$$

Applying (21.9) to (21.8) and replacing $b h \rightarrow A$ and $a \rightarrow L$ to pass to the more usual bar notation, gives the bar stiffness equations as

$$\frac{\hat{E} A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yj} \end{bmatrix} = \begin{bmatrix} -f_x + \frac{(1-\alpha)\nu b}{L} f_y \\ f_x - \frac{(1-\alpha)\nu b}{L} f_y \end{bmatrix} = \begin{bmatrix} f_{xi} \\ f_{xj} \end{bmatrix}, \quad (21.10)$$

in which

$$\hat{E} = \frac{1 - (1 - \alpha^2)\nu^2}{1 - \nu^2} E. \quad (21.11)$$

is an effective axial modulus that reduces to E if either $\nu = 0$ or $\alpha = 0$.

If the transformations in (21.7) and (21.9) are combined one gets

$$\mathbf{T}_{Rb} = \mathbf{T}_{Rd} \mathbf{T}_{db} = \frac{1}{2} \begin{bmatrix} 1 & -\frac{(1-\alpha)\nu b}{a} & -1 & -\frac{(1-\alpha)\nu b}{a} & -1 & \frac{(1-\alpha)\nu b}{a} & 1 & \frac{(1-\alpha)\nu b}{a} \\ 1 & \frac{(1-\alpha)\nu b}{a} & -1 & \frac{(1-\alpha)\nu b}{a} & -1 & -\frac{(1-\alpha)\nu b}{a} & 1 & -\frac{(1-\alpha)\nu b}{a} \end{bmatrix}^T. \quad (21.12)$$

Applying $\mathbf{T}_{Rb}$ to the panel produces (21.10) in one shot, but the two-step morphing process is more illuminating. One could add rigid body motions of $u_x = \frac{1}{2}$ and $u_y = \frac{1}{2}$ to the 2 columns of $\mathbf{T}_{Rb}$, respectively, to produce more zero entries but the morphed bar equation would be the same.

The corresponding process for the non-isotropic case, which is not covered here, is more involved unless material tensor symmetries allow symmetries to be introduced in the first step. Otherwise one has to start with uniform stress assumptions, convert to strains, integrate for displacements and evaluate at nodes.

§21.3.3. Morphing to Simply Supported Beam

We next consider morphing to a prismatic, 2-node, simply-supported, x aligned beam element as diagrammed in Figure 21.6. The material is assumed to be isotropic. The target is the 2-node Bernoulli-Euler beam element shown in Figure 21.6(d), with node displacements and forces

$$\mathbf{u}_{beam} = \begin{bmatrix} \theta_{zi} \\ \theta_{zj} \end{bmatrix}, \quad \mathbf{f}_{beam} = \begin{bmatrix} M_{zi} \\ M_{zj} \end{bmatrix}. \quad (21.13)$$

Here θ_z is the cross section rotation about z , positive CCW and M_z the corresponding generalized force. The element can take only a constant bending moment. As in the previous example the process is carried out in two steps. First the deformational motion sketched in Figure 21.6(b) is assumed. This motion is symmetric about y and antisymmetric about x . Deformational displacements d_x and d_y are defined in that figure. The resulting freedom transformation is

$$\mathbf{u}_{RP} = \mathbf{T}_d \mathbf{d}_{RPbend} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} d_x \\ d_y \end{bmatrix}. \quad (21.14)$$

The transformed stiffness equations are

$$\frac{b h R_{11}}{a} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} f_{x2} - f_{x3} - f_{x1} + f_{x4} \\ f_{y1} + f_{y2} + f_{y3} + f_{y4} \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}. \quad (21.15)$$

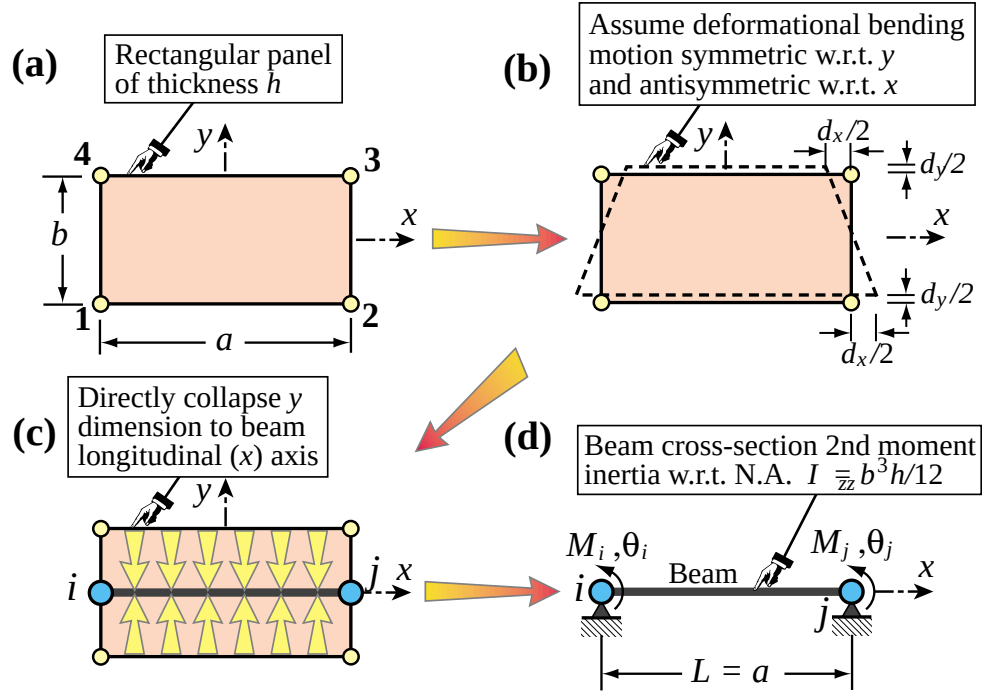


FIGURE 21.6. Morphing of the rectangular panel of Figure 21.4 (source element) to a 2-node, prismatic, simply-supported Bernoulli-Euler beam element in the axial x direction.

Although the transverse (y) motion pictured in Figure 21.6(b) is deformational, arising from Poisson's ratio effects, it appears as a y rigid body to the panel freedoms since $u_{y1} = u_{y2} = u_{y3} = u_{y4} = \frac{1}{2}d_y$. This explains the disappearance of d_y in (21.15). The x deformation d_x may be expressed in term of beam rotational DOF as $d_x/2 = (\theta_{zj} - \theta_{zi}) b/2$, or in matrix form

$$\mathbf{d}_{SSbeam} = [d_x] = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_{zi} \\ \theta_{zj} \end{bmatrix} = \mathbf{T}_{db} \boldsymbol{\theta}. \quad (21.16)$$

Transforming (21.15) as per (21.16), and setting $\frac{1}{12}b^3 h \rightarrow I_{zz}$ and $a \rightarrow L$ to pass to standard beam notation, yields

$$\frac{4R_{11} I_{zz}}{3L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_{xi} \\ \theta_{yj} \end{bmatrix} = \frac{1}{2}b \begin{bmatrix} f_{x1} - f_{x2} + f_{x3} - f_{x4} \\ -f_{x1} + f_{x2} - f_{x3} + f_{x4} \end{bmatrix} = \begin{bmatrix} M_{zi} \\ M_{zj} \end{bmatrix}. \quad (21.17)$$

To make (21.17) agree with the correct Bernoulli-Euler beam stiffness equations, it is necessary to take

$$R_{11} = \frac{1}{3}E. \quad (21.18)$$

This result was established in [,] with other methods. The anisotropic case is treated there.

§21.3.4. Morphing to Spar

We next consider morphing to a prismatic, 2-node, x aligned spar element as diagrammed in Figure 21.7. The material is assumed to be isotropic. The target is the 2-node spar element shown

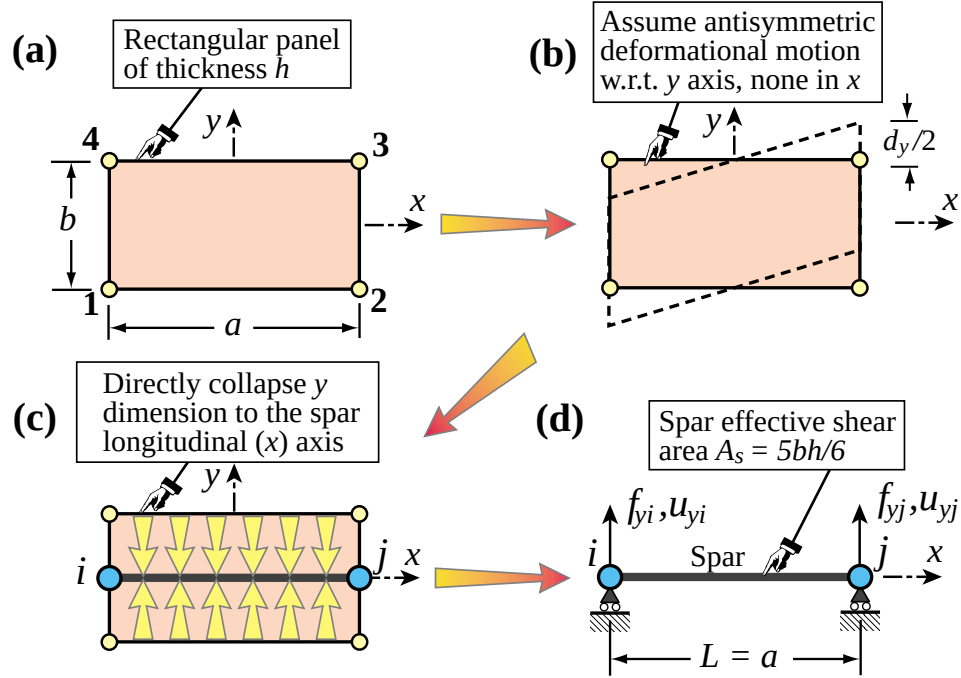


FIGURE 21.7. Morphing of the rectangular panel of Figure 21.4 (source element) to a 2-node, prismatic, spar element in the axial x direction.

in Figure 21.6(d), with node displacements and forces

$$\mathbf{u}_{spar} = \begin{bmatrix} u_{yi} \\ u_{yj} \end{bmatrix}, \quad \mathbf{f}_{spar} = \begin{bmatrix} f_{yi} \\ f_{yj} \end{bmatrix}. \quad (21.19)$$

The spar element can take only a constant shear force. As in the previous 2 examples the process is carried out in two steps. First the deformational motion sketched in Figure 21.7(b) is assumed. This motion is antisymmetric about y and zero along x , since the element deforms only in shear. The deformational displacement d_y is defined in that figure. The resulting freedom transformation is

$$\mathbf{u}_{RP} = \mathbf{T}_d \mathbf{d}_{RPspar} = \frac{1}{2} [0 \quad -1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad -1]^T [d_y]. \quad (21.20)$$

The transformed stiffness equations are

$$\frac{E b}{2(1 + \nu)a} [1] [d_y] = \frac{1}{2} [f_{y2} + f_{y3} - f_{y1} - f_{y4}] = [f_x]. \quad (21.21)$$

The deformation d_y may be expressed in term of spar translations

$$[d_y] = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_{yi} \\ u_{yj} \end{bmatrix} = \mathbf{T}_{ds} \mathbf{u}_{spar}. \quad (21.22)$$

Applying this transformation, and passing to the usual spar element notation: $E/(2(1 + \nu)) \rightarrow G$, $b h \rightarrow A$, $a \rightarrow L$, gives

$$\frac{G A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{yi} \\ u_{yj} \end{bmatrix} = \begin{bmatrix} f_{yi} \\ f_{yj} \end{bmatrix}. \quad (21.23)$$

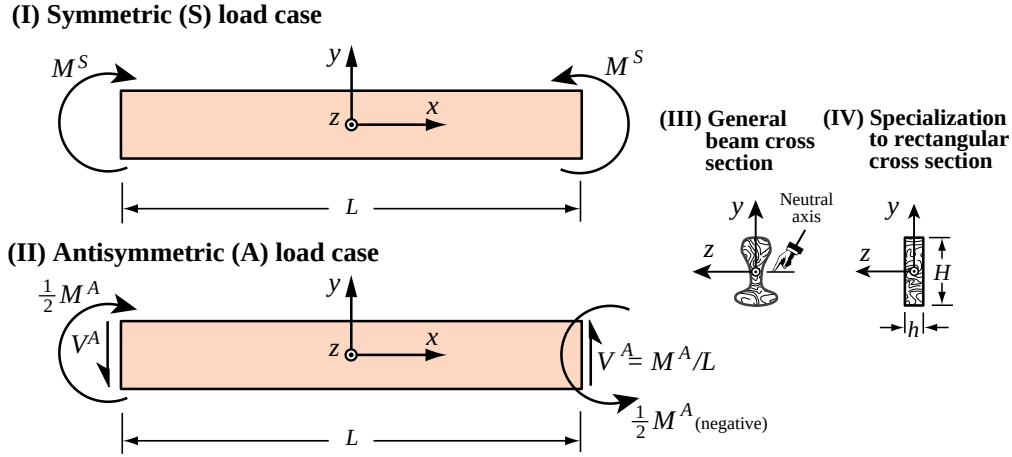


FIGURE 21.8. Benchmark beam solution used for advanced morphing examples.

The result (21.23) agrees with the spar stiffness equations derived with Mechanics of Materials methods (for example, in Chapter 5 of the Introduction to Finite Elements Notes) except for one detail: the transverse panel area $A = bh$ should be replaced by the effective shear area. For a narrow rectangle with $h \ll b$, $A_s = 5bh/6$. This discrepancy does not reflect, however, a flaw in the source element. The parabolic shear stress distribution that produces that A_s assumes that σ_{xy} vanishes at the top and bottom spar surfaces $y = \pm \frac{1}{2}b$. This will not generally be the case when the panel is used in a 2D mesh.

§21.4. Benchmark Plane Beam Solutions

To support more advanced morphing examples it is convenient to have a benchmark beam solution that is exact in the sense of plane stress isotropic elasticity, as well as that provided by the shear-flexible Timoshenko beam model. The problem is illustrated in Figure 21.8. The beam has span L and prismatic cross section. Axes $\{x, y, z\}$ are chosen as shown. Equations are initially derived for a general cross section sketched in Figure 21.8(III) and later specialized to the rectangular cross section of height H and width h illustrated in Figure 21.8(IV). For a general cross section, the moment of inertia with respect to the neutral axis z is I_{zz} , the cross section area is A . The internal bending moment is M_z and the transverse (resultant) shear force is V_y , with the usual sign conventions.

The material is isotropic with elastic modulus E , shear modulus G and Poisson's ratio ν and shear modulus $G = E/(2(1 + \nu))$. The in-plane displacements are $\{u_x, u_y\}$ and the infinitesimal rotation about z is $\theta_z = \frac{1}{2}(\partial u_y/\partial x - \partial u_x/\partial y)$. The effective shear area is $A_s = k_s A$, in which k_s is Timoshenko's static shear correction coefficient defined by¹

$$U_S \stackrel{\text{def}}{=} \frac{1}{2} \int_V \frac{V_y^2}{G A_s} dV = \frac{1}{2} \int dx \int \frac{V_y^2}{G k_s A} dA, \quad (21.24)$$

in which U_S is the internal energy due to transverse shear stresses. The following symbols are introduced for later use:

$$r_{Gz}^2 = \frac{I_{zz}}{A}, \quad \Phi = \frac{12 E I_{zz}}{G A_s L^2} = \frac{12 E r_{Gz}^2}{G k_s L^2} = \frac{24(1 + \nu) r_{Gz}^2}{k_s L^2}. \quad (21.25)$$

¹ Some authors use the inverse value: e.g., $k = 1/k_s$ in [34, p. 177] and $m = 1/k_s$ in [18, p. 21].

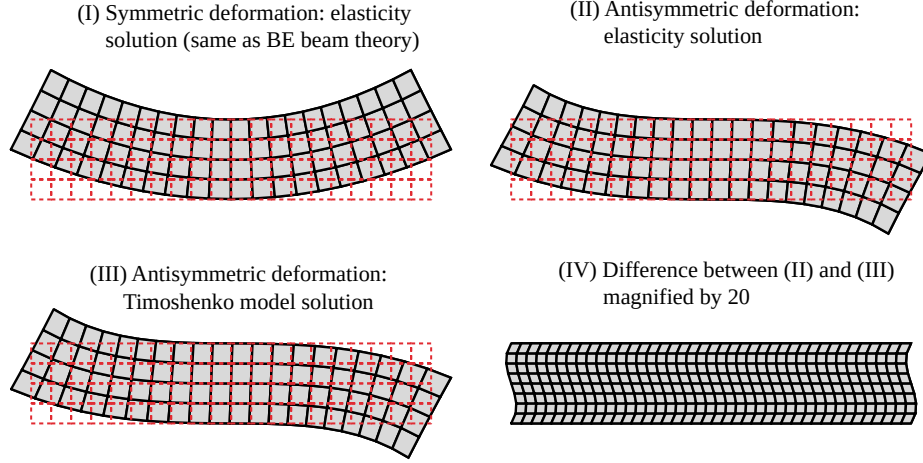


FIGURE 21.9. Deformed shapes for benchmark solutions. Beam of rectangular cross section with $L = 5$, $H = 2$, $h = 1$, $E = 1$, $\nu = 1/4$, $\kappa_S = 1$ and $\kappa^A = 3$.

Here r_{Gz} denotes the radius of gyration of the cross section about the neutral axis z while Φ is a dimensionless measure of the section-averaged shear rigidity of the element, which is used in the Timoshenko model. The last form of Φ given in (21.25) assumes an isotropic material.

Two load systems pictured in Figure 21.8(I,II) are considered:

- (S) Symmetric: pure bending produced by equal and opposite applied end moments M^S . The bending moment $M_z = M_{zS}$ is constant along the beam whereas the transverse shear force V_y is zero.
- (A) Antisymmetric: linearly varying bending produced by equal and opposite shear forces V^A balanced by end moments with total resultant $M^A = V_y L$. The internal bending moment $M_z = -M^A x/L$ varies linearly from $+\frac{1}{2}M^A$ at $x = -\frac{1}{2}L$ through $-\frac{1}{2}M^A$ at $x = \frac{1}{2}L$. The transverse shear force V_y is constant and equal to V^A .

§21.4.1. Symmetric Solution

The symmetric displacement solution is

$$u_x^S = -\kappa_S x y, \quad u_y^S = \frac{1}{2} \kappa_S (x^2 + y^2 \nu), \quad \theta_z^S = \frac{1}{2} \left(\frac{\partial u_y^S}{\partial x} - \frac{\partial u_x^S}{\partial y} \right) = \kappa_S x. \quad (21.26)$$

in which $\kappa_S = M_z^S / (E I_{zz}) = \partial^2 u_y^S / \partial x^2$ is the constant curvature produced by M^S . The deformed shape, pictured in Figure 21.9(I), is symmetric about x and y . The associated strains and stresses are

$$e_{xx}^S = -\kappa_S y, \quad e_{yy}^S = -\nu e_{xx}^S, \quad \gamma_{xy}^S = 0, \quad \sigma_{xx}^S = -E \kappa_S y, \quad \sigma_{yy}^S = 0, \quad \sigma_{xy}^S = 0. \quad (21.27)$$

The stresses (21.27) satisfy identically the plane stress differential equilibrium equations for any plane-symmetric cross section. The internal energy stored in the beam is

$$U = \frac{1}{2} \int_V \sigma_{xx} e_{xx} dV = \frac{1}{2} E L \kappa_S^2 \int_A y^2 dA = \frac{1}{2} E I_{zz} \kappa_S^2 L = \frac{1}{2} M^S \kappa_S L. \quad (21.28)$$

§21.4.2. Antisymmetric Elasticity Solution

The antisymmetric displacement solution for an elastic isotropic material is

$$\begin{aligned} u_x^A &= \frac{\kappa_A y}{6L} (3x^2 - y^2(2 + \nu) + 18r_{Gz}^2(1 + \nu)), \\ u_y^A &= -\frac{\kappa_A x}{6L} (x^2 + 3\nu y^2), \quad \theta_z^A = -\frac{\kappa_A}{2L} ((x^2 - y^2) + 3r_{Gz}^2(1 + \nu)) \end{aligned} \quad (21.29)$$

The deformed shape is pictured in Figure 21.9(II) for a rectangular cross section with the values given there. The associated strains and stresses are

$$\begin{aligned} e_{xx}^A &= \frac{\kappa_A}{L} x y, \quad e_{yy}^A = -\nu e_{xx}^A, \quad \gamma_{xy}^A = \frac{\kappa_A (1 + \nu)}{L} (3r_{Gz}^2 - y^2), \\ \sigma_{xx}^A &= E \frac{\kappa_A}{L} x y, \quad \sigma_{yy}^A = 0, \quad \sigma_{xy}^A = E \frac{\kappa_A}{2L} (3r_{Gz}^2 - y^2) \end{aligned} \quad (21.30)$$

It is easily verified that $\int_A \sigma_{xx}^A dA = E I_{zz} \kappa_A (x/L) = M^A$. To show that $\int_A \sigma_{xy}^A dA = V^A$, integrate over A , use $I_{zz} = \int_A y^2 dA = A r_{Gz}^2$ and $V^A = M^A/L = E I_{zz} \kappa_A/L$. The elasticity equilibrium equations are also satisfied identically. As regards stress boundary conditions, note that σ_{xy}^A vanishes at $y = \pm\sqrt{3}r_{Gz}$. These are not the top and bottom beam fibers unless the cross section is rectangular.

§21.4.3. Antisymmetric Timoshenko Solution

The solution provided by the Timoshenko beam model, which assumes a constant average shear over the cross section is

$$\begin{aligned} u_x^{Timo} &= \frac{\kappa_A y}{6L} (3x^2 + \nu y^2 + 12r_{Gz}^2(1 + \nu)), \\ u_y^{Timo} &= -\frac{\kappa_A x}{6L} (x^2 + 3\nu y^2), \quad \theta_z^{Timo} = -\frac{\kappa_A}{2L} (x^2 + \nu y^2 + 2r_{Gz}^2(1 + \nu)) \end{aligned} \quad (21.31)$$

The deformed shape is pictured in Figure 21.9(III) for a rectangular cross section with the same values used for (II). There is no discernible difference between the elasticity and Timoshenko solutions at the plot scaled used there. The difference is shown Figure 21.9(IV) with displacements magnified by 20. While vertical displacements are the same, the elasticity solution accounts for the warping of the cross sections: $u_x^A - u_x^{Timo} = \kappa_A y (1 + \nu)(2r_{Gz}^2 - y^2)/2L$. The associated strains and stresses are

$$\begin{aligned} e_{xx}^{Timo} &= \frac{\kappa_A}{L} x y, \quad e_{yy}^{Timo} = -\nu e_{xx}^{Timo}, \quad \gamma_{xy}^{Timo} = 2\frac{\kappa_A}{L} r_{Gz}^2 (1 + \nu), \\ \sigma_{xx}^{Timo} &= E \frac{\kappa_A}{L} x y, \quad \sigma_{yy}^{Timo} = 0, \quad \sigma_{xy}^{Timo} = E \frac{\kappa_A}{L} r_{Gz}^2. \end{aligned} \quad (21.32)$$

§21.4.4. Specialization to a Rectangular Cross Section

For the $H \times h$ rectangular cross section, $A = Hh$, $I_{zz} = H^3 h/12$, $r_{Gz}^2 = H^2/12$,

$$\begin{aligned} u_x^S &= -\kappa_S x y, \quad u_y^S = \frac{1}{2}\kappa_S (x^2 + \nu y^2), \quad \theta_z^S = \kappa_S x, \\ u_x^A &= \frac{\kappa_A y}{L} (x^2 - \frac{2+\nu}{3}y^2 + \frac{1+\nu}{3}H^2), \quad u_y^A = -\frac{\kappa_A x}{3L} (x^2 + 3\nu y^2), \quad \theta_z^A = -\frac{\kappa_A}{L} ((x^2 - y^2) + \frac{1+\nu}{6}H^2) \\ u_x^{AT} &= \frac{\kappa_A y}{L} (x^2 + \frac{\nu}{3}y^2 + \frac{1+\nu}{3}H^2), \quad u_y^{AT} = -\frac{\kappa_A x}{3L} (x^2 + 3\nu y^2), \quad \theta_z^{AT} = -\frac{\kappa_A}{L} ((x^2 + \nu y^2) + \frac{1+\nu}{6}H^2). \end{aligned} \quad (21.33)$$

22

Element Fabrication Overview

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This Chapter surveys basic and advanced methods for constructing finite elements. A catalog of various formulations for mechanical elements is given, and approaches that may combine several formulations are outlined.

§22.1. Balancing Physics and Mathematics

Chapter 1 of the IFEM Notes [86] discussed two interpretation of the Finite Element method (FEM): physical and mathematical. Both are given equal time in that course.

The physical interpretation emphasizes the breakdown (disassembly, decomposition, separation, tearing) of a complex system into simpler components that eventually reach a primitive level called *elements*. This interpretation is convenient for engineering users since it simplifies modeling. It is also the basis for implementation of element libraries and assemblers in commercial codes.

The mathematical interpretation views FEM as a spatial discretization method for solving partial differential equations. The concept of breakdown and assembly, so integral to the physical interpretation, are no longer necessary.

This interpretation plays a significant role in establishing the mathematical foundations as well as the extension of FEM applications beyond the original focus on mechanics.

Chapter 1 of [86] says that the two interpretations synergically complement each other. That sweeping statement is appropriate to an introductory exposition, where to lessen confusion things are painted black and white.

A more realistic perspective is the spectrum pictured in Figure 22.1. Therein “physical modeling dominates” indicates that the fit to a particular application (or even a particular physical system) governs element construction. By contrast, “mathematical virtuosity dominates” conveys the opposite: the use of advanced functional analysis tools takes the upper hand while applications, if any, are downplayed.

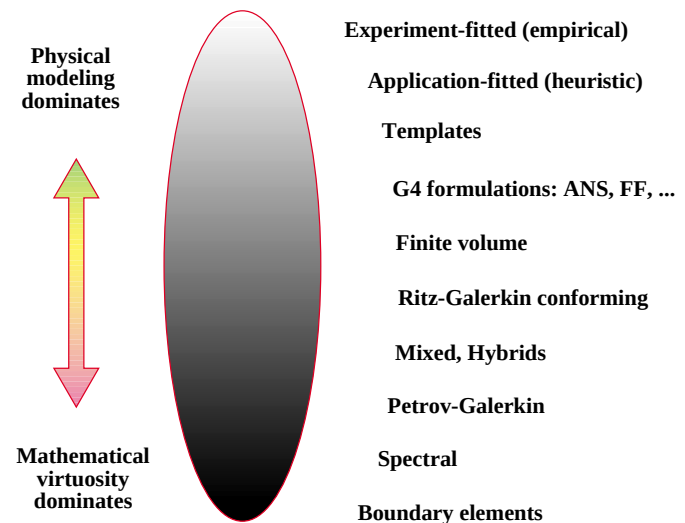


FIGURE 22.1. The spectrum of finite element fabrication methods. Toward the top the physical interpretation, stressing modeling and customized fit-to-application, dominates. Toward the bottom the mathematical interpretation, stressing the use of advanced tools, dominates.

Elements that rely heavily on physics and observations¹ tend to be interesting because developers often have to negotiate tradeoffs, and agreement with experience and observation tends to keep them honest.

At the very top of the figure are “experimental elements” defined entirely by test data. For example, a dry-friction contact-impact element with behavior defined entirely by digitized tables, or an actuator

¹ The disconnection from mathematics should not be taken too literally. A finite element always obeys some rules of mathematics such as matrix algebra and calculus.

element defined by force-extension-rate response functions provided by the manufacturer.²

Next come elements based on rules that have not been fully mathematically blessed, such as the patch test covered on the next Chapter. These includes templates³ as well as high-performance elements of Generation 4 (1980-date) such as Assumed Natural Strain (ANS) and the Free Formulation (FF). Elements that directly rely on conservation laws, such as finite volume models for Euler and Navier-Stokes fluid flows are placed somewhat closer to the physical side.

In the middle of the spectrum one finds elements based on conventional variational principles (Total Potential Energy, etc). that may be collectively labeled “Ritz-Galerkin conforming.” Subclasses of these, such as the isoparametric family, are covered in introductory FEM courses.

Hybrid elements are based on variational principles that are somewhat unconventional because they were designed specifically for FEM. As such, the mathematical tools required are a bit more advanced and have not been studied as extensively as conventional variational formulations.

If no variational principle is available, elements may be still constructed on the basis of the weak form, using the Method of Weighted Residuals (MWR). Although MWR spans a vast class of formulations (including Ritz-Galerkin, finite volumes, least-squares, collocation, subdomain, etc.) a popular choice when a functional is unavailable is Petrov-Galerkin. The mathematical tools needed here are more exotic because lack of self-adjointness removes the “Ritz safety blanket” and physical transparency. Spectral elements are also placed in this region of Figure 22.1 as their formulation is done in transform spaces unrelated to physics.

Boundary elements, if viewed as special case of FEM, are placed at the bottom of Figure 22.1 since they rely on esoteric mathematical tools (for example source distributions) only indirectly verifiable by observation. In general an element constructed to numerically solve a PDE completely devoid of physics would be placed at this end.

§22.2. A Catalog of Formulations for Mechanical Elements

Even restricting attention to solid and structural mechanics, the number of finite element formulations has steadily grown over the past five decades. To throw some light into this melange, the following collection outlines several important methodologies for that application, noting advantages and limitations.

§22.2.1. Isoparametric (isoP) Displacement Formulation

<i>Source</i>	Outgrow of Taig’s quadrilateral presented in [237]. Extended by Irons to arbitrary geometries [143,147].
<i>Master field(s)</i>	One internal: displacements. Strains and stresses are slaves.
<i>Var. principle</i>	Total Potential Energy (TPE)

² In commercial codes these are often implemented as “user-defined elements.” The idea of having user entry points to the element library to accomodate unpredictable modeling needs is surprisingly old: it was first discussed by Turner, Martin and Weikel in a landmark 1964 paper [257] that sets out the definitive exposition of the Direct Stiffness Method.

³ Templates are parametrized algebraic forms of element operators that include specific elements as instances. Templates for certain applications are discussed later.

<i>Description</i>	Geometry and displacements interpolated by same shape functions. Numerical integration by Gauss quadrature.
<i>Applicability</i>	Problems governed by functional with variational index one. Only displacements as degrees of freedom.
<i>Popularity</i>	Huge despite limitations noted below.
<i>Strengths</i>	Well established (nearly 40 years old in 2006), widely implemented. Advantages and shortcomings well known by now. Systematic rules, valid for any dimensionality and element complexity. Naturally handles elements with curved sides/faces. Technique easily extended to non-structural elements (thermal, fluids, EM, etc) as long as variational index in the primary variable stays one. Shape functions useful to express interpolation rules for other applications, such as data fitting.
<i>Limitations</i>	Low order isoP models may have poor performance in terms of locking and distortion sensitivity. Some fixed-up devices are described in §22.2.2. Simplex elements of this type (linear triangles and tetrahedra) cannot be improved, however, by any device. Not applicable to problems with variational index greater than one in displacements, such as Bernoulli-Euler beams, Kirchhoff plates and thin shells. Cannot handle elements with rotational or derivative-type freedoms without substantial modifications that usually involve restriction to the subP formulation outlined in §22.2.3. “Blows up” for incompressible plane strain, axisymmetric and solid elements.

§22.2.2. Fixed-Up Isoparametric Formulation

<i>Source</i>	Bag of tricks emerging over period 1969–75: [60,239,268,278]. Equivalence to mixed methods: [137]. Comprehensive exposition in Hughes’ book [141].
<i>Master field(s)</i>	Same as isoP.
<i>Var. principle</i>	Total Potential Energy (TPE)
<i>Description</i>	This category embraces a number of ad-hoc devices introduced to fix or at least alleviate the poor performance of certain isoparametric models, especially low-order ones, as regards locking (extreme over stiffness) and distortion sensitivity. Most common: (1) reduced integration, (2) selective integration, (3) directional integration, and (4) incompatible displacement modes.
<i>Applicability</i>	Same as isoP formulation.
<i>Popularity</i>	High because fixed-up devices are easy to implement. Expected knowledge does not rise to level of variational techniques used in mixed and hybrid formulations.
<i>Strengths</i>	Easy to implement. Software reuse (for example, of shape function modules) is facilitated. This is particularly important in legacy and nonlinear codes.
<i>Limitations</i>	Erratic backstabbing effects. As more tricks are introduced, unpleasant surprises can happen, such as hourglassing and invariance loss. Repair may produce more unexpected side effects: “turn on the light, the shower goes on” (technically these are called regression bugs). Eventually legacy software adorned with these accoutrements may become untouchable for fear of additional side effects.

§22.2.3. Subparametric (subP) Displacement Formulation

<i>Source</i>	Assumed-displacement models preceding the isoP formulation, as in [8,177,178].
<i>Master field(s)</i>	One internal: displacements. Strains and stresses are slaves.
<i>Var. principle</i>	Total Potential Energy (TPE)
<i>Description</i>	Element geometry kept simple, typically restricted to the simplest possible shapes. Displacements interpolated by equal or higher order functions than geometry.
<i>Applicability</i>	Problems governed by functionals of any variational index. Handles elements with rotational or derivative-type freedoms.
<i>Popularity</i>	Medium. Easy to present and teach.
<i>Strengths</i>	Applicability wider than that of isoP elements. Can directly handle C^1 beams, plates and shells.
<i>Limitations</i>	Element geometry is restricted, restricting modeling flexibility. No systematic construction rules. As in the isoP case, performance of low-order elements may be poor. For some configurations full interelement compatibility may be difficult or even impossible to attain.

§22.2.4. Mixed Hellinger-Reissner (HR) Formulation

<i>Source</i>	First FEM use proposed by Herrmann [129] for incompressible and nearly-incompressible elasticity (motivation was modeling of solid rocket propellants).
<i>Master field(s)</i>	Two internal: displacements and stresses. Two strain slaves.
<i>Var. principle</i>	Hellinger-Reissner (HR)
<i>Description</i>	Displacement master field may be interpolated with shape functions. Separate master stress interpolation (or pressure) usually done in Cartesian coordinates. Stress DOFs (or stress parameters) may be condensed out at element level if (1) stress (or pressure) may jump between adjacent elements and (2) material is compressible. Else stress (or pressure) DOFs must go into the assembly.
<i>Applicability</i>	Two major uses: (1) treating incompressible or nearly-incompressible materials, and (2) improving the performance of low-order displacement models. For case (1) pressure is assumed in addition to displacements.
<i>Popularity</i>	Low since it requires advanced knowledge. Hampered by the discovery of fixed-up devices for isoP elements, which under certain restrictions effectively produce the equivalent of mixed elements with simpler tricks.
<i>Strengths</i>	Possible way to improve element performance. Proven useful for the incompressible case. If stress DOFs can be eliminated, condensed model looks like a displacement element to the assembler. Comparing strains from displacements and stresses allows easy construction of an element-level error measure.
<i>Limitations</i>	Requires knowledge of mixed variational principles, not an easy subject. Prone to failure because of rank deficiency if stress (or pressure) and displacement

assumptions violated certain stability conditions.⁴ Even if element is stable, expected improvements in performance may not necessarily materialize: “mixed elements lead to mixed results” (G. Strang).

§22.2.5. Equilibrium-Stress Hybrid Formulation

<i>Source</i>	Pian [194,195], variational basis by Pian and Tong [196].
<i>Master field(s)</i>	Over element: internal equilibrium stress field, interelement-compatible boundary displacements. Slave Strains. Displacement field inside element known only weakly.
<i>Var. principle</i>	Total Potential Complementary Energy (TCPE) augmented by dislocation potential.
<i>Description</i>	Ingenious motivation: facilitate interelement compatibility by assuming only boundary displacements while maintaining internal equilibrium, which should result in better stress recovery.
<i>Applicability</i>	Applicable in principle to any geometrically linear element.
<i>Popularity</i>	Medium. Despite age ⁵ requires advanced knowledge.
<i>Strengths</i>	If workable, a proven way to improve element stress-recovery performance. Especially useful when assumed-displacement interelement compatibility is hard to achieve, as in thin plates and shells. Less prone to stability limitations than HR elements.
<i>Limitations</i>	Requires knowledge of hybrid variational principles, a tough subject. Construction of an invariant equilibrated stress field may be difficult or even impossible for arbitrary geometries. ⁶ Difficult to extend to geometrically nonlinear problems.

Remark 22.1. There is a large number of possible hybrid element formulations. They share a common feature: different master fields are assumed over the interior and boundary of the element. The foregoing case is just one of many possibilities. It is listed here on account of historical importance and its implicit presence in FF, ANDES and template formulations.

§22.2.6. Assumed Strain Formulations

<i>Source</i>	For assumed Cartesian strains: MacNeal [161]. For assumed natural strains (ANS): Bathe-Dvorkin [17] for plates, Park-Stanley [190] and Huang-Hinton [136] for shells.
<i>Master field(s)</i>	Varies with author. Common feature is that element strain variation is assumed in some form.

⁴ Many papers have been written on this topic, called the Babuska-Brezzi (BB) condition. Of these, 99.9% are forgettable fillers. The physical interpretation of this condition by Fraeijs de Veubeke is called the “limitation principle” [101].

⁵ Over 40+ years since original publications. In fact came a bit earlier than the more popular isoP formulation.

⁶ This difficulty (lack of observer invariance) held up advances in extending Pian’s original results for 20 years until the landmark paper by Pian and Sumihara [197].

<i>Var. principle</i>	Varies with author. Most use Total Potential Energy (TPE) with various forms of strain assumptions for certain energy components.
<i>Description</i>	The slave connection between displacements and stresses is selectively broken to alleviate locking problems. In the Assumed Natural Strain variant (the most powerful variant) the strain field is expressed in natural coordinate directions, but not necessarily in tensorial form.
<i>Applicability</i>	Primarily useful for plates and shells fabricated from degenerated solid elements.
<i>Popularity</i>	Medium, since applicability is restricted.
<i>Strengths</i>	When done correctly it produces high performance plates and shell elements. Main advantages arise in geometrically nonlinear analysis of shells.
<i>Limitations</i>	To make it work correctly requires substantial physical insight, a dab of black magic incantations and plenty of luck.

§22.2.7. Free Formulation (FF)

<i>Source</i>	Bergan and coworkers [26,27].
<i>Master field(s)</i>	Element response behavior split into basic and higher order. For basic response, same masters as constant-stress equilibrium hybrids. For higher order response, assumed displacements in generalized coordinates. These are not usually interelement compatible. ⁷
<i>Var. principle</i>	Not provided in original formulation. Shown to be a mixture of hybrid and TPE functionals in [73].
<i>Description</i>	Fundamental idea is splitting of element response. Each component is assigned different but complementary roles. Basic component takes care of convergence and mixability. Higher order component provides stability (rank sufficiency) and accuracy. Orthogonality conditions between those components insure <i>a priori</i> satisfaction of the Individual Patch Test (IPT) of [25].
<i>Applicability</i>	Any mechanical element.
<i>Popularity</i>	Low. Splitting concept is at odds with historical tradition, variational formulation is highly unconventional, and use of generalized coordinates is unfamiliar to many developers.
<i>Strengths</i>	Provides high performance elements. Basing the higher order component on assumed displacements taps a well studied subject.
<i>Limitations</i>	Requires substantial physical insight on the part of a developer. Construction of the higher order component not easy, involving many trials to balance orthogonality conditions with rank sufficiency. Mathematical basis incomplete.

⁷ The qualifier “free” is intended to reflect the fact that these higher order displacement functions need not satisfy interelement compatibility restrictions as in the case of subP and isoP elements, and thus enjoy greater freedom of choice than those required to do so. It is not used in the English sense of “free of charge.”

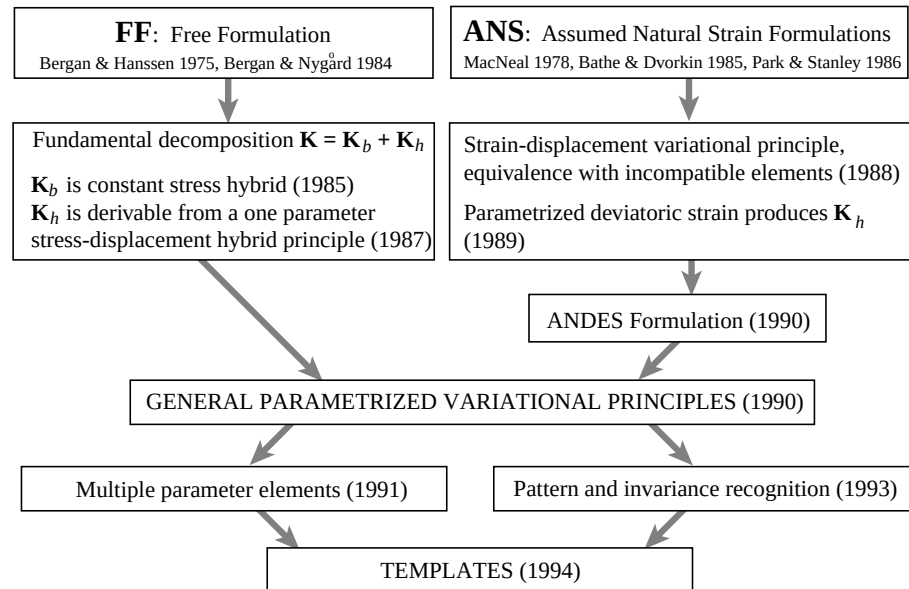


FIGURE 22.2. The road to templates.

§22.2.8. Assumed Natural Deviatoric Strain (ANDES) Formulation

<i>Source</i>	Felippa and Militello [75,182].
<i>Master field(s)</i>	Element response behavior split into basic and higher order. For basic response, same masters as constant-stress equilibrium hybrid. For higher order response, assumed natural strains.
<i>Var. principle</i>	Falls in the context of parametrized variational principles [77].
<i>Description</i>	Combines ideas from the Free Formulation (FF) and the Assumed Natural Strain (ANS) formulation. Same response splitting as in FF, and identical basic component. Higher order component built with ANS ideas using assumed deviatoric strains (strains that deviate from a constant state) instead of total strains.
<i>Applicability</i>	Any mechanical element.
<i>Popularity</i>	Low, as in the case of the FF. Deviates too much from tradition. Particular case of templates.
<i>Strengths</i>	Similar to FF, but construction of the higher order part allows more easy parametrization on the way to templates.
<i>Limitations</i>	Requires substantial expertise and a dose of luck since methodology is very new. Trial and error mandatory since mathematical basis is incomplete.

§22.2.9. Templates

<i>Source</i>	First mentioned in [77]. Developed in [83,90]. Recent tutorial: [91]. The roadmap to templates is flowcharted in Figure 22.2.
<i>Master field(s)</i>	Element response split as in case of FF and ANDES. Basic part is constant equilibrium-stress hybrid. For higher order component, no predefined masters.

<i>Var. principle</i>	For basic component, equilibrium stress hybrid principle. For higher order component, no predefined principle.
<i>Description</i>	A template is a parametrized algebraic form of element operators (stiffness, mass, etc). This produces an infinity of possible elements that <i>a priori</i> satisfy the Individual Element Test (IET). Specific element instances obtained by assigning numerical values to parameters. A <i>universal template</i> is one that includes all possible elements that pass the IET.
<i>Applicability</i>	In principle any element. But see <i>Limitations</i> below.
<i>Popularity</i>	None, as it is a recent idea barely off the ground.
<i>Strengths</i>	Includes an infinite number of possible elements. Instances can be customized to produce optimal results for envisioned use.
<i>Limitations</i>	Development impossible by hand since one must carry along element properties (geometry, constitutive, fabrication, ...) in symbolic form, along with free parameters. Use of a computer algebra system (CAS) mandatory. Limits in current CAS power, however, has restricted the idea to 1D and 2D elements of simple geometry.

§22.3. *Approaches to Element Construction

The term *approach* is taken here to mean a combination of methods and empirical tools to achieve a given objective. In FEM work, isoparametric, stress-assumed-hybrid and ANS (Assumed Natural Strain) formulations are methods and not approaches. An approach may zig-zag through several methods. FEM approaches range from heuristic to highly analytical.

Figure 22.3 makes an implicit assumption: the performance of an element of given geometry, node and freedom configuration can be improved. There are obvious examples where this is not possible. For example, constant-strain elements with translational freedoms only: 2-node bar, 3-node membrane triangle and 4-node elasticity tetrahedron. Those cases are excluded because it makes no sense to talk about high performance or optimality under those conditions.

§22.3.1. *Fixing Up

As noted in §22.2 certain popular element construction methods, such as the isoP formulation, may produce bad or mediocre low-order elements. If that is the case two questions may be raised:

- (i) Can the element be improved?
- (ii) Is the improvement worth the trouble?

If the answer to both is yes, the fix-up approach tries to improve the performance by an array of remedies that may be collectively called the FEM pharmacy. Cures range from heuristic tricks such as reduced and selective integration to more scientifically based concoctions.

This approach accounts for most of the current publications in finiteelementology. Playing doctor can be fun. But also frustrating, as trying to find a black cat in a dark cellar at midnight. Inject these incompatible modes: oops! the patch test is violated. Make the Jacobian constant: oops! it locks in distortion. Reduce the integration order: oops! it lost rank sufficiency. Split the stress-strain equations and integrate selectively: oops! it is not observer invariant. And so on.

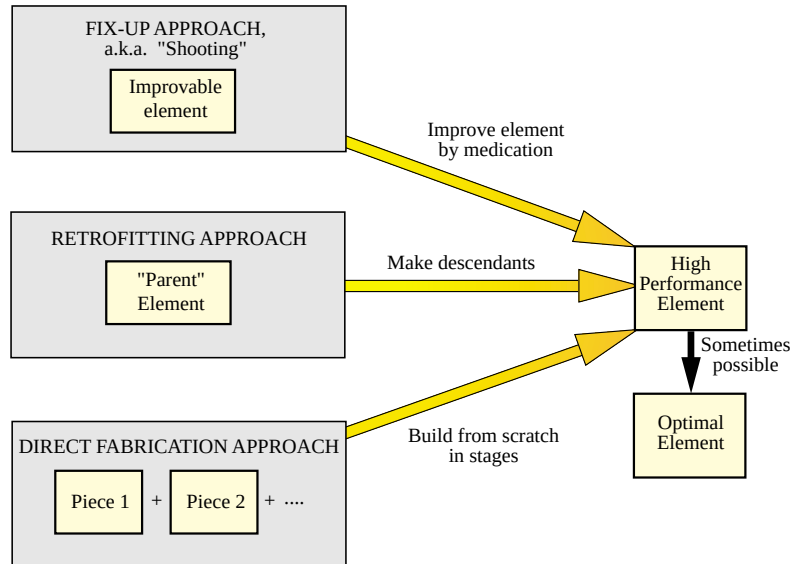


FIGURE 22.3. Three approaches to element construction.

§22.3.2. *Retrofitting

Retrofitting is a more sedated activity. One begins with a irreproachable parent element, free of obvious defects. Typically this is a higher order iso-P element constructed with a complete or bicomplete polynomial; for example the 6-node quadratic triangle or the 9-node Lagrange quadrilateral. The parent is fine but too complicated to be an HP element. Complexity is reduced by master-slave constraint techniques so as to fit the desired node-freedom configuration pattern.

This approach commonly makes use of node and freedom migration techniques. For example, drilling freedoms may be defined by moving translational midpoint or thirdpoint freedoms to corner rotations by kinematic constraints. Discrete Kirchhoff constraints and degeneration (3D→2D) for plate and shell elements provides another example. Retrofitting has the advantage of being easy to understand and teach. It occasionally produces useful elements but rarely high performance ones.

§22.3.3. *Direct Fabrication

This approach relies on divide and conquer. To give an analogy: upon short training a FEM novice knows that a discrete system is decomposed into elements, which interact only through common freedoms. Going deeper, an element can be constructed as the superposition of components or pieces, with interactions limited through appropriate orthogonality conditions. Components are invisible to the user once the element is implemented.

Fabrication is done in stages. At the start there is nothing: the element is without form, and void. At each stage the developer injects another component (= subspace). Components may be done through different methods. The overarching principle is correct performance after each stage. If at any stage the element has problems (for example: it locks) no retroactive cure is attempted as in the fix-up approach. Instead the component is trashed and another one picked. One never uses more components than strictly needed: condensation is forbidden. Components may contain free parameters, which may be used to improve performance and eventually to try for optimality. One general scheme for direct fabrication is the template formulation outlined in §22.2.9.

All applications of the direct fabrication method to date have been done in two stages, separating the element response into basic and higher order. This process is further elaborated in further Chapters.

References

Referenced items have been moved to Appendix R.

Homework Exercises for Chapter 22**Element Fabrication Overview**

EXERCISE 22.1 [D:5] What are the main restrictions for use of conventional isoparametric elements? In particular, can they be used directly (meaning without modifications) to model Bernoulli-Euler beams and thin plates/shells?

EXERCISE 22.2 [D:5] A “bumper element” for a high speed train is developed from recorded low-speed crash data. Where would that fit in Figure 20.1?

23

The Patch Test

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§23.1. Introduction

Chapter 20 tangentially mentioned High Performance (HP) elements. These were defined in [72] as “simple elements that deliver engineering accuracy with arbitrary coarse meshes.” A major goal of the post-1980 element formulation reviewed in Chapter 20 is to produce such elements.

A key tool used in these advanced formulations is the *patch test* and a specialization thereof called the *Individual Element Test* or IET. These tools are outlined in this Chapter. Some of this material is taken verbatim from [78].

§23.2. Motivation

The historical origins of the patch test are detailed at the end of this Chapter. Following is a summary of the motivation behind it.

§23.2.1. The Black Cat

By the end of FEM Generation 1 (1952-1962), closed by Melosh’s thesis [177] it was generally recognized that the *displacement-assumed* form could be viewed as variation of the century-old Rayleigh-Ritz direct variational method. The new idea introduced by FEM was that trial functions had local support extending over element patches. To guarantee convergence as the mesh was refined, those functions ought to satisfy the conditions discussed in Chapter 19 of [86]: continuity, completeness and stability¹.

Understanding of those conditions was reached only gradually and published in fits and starts. Continuity requirements were quickly found because for displacement elements they are easily visualized: noncompliance means materials gaps or interpenetration. They were first unequivocally stated in [177,178].

Completeness was understood in stages: first rigid body modes and then constant strain (or curvature) states. Stability in terms of rank sufficiency was mentioned occasionally but often dismissed as comparatively unimportant, until the early 1980.

The rudimentary state of the mathematical theory did not impede progress. All the contrary. The years of Generation 2: 1963–1972 marked a period of exuberant productivity and advances in FEM for the Direct Stiffness Method: solid elements, new plates and shell models, as well as the advent of isoparametric elements with numerical quadrature. It was also a period of mystery: FEM was a black cat in a dark cellar at midnight. The patch test was the first device to throw some light on that cellar.

§23.2.2. Variational Crimes

The equivalence between FEM and classical Ritz as understood by 1970 is well described by Strang in a paper [226] that introduced the term “variational crimes”. Here is an excerpt from the Introduction of that paper:

“The finite element method is nearly a special case of the Rayleigh-Ritz technique. Both methods begin with a set of trial functions $\phi_1(x), \dots, \phi_N(x)$; both work with the space of linear combinations $v^h = \sum q_i \phi_i$; and both chose the particular combination (we will call it u^h) which minimizes

¹ Stability includes rank sufficiency and positive Jacobian determinant

a given quadratic functional. In many applications this functional corresponds to potential energy. The convergence of the Ritz approximation is governed by a single fundamental theorem: *if the trial functions ϕ_j are admissible in the original variational principle, then u^h is automatically the combination which is closest (in the sense of strain energy), to the true solution u .* Therefore convergence is proved by establishing that as $N \rightarrow \infty$, the trial functions fill out the space of all admissible functions. The question of *completeness* dominates the classical Ritz theory.

For finite elements a new question appears. Suppose the basic rule of the Ritz method is broken, and the trial functions are not quite admissible in the true variational problem: it is still possible to prove that u^h converges to u , and to estimate the rate of convergence? This question is inescapable if we want to analyze the method as *it is actually used* [my italics]. We regard the convenience and effectiveness of the finite element technique as conclusively established; it has brought a revolution in the calculations of structural mechanics, and other applications are rapidly developing [this was written in the early 1970s]. Our goal is to examine the modification of the Ritz procedure which have been made (quite properly, in our view) in order to achieve an efficient finite element system. The hypotheses of the classical Ritz theory are violated because of computational necessity, and we want to understand the consequences.”

Later in this paper Strang introduces the term *variational crimes* to call attention to the fact that many existing elements known at that time did not comply with the laws of the classical Ritz method. In decreasing order of seriousness the violations are:

The Sins of Delinquent Elements
(1) Lack of completeness (2) Lack of invariance: element response depends on observer frame (3) Rank deficiency (4) Nonconformity: violation of interelement continuity required by variational index (5) Inexact treatment of curved boundaries and essential BCs (6) Inexact, but rank sufficient, numerical integration

(23.1)

The term *delinquent element* was coined by Bruce Irons to denote one guilty of one or more of these sins. He also used *naughty element*.²

In Chapter 4 (“Variational Crimes”) of the Strang and Fix monograph [227] attention is focused on the last three items in (23.1). The sensational title served the intended purpose: it attracted interdisciplinary attention from both mathematicians and the FEM community. But thirty years later we take the label as too sweeping. Some “crimes”, notably (4) and (6), may be actually virtues, that is, beneficial to performance when honestly acknowledged.³ That chapter actually says little about items (1) through (3), which are the truly evil ones.

Remark 23.1. What does Strang mean by “classical Ritz theory”? A bit of history is appropriate here. Lord Rayleigh in his *Theory of Sound* masterpiece calculated approximate vibration frequencies by assuming the form of a vibration mode as a polynomial satisfying kinematic BCs and endowed with a free coefficient, say c . Inserting this into the functional of acoustics produces an algebraic function $\Pi(c)$ from which c is determined

² An old Middle English word: “So shines a good deed in a naughty world” (William Shakespeare, *Merchant of Venice*).

³ “To live outside the law, you must be honest.” (Bob Dylan, *Absolutely Sweet Marie*).

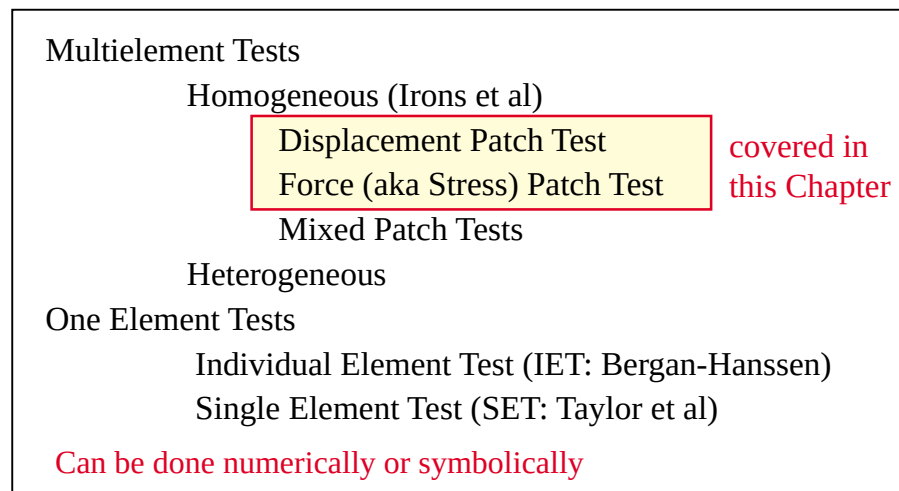


FIGURE 23.1. Variants of the patch test.

by minimization: $\partial \Pi / \partial c = 0$. Inserting into the Rayleigh quotient gives the squared frequency. This is called a *direct variational method*.

The Swiss mathematician Ritz extended in 1908 this idea to multiple coefficients by taking a linear combination of suitable admissible functions. These are now called *trial functions*. In the original (= classical) Ritz method, trial functions are infinitely smooth (typically polynomials) over the computational domain and satisfy the essential BCs exactly. The necessary integrals were done analytically over the computational domain: no numerical quadrature was used.

§23.2.3. Has the Jury Reached a Verdict?

In the present state of knowledge only the first two items in the alleged–crimes list (23.1) are truly serious. Certainly (1) is a capital offense. Not only it precludes convergence, but may cause convergence to the solution of the wrong problem, misleading unsuspecting users. Lack of invariance (2) can also lead to serious mistakes.

The effect of rank deficiency (3) is very problem dependent (dynamics versus statics, boundary conditions, etc.). While acceptable in special codes used by experts⁴ it should be avoided in elements intended for general use. Verdict: a rank-deficient element should be kept on probation.

The effect of nonconformity (4) was unpredictable before the development of the patch test. By now the consequences are well understood. In the hands of experts it has become a useful tool for fabricating high performance elements.

Inexact but rank sufficient integration (5) is often beneficial so its presence in that list is highly questionable. Finally the effect of (6) is generally minor but may require some attention when boundary layers may occur, as in the Reissner-Mindlin plate model.

⁴ For example, hydrocodes used for simulating a rapid transient response.

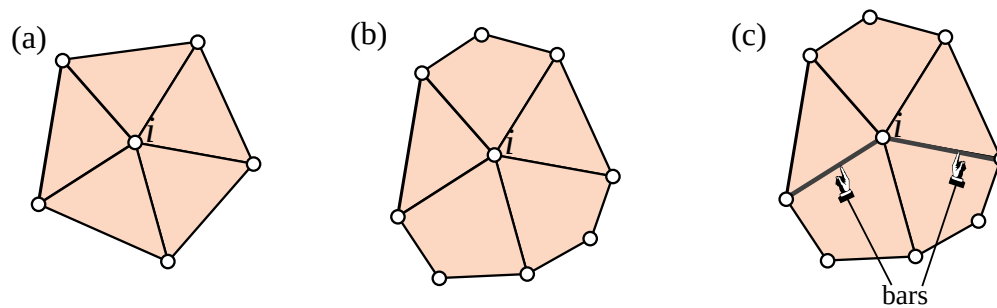


FIGURE 23.2. An element patch is the set of all elements attached to a patch node, herein labeled i . (a) illustrates a patch of triangles; (b) a mixture of triangles and quadrilaterals; (c) a mixture of triangles, quadrilaterals, and bars.

§23.3. Terminology

The original patch test form, historically outlined in §23.7, has evolved since into a great number of variants, summarized in Figure 23.1. Of these only the multielement homogeneous patch test in the displacement and force forms will be covered in detail here. Other versions will be described briefly with appropriate references if available. Before explaining the test procedure for the multielement versions, it is appropriate to introduce the concept of patch.

§23.3.1. Patches

We recall here the concept of *element patch*, or simply *patch*, introduced in Chapter 19 of IFEM. This is the set of all elements attached to a given node. This node is called the *patch node*. The definition is illustrated in Figure 23.2, which depicts three different kind of patches attached to patch node i in a plane stress problem. The patch of Figure 23.2(a) contains only one type of element: 3-node linear triangles. The patch of Figure 23.2(b) mixes two plane stress element types: 3-node linear triangles and 4-node bilinear quadrilaterals. The patch of Figure 23.2(c) combines three element types: 3-node linear triangles, 4-node bilinear quadrilaterals, and 2-node bars. Patches may also include a mixture of widely different element types, as illustrated for three dimensional space in Figure 23.2.

If all elements of the patch are of assumed-displacement type and hence defined via shape functions, we can define a *patch trial function* (PTF) the union of shape functions activated by setting a degree of freedom (DOF) at the patch node to unity, while all other freedoms are zero. If the elements are conforming, a patch trial function “propagates” only over the patch, and is zero beyond it.⁵

If one or more of the patch elements are not of displacement type, shape functions generally do not exist. Still the patch will respond in some way (strains, stresses, etc) to the activation of a patch-node DOF. That response, however, might not be so easily visualized.

§23.3.2. Generalizations

The definition of patch as “all elements attached to a node” is a matter of convenience. It simplifies some theoretical interpretations as well as the description of the physical test. But the concept can

⁵ If the elements are nonconforming, the PTF is by convention truncated at the patch boundary.

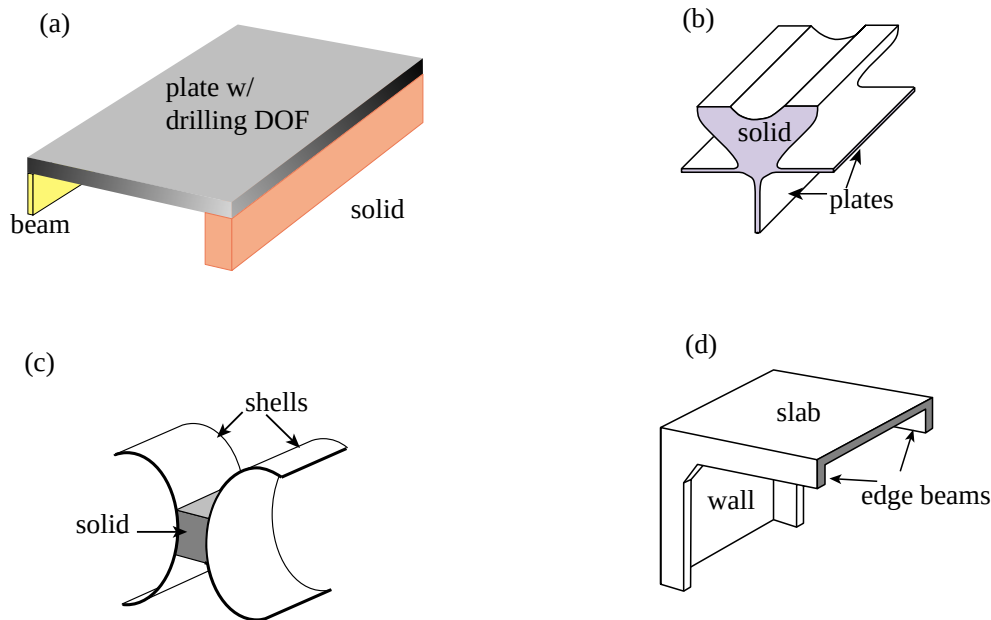


FIGURE 23.3. Heterogeneous patch test test the mixability of elements of different types, as illustrated in (a) through (d). These combinations test the *mixability* of elements.

be generalized to more general assemblies. Recall from Chapter 11 of IFEM that a *superelement* was defined as any grouping of elements that possesses no kinematic deficiencies other than rigid body modes. Not all superelements, however, are suitable for the test. The following definition characterizes a mild generalization:

Any superelement consisting of two or more elements can be used as a patch if it can be viewed as the limit of a mesh refinement process

This “limit mesh condition” characterizes *homogeneous patches*, that is, made up of the same element and having the same material and fabrication properties. In particular, no holes or cracks are permitted. On the other hand, the number of interior nodes can be arbitrary.

The restriction to at least two elements is to have an *interface* on which to test interelement nonconformity effects. However, the test can be reduced to *one element* under some assumptions discussed in later Chapters. Those are called *individual element tests* and are briefly covered in §23.6.

Heterogeneous patches are those including different element types to test element mixability, as in Figure 23.3. Those are not considered in this Chapter.

§23.4. The Physical Patch Test

Like FEM, the patch test has both a physical and a mathematical interpretation. Both are practically useful. The physical form was originally put forward by Irons in the Appendix to a 1965 paper [18]. It was described in more detail in later publications [144,147]. More historical details are provided at the end of the Chapter.

The essential idea behind the physical patch test is: *a good element must solve simple problems exactly* whether individually, or as component of arbitrary patches. The test has two *dual* forms:

Displacement Patch Test or DPT: applies boundary displacements to patch and verifies that the patch response reproduces exactly rigid body modes and constant strain states.

Force Patch Test or FPT: applies boundary forces to patch and verifies that the patch response reproduces exactly constant stress states.

There are also *mixed patch tests* that incorporate both force and displacement BCs. These will not be treated here to keep things simple.

§23.4.1. The Displacement Test Space

What is the “simple problem” alluded to above? For the displacement version, Irons reasoned as follows. In the limit of a mesh refinement process, the state of strain inside each displacement-assumed element is sensibly constant. (For bending models, strains are replaced by curvatures.) The associated displacement states are therefore linear in elements modeling bars, plane elasticity or 3D elasticity, and quadratic in beams, plates and shells. These limit displacement states are of two types: *rigid body modes* or *r-modes*, and *constant strain modes* or *c-modes* (constant curvatures in bending models). For brevity this set will be called the *displacement test space*.

In rectangular Cartesian coordinates the limit displacement fields are represented by *polynomials* in the coordinates. If the variational index is m and the number of space dimensions n , those polynomials have degree $\leq m$ in n variables. That set of polynomials is called $\mathcal{P}_n^m$. For example if $m = 1$ in a plane stress problem ($n = 2$), the displacement test space contains

$$u_x \in \mathcal{P}_2^1, \quad u_y \in \mathcal{P}_2^1, \quad \text{or} \quad \mathbf{u}(x, y) = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} c_{x0} + c_{x1}x + c_{x2}y \\ c_{y0} + c_{y1}x + c_{y2}y \end{bmatrix} \quad (23.2)$$

This test space has obviously dimension six: $u_x = \{1, x, y\}$, $u_y = \{0, 0, 0\}$, $u_y = \{1, x, y\}$, $u_x = \{0, 0, 0\}$. Therefore it is sufficient to test six cases since for linear FEM models any combination thereof will also pass the test. It is customary, however, to test for three rigid-body and three constant-strain modes separately, using the so-called *rc* basis:

$$r \text{ modes: } \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -y \\ x \end{bmatrix}, \quad c \text{ modes: } \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \text{ and } \begin{bmatrix} y \\ x \end{bmatrix}, \quad (23.3)$$

This polynomial space is exactly that which appears in the definition of completeness for individual elements given in Chapter 19 of IFEM. Hence the patch test, administered in this form, can be viewed as a *completeness test on arbitrary patches*.

§23.4.2. The Displacement Patch Test

We are now in a position to apply the so called *displacement patch test* or DPT.⁶ This will be illustrated by the 2D patch shown in Figures 23.3 and 23.4, which pertains to a plane stress problem.

Figure 23.4(a) shows the application of a rigid-body-mode (*r-mode*) test displacement field, $u_x = 1$ and $u_y = 0$. The procedure is as follows. Pick a patch. Evaluate the displacement field at the external nodes of the patch, and apply as prescribed displacements. Set forces at interior DOFs to

⁶ Also known as *kinematic patch test* or *Dirichlet patch test*. The latter name should be palatable to mathematicians.

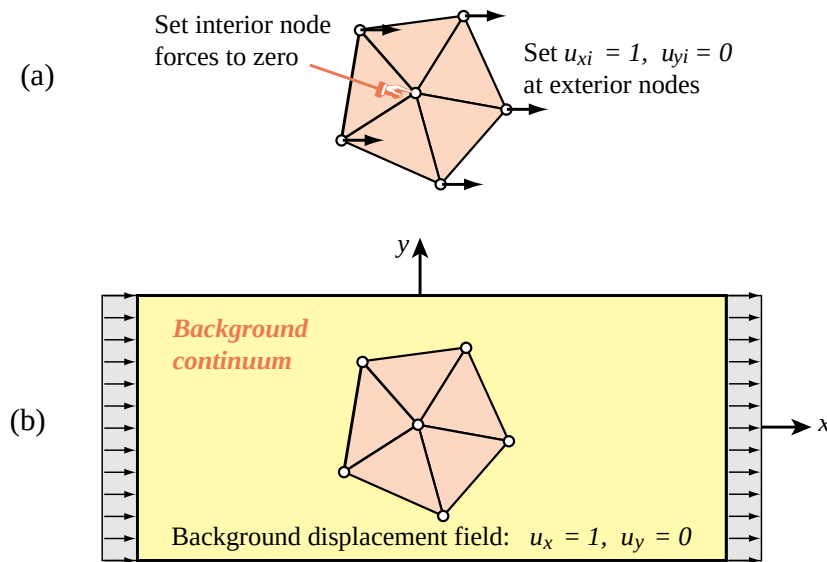


FIGURE 23.4. A two-dimensional rigid-body-mode displacement patch test (DPT). Prescribed displacement field is $u_x = 1, u_y = 0$, which represents a rigid x -motion; (a) shows a patch with external nodes given prescribed forces; (b) interpretation as replacing part of a “background continuum” with the patch.

zero. Solve for the displacement components of the interior nodes (two in the example). These should agree with the value of the displacement field at that node. Recover the strain field over the elements: all components should vanish identically at any point.

Figure 23.5(a) illustrates the application of a constant-strain-mode (c-mode) test displacement field $u_x = x$ and $u_y = 0$. This gives unit x strain $e_{xx} = \partial u_x / \partial x = 1$, others zero. Take the same patch. Evaluate the displacement field at the external nodes of the patch, and apply as prescribed displacements. Set forces at interior DOFs to zero. Solve for the displacement components of the interior nodes (two in the example). These should agree with the value of the displacement field at that node. Recover the strain field over the elements: all components should vanish except $e_{xx} = 1$ at any point.

If all test displacement/strain states are reproduced correctly, the DPT is passed for the selected patch, and failed otherwise. A shortcut possible in symbolic computation is to insert directly the polynomial basis (23.2), keeping the coefficients as variables.

§23.4.3. DPT Q&A

The foregoing subsection gives the recipe for running DPTs. Now for the questions.⁷

To answer most of the questions a thought experiment helps: think of the outside of the patch as a *homogeneous continuum* subject to the given displacement field. You may view the continuum as infinite if that helps. This is called the *background continuum*. This continuum must have exactly

⁷ Some are answered here for the first time. Undoubtedly Irons knew the answers but never bothered to state them, possibly viewing them as too obvious.

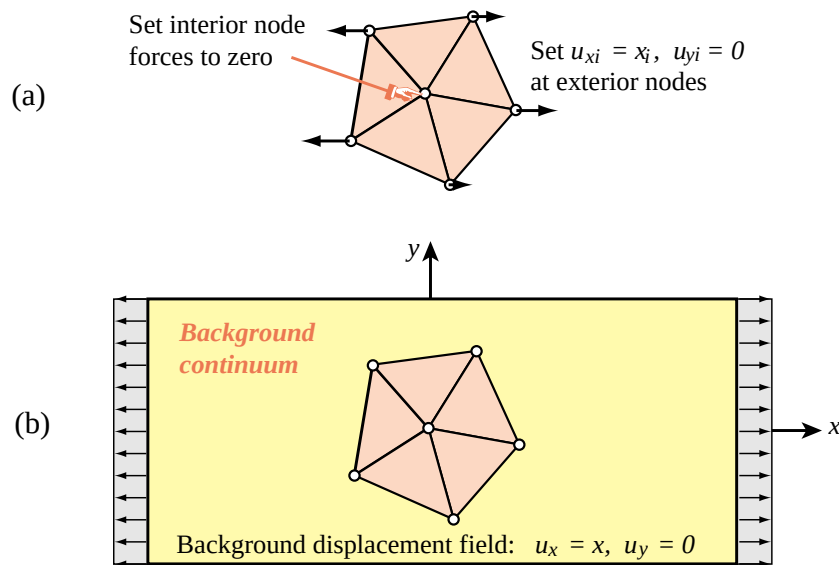


FIGURE 23.5. A two-dimensional constant-strain-mode displacement patch test (DPT). Prescribed displacement field is $u_x = x$, $u_y = 0$, which gives $e_{xx} = 1$, others zero; (a) shows a patch with external nodes given prescribed forces; (b) interpretation as replacing part of a "background continuum" with the patch.

the same material and fabrication properties as the patch. (As shown below, all elements of the patch must have the same properties).

Q1. How arbitrary can the patch be? Cut out a portion of the background continuum as illustrated in Figures 23.4(b) and 23.5(b). Replace by the patch. *Nothing should happen.* If the patch disturbs the given field, it is not admissible. For specific constraints see Q2.

Q2. Can one have elements of different material or fabrication properties in the patch? No. Think of the background continuum under constant strain. Replacing a portion by that kind of patch by a non-homogeneous medium will produce non-uniform strain. (For rigid body mode tests, however, those differences will not have any effect.)

Q3. Why are the displacements applied to exterior nodes only? Because those are the one in contact with the background continuum, which provides the displacement (Dirichlet) boundary conditions for the patch problem. The interior nodes are solved from the FEM equilibrium conditions.

Q4. What happens if all patch nodes and DOF are given displacements? Nothing very useful. The patch will behave as if the elements were disconnected, so there is no difference from testing single elements. The results only would check if completeness is verified in an individual element. This is useful as a strain computation check and nothing more.

Q5. Can elements of different geometry be included in the test? Yes. For example, triangles and quadrilaterals may be mixed in a 2D test as long as material and fabrication properties are the same. Thus the patch of Figure 23.2(b) is acceptable. But that of Figure 23.2(c) is not, because the two bars are a different structural type, which would alter constant strain states.

Q6. Can patch tests be done by hand? Only for some very simple one-dimensional configurations. Otherwise computer use is mandatory. A strong case for the use of computers anyway is that patch

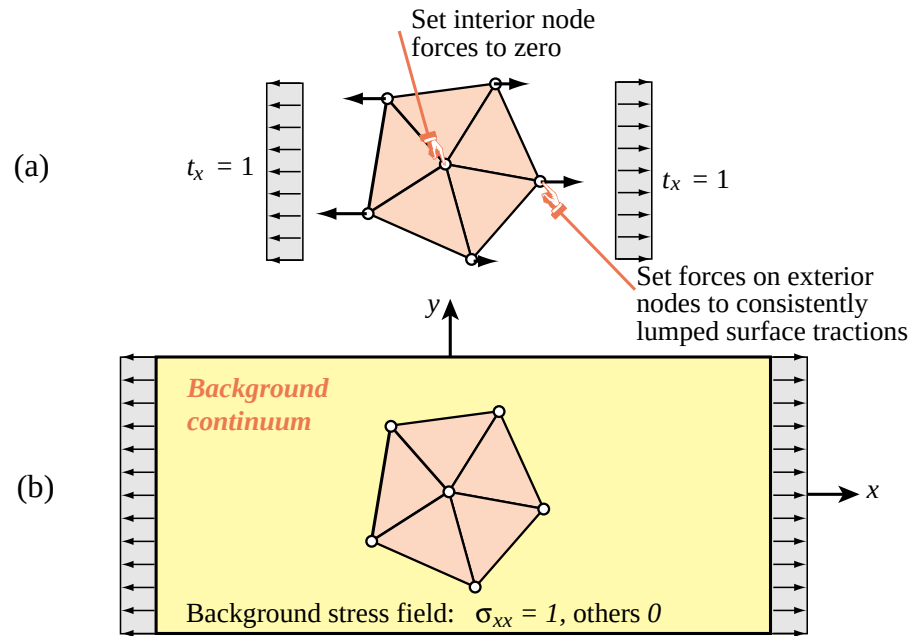


FIGURE 23.6. A two-dimensional force patch test: $\sigma_{xx} = 1$, (a) shows a patch with external patch nodes given prescribed forces; (b) interpretation as replacing part of a stressed “background continuum” with the patch.

tests are often used to verify actual code.

Q7. Are computer patch tests necessarily numeric? Not necessarily. Numerics is of course mandatory is one is verifying element code written in Fortran or C. But the use of computer algebra systems is growing as a tool to *design* elements. In element design it is very convenient to be able to leave geometric, material and fabrication properties as variables. In addition, one can parametrize patch geometries so that a symbolic test is equivalent to an infinite number of numeric tests.

Q8. The applied strain $e_{xx} = 1 = 100\%$ is huge. Finite elements with linearized kinematics are valid only for very small strains. Why are the DPT results correct? This is posed as a Homework exercise.

§23.4.4. The Stress Test Space

For the force version of the physical patch test one needs to reword part of the discussion. Strains become stresses, and boundary displacements become surface tractions.

To create “simple problems” again one thinks of a mesh refinement process. In the limit the state of stress inside each element is sensibly constant. The test space is that of *constant stress modes*. (For elements that model bending, stresses are replaced by moments.) For a two-dimensional plane stress problem, the test space is spanned by two axial stresses and a shear stress

$$\sigma_{xx}, \sigma_{yy}, \sigma_{xy} \quad (23.4)$$

since the stress in any other direction is a linear combination of these. Alternatively one could run

three axial stresses in three independent directions:

$$\sigma_1, \sigma_2, \sigma_3 \quad (23.5)$$

for example at directions 0° , 90° and 45° from x . This avoids the shear patch test, which is more difficult to program.

In 1D, or 2D models integrated through the thickness, one applies stress resultants, such as membrane or axial forces and moments.

For the DPT we distinguished between rigid-body and constant strain modes. The distinction is not necessary for stress modes. This renders this test form simpler in some ways but more complicated in others, in that rigid-body motions must be precluded to avoid singularities.

§23.4.5. The Force Patch Test

The procedure for the force *force patch test* or FPT⁸ will be illustrated by the 2D patch shown in Figure 23.6, which pertains to a plane stress problem.

Figure 23.6(a) shows the FPT administered for uniform stress $\sigma_{xx} = 1$, others zero. On the patch boundary of the patch apply a uniform surface traction $t_x = \sigma_x x$. Convert this to nodal forces using a consistent force lumping approach. Forces at interior DOFs should be zero. Apply a minimal number of displacement BC to eliminate rigid body motions (more about this below). Solve for the displacement components, and recover strains and stresses over the elements. The computed stresses should recover exactly the test state.

If all test states are correctly reproduced, the FPT is passed, and failed otherwise. Note that there is no such thing as “PT passed 90%”. In this form the test is always pass-or-fail (ex

§23.4.6. FPT Q&A

Q1. How arbitrary can the patch be? The “background check” also applies. Nothing should happen to the continuum upon inserting the patch.

Q2. Can one have elements of different material or fabrication properties in the patch? There is a bit more leeway here than in the DPT. The flux of stress, or of stress resultants, should be preserved. This is easier under statically determinate conditions. For example, in the case of a plane-beam test under pure moment, a patch of two beams with different cross sections should work. There are more restrictions in 2D and 3D because of statically indeterminacy cross-effects, such as Poisson’s ratio. In practice most tests of this type are done under the same restrictions as DPT.

Q3. Can singular-stiffness problems arise? Certainly. The test is supposed to be conducted on a free-free patch under equilibrium conditions, but the elements to be checked are likely to be of displacement type. Unsuppressed rigid-body modes (RBMs) will result in a singular patch stiffness, impeding the computation of node displacements. There are basically four ways around this:

1. Apply the *minimum* number of independent displacement BC to suppress the RBMs. It is important not to overconstrain the system, as that may cause force flux perturbations. The way to check that you are doing the right thing is to compute the reaction forces: *they must be*

⁸ Also known as *static patch test* or *flux patch test*.

identically zero since the stress states are in self-equilibrium. The test now becomes of *mixed force-displacement type* since some nodal displacements must be prescribed.

2. Use an equation solver that automatically removes the RBMs. This can be tricky in floating-point work. Unfeasible if the solver comes from a standard library, as in commercial packages.
3. Project the patch stiffness matrix onto the space of deformation modes. This technique is tricky and difficult to use.
4. Use the free-free flexibility technique of [89]. Unlike the preceding one this is foolproof but since the technique is very recent, it has not been used for this purpose.

Q4. When should this test be applied? Ideally both DPT and FPT should be administered when one is checking element code. For example, DPT directly checks rigid body motions but FPT does not. For single-master displacement-assumed elements there is overlap in the strain and stress checks if the stresses are recovered from strains.

§23.5. One Element Tests: IET and SET

Because of practical difficulties incurred in testing all possible patches there have been efforts directed toward translating the original test into statements involving a single element. These will be collectively called *one-element tests*.

The first step along this path was taken by Strang [227], who using integration by parts recast the original test in terms of “jump” contour integrals over element interfaces. An updated account is given by Griffiths and Mitchell [116], who observe that Strang’s test can be passed in three different ways.

- JCS: Jump integrals cancel over common sides of adjacent elements. Examples: Fraeijs de Veubeke’s 3-midside-node triangle [101], Morley’s constant-moment plate element [184].
- JOS: Jump integrals cancel over opposite element sides. Example: Wilson’s incompatible plane rectangle [268].
- JEC: Jump integrals cancel over the element contour. Examples are given in the aforementioned article by Griffiths and Mitchell [116].

Another consequential development, not so well publicized as Strang’s, was undertaken by Bergan and coworkers at Trondheim over the decade 1974–1984. The so called *individual element test*, or IET, was proposed by Bergan and Hanssen [25]. The underlying goal was to establish a test that could be directly carried out on the stiffness equations of a single element — an obvious improvement over the multielement form. In addition the test was to be *constructive*, that is, used as a guide during element formulation, rather than as *a posteriori* check.

The IET has a simple physical motivation: to demand pairwise cancellation of tractions among adjacent elements that are subjected to a common uniform stress state. This is in fact the ‘JCS’ case of the Strang test noted above. Because of this inclusion, the IET is said to be a *strong* version of the patch test in the following sense: any element passing the IET also verifies the conventional multielement form of the patch test, but the converse is not necessarily true.

The IET goes beyond Strang’s test, however, in that it provides *a priori* rules for constructing finite elements. These rules have formed the basis of the Free Formulation (FF) developed by Bergan

and Nygård [27]. The test has also played a key role in the development of high performance finite elements undertaken by the writer.

In an important paper written in response to Stummel's counterexample, Taylor et al [240] defined multielement patch tests in more precise terms, introducing the so-called A, B and C versions. They also discussed a one-element test called the "single element test," herein abbreviated to SET. They used the BCIZ plate bending element [18] to show that an element may pass the SET but fail multielement versions, and consequently that tests involving single elements are to be viewed with caution.

§23.6. *Historical Background

The patch test for convergence is a fascinating area in the development of nonconforming finite element methods. It grew up of the brilliant intuition of Bruce Irons. Initially developed in the mid-1960s at Rolls Royce and then at the Swansea group headed by O. C. Zienkiewicz, by the early 1970s the test had become a powerful and practical tool for evaluating and checking nonconforming elements. And yet today it remains a controversial issue: accepted by many finite element developers while ignored by others, welcomed by element programmers, distrusted by mathematicians. For tracing down the origins of the test there is no better source than a 1973 survey article by the Irons and Razzaque [144]. Annotations to the quoted material are inserted in square brackets, and reference numbers have been altered to match those of the present Chapter.

"Origins of the Patch Test"

In 1965 even engineering intuition dared not predict the behavior of certain finite elements. Experience forced those engineers who doubted it to admit that interelement continuity was important: the senior author [Bruce Irons] believed that it was necessary for convergence. It is not known which ideas inspired a numerical experiment by Tocher and Kapur [248] which demonstrated convergence within 0.3% in a biharmonic problem of plate bending, using equal rectangular elements with 1, x , y , x^2 , xy , y^2 , x^3 , x^2y , xy^2 , y^3 , x^3y and xy^3 , as functional basis. The nodal variables of this Ari Adini rectangle [3] are w , $\partial w/\partial x$ and $\partial w/\partial y$ at the four corners, and this element guarantees only C^0 conformity.

Some months later, research at Rolls-Royce on the Zienkiewicz nonconforming triangle [18] — a similar plate-bending element — clarified the situation. [This element is that identified by 'BCIZ' in the sequel.] Three elements with C^1 continuity were simultaneously available, and, because the shape function subroutine used for numerical integration had been exhaustively tested, the results were trustworthy. It was observed: (a) that every problem giving constant curvature over the whole domain was accurately solved by the conforming elements whatever the mesh pattern, as was expected, and (b) that the nonconforming element was also successful, but only for one particular mesh pattern. [The bending element test referred to in this passage is described in the Addendum to [18]. This Addendum was not part of the original paper presented at the First Wright-Patterson Conference held in September 1965; it was added to the Proceedings that appeared in 1966. The name "patch test" will not be found there; see the Appendix of the Strang-Fix monograph [227] for further historical details.]

Thus the patch test was born. For if the external nodes of any sub-assembly of a successful assembly of elements are given prescribed values corresponding to an arbitrary state of constant curvature, then the internal nodes must obediently take their correct values. (An internal node is defined as one completely surrounded by elements.) Conversely, if two overlapping patches can reproduce any given state of constant curvature, they should combine into a larger successful patch, provided that every external node lost is internal to one of the original patches. For such nodes are in equilibrium at their correct values, and should behave correctly as internal nodes of the extended patch. In an unsuccessful patch test, the internal nodes take unsuitable values, which introduce

interelement discontinuities. The errors in deflection may be slight, but the errors in curvature may be $\pm 20\%$. We must recognize two distinct types of errors:

- (i) The finite element equations would not be exactly satisfied by the correct values at the internal nodes — in structural terms, we have disequilibrium;
- (ii) The answers are nonunique because the matrix of coefficients $\mathbf{K}$ is semidefinite.

Role of the Patch Test

Clearly the patch test provides a *necessary* condition for convergence with fine mesh. We are less confident that it provides a *sufficient* condition. The argument is that if the mesh is fine, the patches are also small. Over any patch the correct solution gives almost uniform conditions to which the patch is known to respond correctly — provided that the small perturbations from uniform conditions do not cause a disproportionate response in the patch: we hope to prevent this by insisting that $\mathbf{K}$ is positive definite. [Given later developments, this was an inspired guess.]

The patch test is invaluable to the research worker. Already, it has made respectable

- (i) Elements that do not conform,
- (ii) Elements that contain singularities,
- (iii) Elements that are approximately integrated,
- (iv) Elements that have no clear physical basis.

In short, the patch test will help a research worker to exploit and justify his wildest ideas. It largely restores the freedom enjoyed by the early unsophisticated experimenters.”

The late 1960s and early 1970s were a period of unquestionable success for the test. That optimism is evident in the article quoted above, and prompted Gilbert Strang to develop a mathematical version popularized in the Strang-Fix monograph [227].

Confidence was shaken in the late 1970s by several developments. Numerical experiments, for example, those of Sander and Beckers [215] suggested that the test is not necessary for convergence, thus disproving Irons’ belief stated above. Then a counterexample by Stummel [233] purported to show that the test is not even sufficient. This motivated defensive responses by Irons [148] shortly before his untimely death and by Taylor et al. [240] These papers tried to set out the engineering version of the test on a more precise basis.

Despite these ruminations many questions persist, as noted in the lucid review article by Griffiths and Mitchell [116]. The most important ones are listed below.

- Q1. What is a patch? Is it the ensemble of all possible meshes? Are some meshes excluded? Can these meshes contain different types of elements?
- Q2. The test was originally developed for harmonic and biharmonic problems of compressible-elasticity, for which the concept of “constant strains” or “constant curvatures” is unambiguous. But what is the equivalent concept for arches and shells, if one is unwilling to undergo a limiting process?
- Q3. What are the modifications required for incompressible media? Is the test applicable to dynamic or nonlinear problems?
- Q4. Are single-element versions of the test equivalent to the conventional, multielement versions?
- Q5. Is the test restricted to nonconforming assumed-displacement elements? Can it be extended to encompass assumed-stress or assumed-strain mixed and hybrid elements? Initial attempts in this direction were made by Fraeijs de Veubeke in 1974.

Remark 23.2. Stummel has constructed [234] a *generalized patch test*, which is mathematically impeccable in that it provides necessary and sufficient conditions for convergence. Unfortunately such test lacks practical side benefits of Irons’ patch test, such as element checkout by computer (either numerically or symbolically), because it is administered as a mathematically limiting process in function spaces. Furthermore, it does not apply to a mixture of different element types, which is of crucial importance in complex physical models.

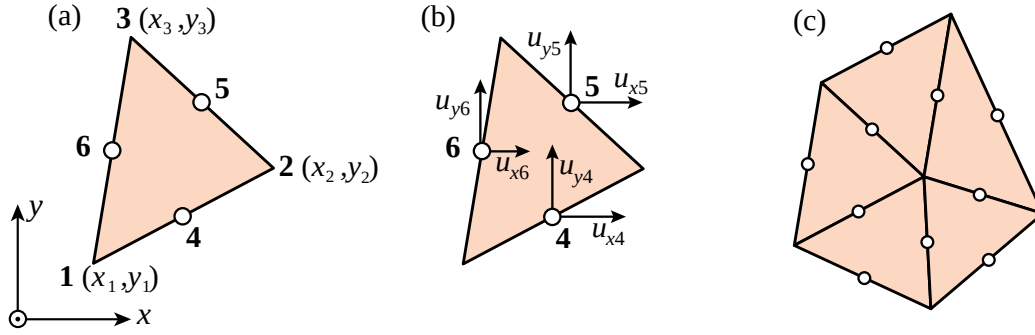


FIGURE 23.7. The Veubeke equilibrium triangle: (a) geometric definition; (b) degree-of-freedom configuration; (c) element patch showing how triangles are connected at the midpoints.

§23.7. The Veubeke Equilibrium Triangle

The Veubeke equilibrium triangle is briefly described below so as to provide background material for applying the multielement patch test in Exercise E23.2. This element differs from the conventional 3-node plane stress triangle, also known as *Turner triangle*, in the degree-of-freedom configuration. As illustrated in Figure 23.7, freedoms are moved to the midpoints {4, 5, 6} while the corner nodes {1, 2, 3} still define the geometry of the element. In the FEM terminology introduced in Chapter 6, the geometric nodes {1, 2, 3} and the connection nodes {4, 5, 6} no longer coincide. The node displacement vector collects the freedoms shown in Figure 23.7(b):

$$\mathbf{u}^e = [u_{x4} \quad u_{y4} \quad u_{x5} \quad u_{y5} \quad u_{x6} \quad u_{y6}]^T. \quad (23.6)$$

The quickest way to formulate the stiffness matrix of this element is to relate (23.6) to the node displacements of the Turner triangle, identified as

$$\mathbf{u}_T^e = [u_{x1} \quad u_{y1} \quad u_{x2} \quad u_{y2} \quad u_{x3} \quad u_{y3}]^T. \quad (23.7)$$

§23.7.1. Kinematic Relations

The node freedom vectors (23.6) and (23.7) are easily related since by linear interpolation along the sides one obviously has $u_{x4} = \frac{1}{2}(u_{x1} + u_{x2})$, $u_{y4} = \frac{1}{2}(u_{y1} + u_{y2})$, etc. Expressing those links in matrix form gives

$$\begin{bmatrix} u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \\ u_{x6} \\ u_{y6} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}, \quad \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \\ u_{x6} \\ u_{y6} \end{bmatrix}. \quad (23.8)$$

In compact form: $\mathbf{u}^e = \mathbf{T}_{VT} \mathbf{u}_T^e$ and $\mathbf{u}_T^e = \mathbf{T}_{TV} \mathbf{u}^e$, with $\mathbf{T}_{VT} = \mathbf{T}_{TV}^{-1}$. The shape functions are

$$N_4 = \zeta_1 + \zeta_2 - \zeta_3, \quad N_5 = -\zeta_1 + \zeta_2 + \zeta_3, \quad N_6 = \zeta_1 - \zeta_2 + \zeta_3. \quad (23.9)$$

Calling the Turner triangle strain-displacement matrix of as $\mathbf{B}_T$, the corresponding matrix that relates $\mathbf{e} = \mathbf{B} \mathbf{u}^e$ in the Veubeke equilibrium triangle becomes

$$\mathbf{B} = \mathbf{B}_T \mathbf{T}_{TV} = \frac{1}{A} \begin{bmatrix} y_{21} & 0 & y_{32} & 0 & y_{13} & 0 \\ 0 & x_{12} & 0 & x_{23} & 0 & x_{31} \\ x_{12} & y_{21} & x_{23} & y_{32} & x_{31} & y_{13} \end{bmatrix} \quad (23.10)$$

§23.7.2. Stiffness Matrix

As usual the element stiffness matrix is given by $\mathbf{K}^e = \int_{\Omega^e} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega$. For constant plate thickness h one obtains the closed form

$$\mathbf{K}^e = A h \mathbf{B}^T \mathbf{E} \mathbf{B} = \frac{h}{A} \begin{bmatrix} y_{21} & 0 & x_{12} \\ 0 & x_{12} & y_{21} \\ y_{32} & 0 & x_{23} \\ 0 & x_{23} & y_{32} \\ y_{13} & 0 & x_{31} \\ 0 & x_{31} & y_{13} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{21} & 0 & y_{32} & 0 & y_{13} & 0 \\ 0 & x_{12} & 0 & x_{23} & 0 & x_{31} \\ x_{12} & y_{21} & x_{23} & y_{32} & x_{31} & y_{13} \end{bmatrix}. \quad (23.11)$$

§23.7.3. Implementation

The implementation of the Veubeke equilibrium triangle as a *Mathematica* module that returns $\mathbf{K}^e$ is shown in Figure 23.8. It needs only 8 lines of code. It is invoked as

$$\text{Ke} = \text{Trig3VeubekeMembraneStiffness}[\text{ncoor}, \text{Emat}, h, \text{numer}]; \quad (23.12)$$

where the arguments are

- ncoor Element corner coordinates, arranged as a list: $\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}\}$.
- Emat A two-dimensional list storing the 3×3 plane stress matrix of elastic moduli as $\{\{E_{11}, E_{12}, E_{13}\}, \{E_{12}, E_{22}, E_{23}\}, \{E_{13}, E_{23}, E_{33}\}\}$.
- h Plate thickness, assumed uniform over the triangle.
- numer A logical flag: True to request floating-point computation, else False.

This module is exercised by the statements listed at the top of Figure 23.9, which forms a triangle with corner coordinates $\{\{0, 0\}, \{3, 1\}, \{2, 2\}\}$, isotropic material matrix with $E_{11} = E_{22} = 32$, $E_{12} = 8$, $E_{33} = 16$, others zero, and unit thickness. The results are shown at the bottom of Figure 23.9. Note that the element is rank sufficient.

```
Trig3VeubekeMembraneStiffness[ncoor_, Emat_, h_, numer_] := Module[{
  x1, x2, x3, y1, y2, y3, x12, x23, x31, y21, y32, y13, A, Be, Te, Ke},
  {{x1, y1}, {x2, y2}, {x3, y3}} = ncoor;
  A = Simplify[(x2*y3 - x3*y2 + (x3*y1 - x1*y3) + (x1*y2 - x2*y1))/2];
  {x12, x23, x31, y21, y32, y13} = {x1 - x2, x2 - x3, x3 - x1, y2 - y1, y3 - y2, y1 - y3};
  Be = {{y21, 0, y32, 0, y13, 0}, {0, x12, 0, x23, 0, x31},
        {x12, y21, x23, y32, x31, y13}}/A;
  If[numer, Be = N[Be]]; Ke = A*h*Transpose[Be].Emat.Be;
  Return[Ke];
```

FIGURE 23.8. Implementation of Veubeke equilibrium triangle stiffness matrix as a *Mathematica* module.

```

ncoor={{0,0},{3,1},{2,2}}; Emat=8*{{8,2,0},{2,8,0},{0,0,3}};
Ke=Trig3VeubekeMembraneStiffness[ncoor,Emat,1,False];
Print["Ke=",Ke//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Ke]]]];

```

$$Ke = \begin{pmatrix} 140 & -60 & -4 & -28 & -136 & 88 \\ -60 & 300 & -12 & -84 & 72 & -216 \\ -4 & -12 & 44 & 20 & -40 & -8 \\ -28 & -84 & 20 & 44 & 8 & 40 \\ -136 & 72 & -40 & 8 & 176 & -80 \\ 88 & -216 & -8 & 40 & -80 & 176 \end{pmatrix}$$

eigs of Ke={557.318, 240., 82.6816, 0, 0, 0}

FIGURE 23.9. Test statements to exercise the module of Figure 23.8, and outputs.

§23.7.4. Patch Test Configuration

The Veubeke equilibrium triangle is complete since the sum of shape functions of (23.9) is unity: $N_4 + N_5 + N_6 = \zeta_1 + \zeta_2 - \zeta_3 - \zeta_1 + \zeta_2 + \zeta_3 + \zeta_1 - \zeta_2 + \zeta_3 = \zeta_1 + \zeta_2 + \zeta_3 = 1$. But it is nonconforming (interelement discontinuous) because the variation of displacements along each side is linear, but there is only one connection node at the midpoint. Thus a patch test is a good idea.

The multielement patch test is to be carried out for the two-triangle patch shown in Figure 23.10. The patch geometry is rectangular for simplicity. The patch exterior connector nodes are 5,6,7,8 and the interior node is 9. Test details are specified in Exercise E23.2.

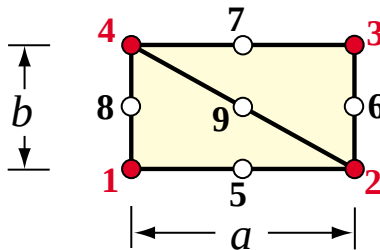


FIGURE 23.10. Two-element patch for testing Veubeke equilibrium triangle. Red filled circles and white filled circles denote geometric and connection nodes, respectively.

Homework Exercises for Chapter 23**The Patch Test**

EXERCISE 23.1 [A/D:15] Answer Q8 of §23.3.3.

EXERCISE 23.2 [C:25] Carry out both displacement and force patch tests for the patch of Figure 23.10, which is fabricated with two Veubeke equilibrium triangles. Take $a = 2$, $b = 1$, $E = 10$, $\nu = 0$ and unit thickness. If a patch fails, do not proceed further, and try to explain what went wrong.

EXERCISE 23.3 [A:50] (Possibly worth a Fields Medal Prize, worth \$500,000). Develop a practical patch test executable on the computer, that is necessary and sufficient for convergence.

24

Kirchhoff Plates: Field Equations

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§24.1. Introduction

Multifield variational principles were primarily motivated by “difficult” structural models, such as plate bending, shells and near incompressible behavior. In this Chapter we begin the study of one of those difficult problems: plate bending.

Following a review of the wide spectrum of plate models, attention is focused on the Kirchhoff model for bending of thin (but not too thin) plates. The field equations for isotropic and anisotropic plates are then discussed. The Chapter closes with an annotated bibliography.

§24.2. Plates: Basic Concepts

In the IFEM course [86] a *plate* was defined as a three-dimensional body endowed with special geometric features. Prominent among them are

Thinness: One of the plate dimensions, called its *thickness*, is much smaller than the other two.

Flatness: The *midsurface* of the plate, which is the locus of the points located half-way between the two plate surfaces, is a plane.

In Chapter 14 of that course we studied plates in a *plane stress* state, also called *membrane* or *lamina* state in the literature. This state occurs if the external loads act on the plate midsurface as sketched in Figure 24.1(a). Under these conditions the distribution of stresses and strains across the thickness may be viewed as uniform, and the three dimensional problem can be reduced to two dimensions. If the plate displays linear elastic behavior under the range of applied loads then we have effectively reduced the problem to one of two-dimensional elasticity.

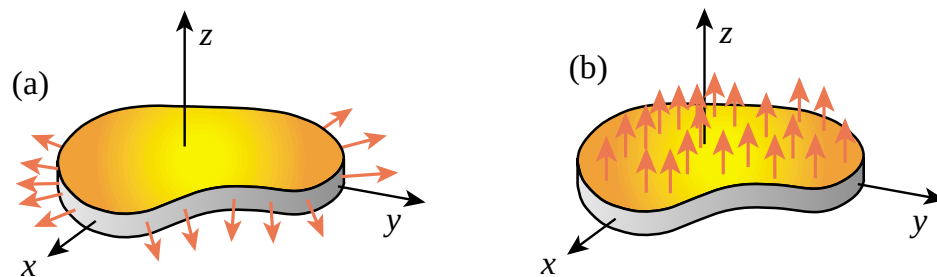


FIGURE 24.1. A flat plate structure in: (a) plane stress or membrane state, (b) bending state.

§24.2.1. Function

In this Chapter we study plates subjected to *transverse loads*, that is, loads normal to its midsurface as sketched in Figure 24.1(b). As a result of such actions the plate displacements out of its plane and the distribution of stresses and strains across the thickness is no longer uniform. Finding those displacements, strains and stresses is the problem of *plate bending*.

Plate bending components occur when plates function as shelters or roadbeds: flat roofs, bridge and ship decks. Their primary function is to carry out lateral loads to the support by a combination of moment and shear forces. This process is often supported by integrating beams and plates. The beams act as stiffeners and edge members.

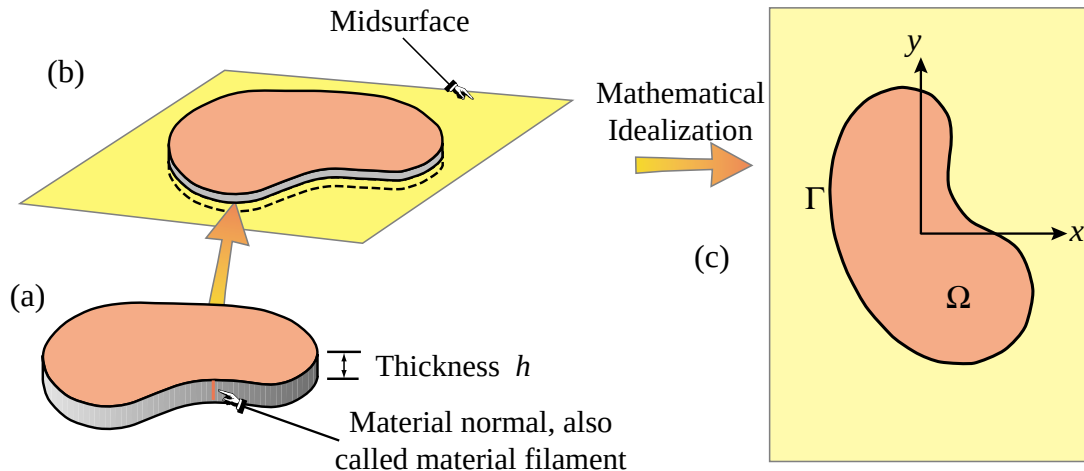


FIGURE 24.2. Idealization of plate as two-dimensional mathematical problem.

If the applied loads contain both loads and in-plane components, plates work simultaneously in membrane and bending. An example where that happens is in *folding plate* structures common in some industrial buildings. Those structures are composed of repeating plates that transmit roof loads to the edge beams through a combination of bending and “arch” actions. If so all plates experience both types of action. Such a combination is treated in finite element methods by *flat shell* models: a superposition of flat membrane and bending elements. Plates designed to resist both membrane and bending actions are sometimes called *slabs*, as in pavements.

§24.2.2. Terminology

This subsection defines a plate structure in a more precise form and introduces terminology.

Consider first a flat surface, called the plate *reference surface* or simply its *midsurface* or *midplane*. See Figure 24.2. We place the axes x and y on that surface to locate its points. The third axis, z is taken normal to the reference surface forming a right-handed Cartesian system. Axis x and y are placed in the midplane, forming a right-handed Rectangular Cartesian Coordinate (RCC) system. If the plate is shown with an horizontal midsurface, as in Figure 24.2, we shall orient z upwards.

Next, imagine *material normals*, also called *material filaments*, directed along the *normal* to the reference surface (that is, in the z direction) and extending $h/2$ above and $h/2$ below it. The magnitude h is the *plate thickness*. We will generally allow the thickness to be a function of x, y , that is $h = h(x, y)$, although most plates used in practice are of uniform thickness because of fabrication considerations. The end points of these filaments describe two bounding surfaces, called the *plate surfaces*. The one in the $+z$ direction is by convention called the top surface whereas the one in the $-z$ direction is the bottom surface.

Such a three dimensional body is called a plate if the thickness dimension h is everywhere *small*, but not too small, compared to a *characteristic length* L_c of the plate midsurface. The term “small” is to be interpreted in the engineering sense and not in the mathematical sense. For example, h/L_c is typically 1/5 to 1/100 for most plate structures. A paradox is that an extremely thin plate, such as the fabric of a parachute or a hot air balloon, ceases to structurally function as a thin plate!

A plate is *bent* if it carries loads normal to its midsurface as pictured in Figure 24.1(b). The resulting problems of structural mechanics are called:

Inextensional bending: if the plate does not experience appreciable stretching or contractions of its midsurface. Also called simply *plate bending*.

Extensional bending: if the midsurface experiences significant stretching or contraction. Also called *combined bending-stretching*, *coupled membrane-bending*, or *shell-like behavior*.

The bent plate problem is reduced to two dimensions as sketched in Figure 24.2(c). The reduction is done through a variety of mathematical models discussed next.

§24.2.3. Mathematical Models

The behavior of plates in the membrane state of Figure 24.1(a) is adequately covered by two-dimensional continuum mechanics models. On the other hand, bent plates give rise to a wider range of physical behavior because of possible coupling of membrane and bending actions. As a result, several mathematical models have been developed to cover that spectrum. The more important models are listed next.

Membrane shell model: for extremely thin plates dominated by membrane effects, such as inflatable structures and fabrics (parachutes, sails, balloons, tents, inflatable masts, etc).

Von-Karman model: for very thin bent plates in which membrane and bending effects interact strongly on account of finite lateral deflections.¹ Important model for post-buckling analysis.

Kirchhoff model: for thin bent plates with small deflections, negligible shear energy and uncoupled membrane-bending action.²

Reissner-Mindlin model: for thin and moderately thick bent plates in which first-order transverse shear effects are considered.³ Particularly important in dynamics as well as honeycomb and composite wall constructions.

High order composite models: for detailed (local) analysis of layered composites including inter-lamina shear effects.⁴

Exact models: for the analysis of additional effects using three dimensional elasticity.⁵

The first two models are *geometrically nonlinear* and thus fall outside the scope of this course. The last four models are *geometrically linear* in the sense that *all governing equations are set up in the*

¹ Th. v. Kármán, Festigkeitsprobleme im Maschinenbau., *Encyklopädie der Mathematischen Wissenschaften*, **4/4**, 311–385, 1910.

² G. Kirchhoff, Über das Gleichgewicht und die Bewegung einer elastischen Scheibe, *Crelles J.*, **40**, 51–88, 1850. Also his Vorlesungen über Mathematischen Physik, *Mechanik*, 1877., translated to French by Clebsch.

³ E. Reissner, The effect of transverse shear deformation on the bending of elastic plates, *J. Appl. Mech.*, **12**, 69–77, 1945; also E. Reissner, On bending of elastic plates, *Quart. Appl. Math.*, **5**, 55–68, 1947. Mindlin's version, intended for dynamics, was published in R. D. Mindlin, Influence of rotary inertia and shear on flexural vibrations of isotropic, elastic plates, *J. Appl. Mech.*, **18**, 31–38, 1951. Timoshenko and Woinowsky-Krieger, cited in §24.5, follow A. E. Green, On Reissner's theory of bending of elastic plates, *Quart. Appl. Math.*, **7**, 223–228, 1949.

⁴ See for example the book by J. N. Reddy cited in §24.5.

⁵ The book of Timoshenko and Woinowsky-Krieger cited in §24.5 contains a brief treatment of the exact analysis in Ch 4.

reference or initially-flat configuration. The last two models are primarily used in detailed or local stress analysis near edges, point loads or openings.

All models may incorporate other types of nonlinearities due to, for example, material behavior, composite fracture, cracking or delamination, as well as certain forms of boundary conditions. In this course, however, we shall look only at the linear elastic versions. Furthermore, we shall focus attention only on the Kirchhoff and Reissner-Mindlin plate models because these are the most commonly used in statics and vibrations, respectively.

§24.3. The Kirchhoff Plate

Historically the first model of thin plate bending was developed by Lagrange, Poisson and Kirchhoff. It is known as the Kirchhoff plate model, of simply *Kirchhoff plate*, in honor of the German scientist who “closed” the mathematical formulation through the variational treatment of boundary conditions. In the finite element literature Kirchhoff plate elements are often called C^1 plate elements because that is the continuity order nominally required for the transverse displacement shape functions.

The Kirchhoff model is applicable to elastic plates that satisfy the following conditions.

- The plate is *thin* in the sense that the thickness h is small compared to the characteristic length(s), but not so thin that the lateral deflection w becomes comparable to h .
- The plate thickness is either uniform or varies slowly so that three-dimensional stress effects are ignored.
- The plate is symmetric in fabrication about the midsurface.
- Applied transverse loads are distributed over plate surface areas of dimension h or greater.⁶
- The support conditions are such that no significant extension of the midsurface develops.

We now describe the field equations for the Kirchhoff plate model.

§24.3.1. Kinematic Equations

The kinematics of a Bernoulli-Euler beam, as studied in Chapter 12 of [86], is based on the assumption that *plane sections remain plane and normal to the deformed longitudinal axis*. The kinematics of the Kirchhoff plate is based on the extension of this to *biaxial bending*:

“Material normals to the original reference surface remain straight and normal to the deformed reference surface.”

This assumption is illustrated in Figure 24.3. Upon bending, particles that were on the midsurface $z = 0$ undergo a deflection $w(x, y)$ along z . The slopes of the midsurface in the x and y directions are $\partial w / \partial x$ and $\partial w / \partial y$. The *rotations* of the material normal about x and y are denoted by θ_x and

⁶ The Kirchhoff model can accept point or line loads and still give reasonably good deflection and bending stress predictions for homogeneous wall constructions. A detailed stress analysis is generally required, however, near the point of application of the loads using more refined models; for example with solid elements.

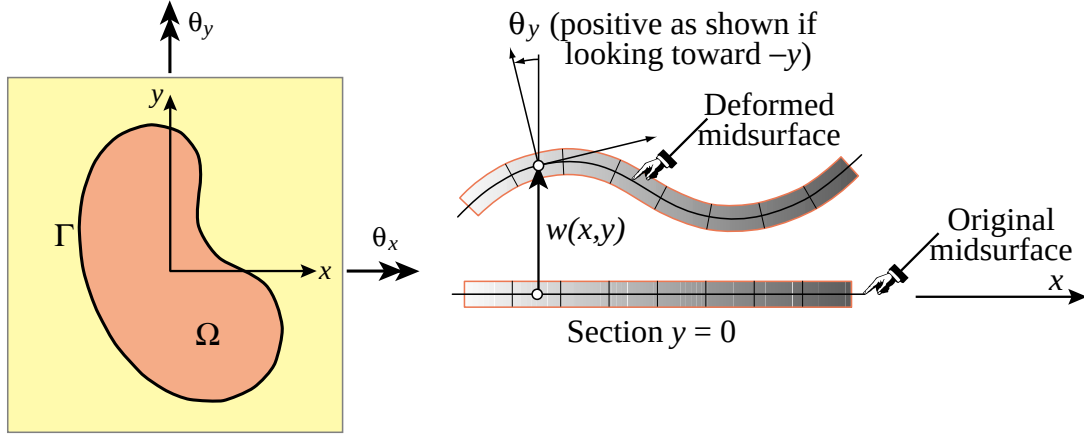


FIGURE 24.3. Kinematics of Kirchhoff plate. Lateral deflection w greatly exaggerated for visibility. In practice $w \ll h$ for the Kirchhoff model to be valid.

θ_y , respectively. These are positive as per the usual rule; see Figure 24.3. For small deflections and rotations the foregoing kinematic assumption relates these rotations to the slopes:

$$\theta_x = \frac{\partial w}{\partial y}, \quad \theta_y = -\frac{\partial w}{\partial x}. \quad (24.1)$$

The displacements $\{u_x, u_y, u_z\}$ of a plate particle $P(x, y, z)$ not necessarily located on the midsurface are given by

$$u_x = -z \frac{\partial w}{\partial x} = z\theta_y, \quad u_y = -z \frac{\partial w}{\partial y} = -z\theta_x, \quad u_z = w. \quad (24.2)$$

The strains associated with these displacements are obtained from the elasticity equations:

$$\begin{aligned} e_{xx} &= \frac{\partial u_x}{\partial x} = -z \frac{\partial^2 w}{\partial x^2} = -z \kappa_{xx}, \\ e_{yy} &= \frac{\partial u_y}{\partial y} = -z \frac{\partial^2 w}{\partial y^2} = -z \kappa_{yy}, \\ e_{zz} &= \frac{\partial u_z}{\partial z} = -z \frac{\partial^2 w}{\partial z^2} = 0, \\ 2e_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -2z \frac{\partial^2 w}{\partial x \partial y} = -2z \kappa_{xy}, \\ 2e_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0, \\ 2e_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0. \end{aligned} \quad (24.3)$$

Here

$$\kappa_{xx} = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_{yy} = \frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y}. \quad (24.4)$$

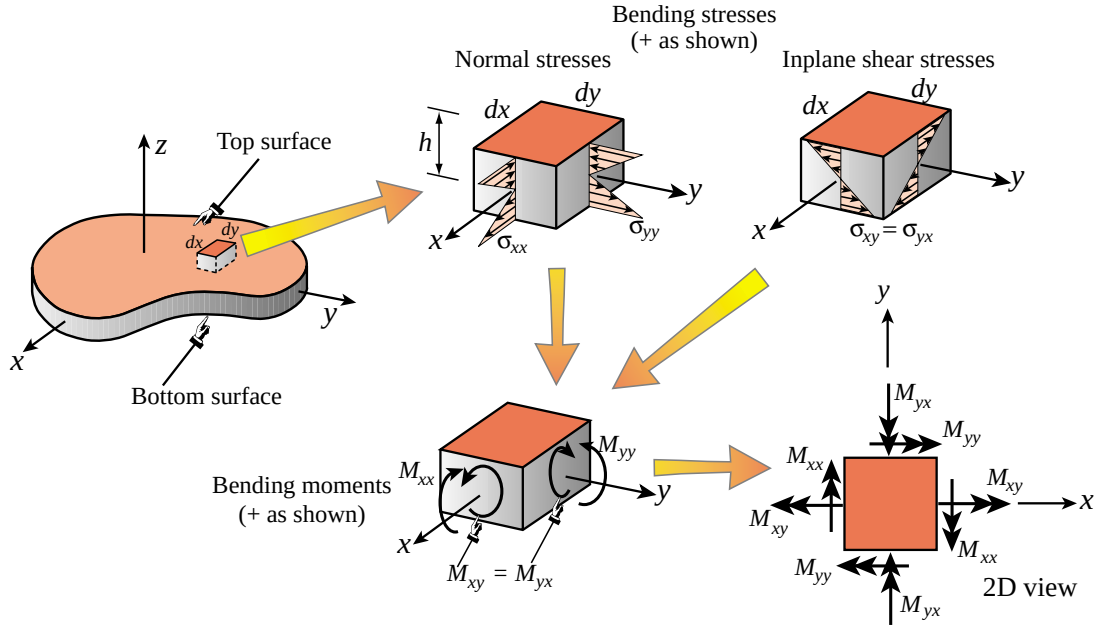


FIGURE 24.4. Bending stresses and moments in a Kirchhoff plate, illustrating sign conventions.

are the *curvatures* of the deflected midsurface. It is seen that the entire displacement and strain field are fully determined if $w(x, y)$ is given.

Remark 24.1. Many entry-level textbooks on plates introduce the foregoing relations gently through geometric arguments, because students may not be familiar with 3D elasticity theory. The geometric approach has the advantage that the kinematic limitations of the plate bending model are more easily visualized. On the other hand the direct approach followed here is more compact.

Remark 24.2. Some inconsistencies of the Kirchhoff model emerge on taking a closer look at (24.3). For example, the transverse shear strains are zero. If the plate is isotropic and follows Hooke's law, this implies $\sigma_{xz} = \sigma_{yz} = 0$ and consequently there are no transverse shear forces. But these forces appear necessarily from the equilibrium equations as discussed in §24.3.4.

Similarly, $e_{zz} = 0$ says that the plate is in plane strain whereas plane stress: $\sigma_{zz} = 0$, is a closer approximation to the physics. For a homogeneous isotropic plate, plane strain and plane stress coalesce if and only if Poisson's ratio is zero. Both inconsistencies are similar to those encountered in the Bernoulli-Euler beam model, and have been the topic of hundreds of learned papers.

§24.3.2. Moment-Curvature Relations

The nonzero bending strains e_{xx} , e_{yy} and e_{xy} produce bending stresses σ_{xx} , σ_{yy} and σ_{xy} as depicted in Figure 24.4. The stress $\sigma_{xy} = \sigma_{yx}$ is sometimes referred to as the in-plane shear stress or the bending shear stress, to distinguish it from the transverse shear stresses σ_{xz} and σ_{yz} studied later.

To establish the plate constitutive equations in moment-curvature form, it is necessary to make several assumptions as to plate material and fabrication. We shall assume here that

- The plate is *homogeneous*, that is, fabricated of the same material through the thickness.
- Each plate lamina $z = \text{constant}$ is in plane stress.

- The plate material obeys Hooke's law for plane stress, which in matrix form is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = -z \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}. \quad (24.5)$$

The bending moments M_{xx} , M_{yy} and M_{xy} are stress resultants with dimension of moment per unit length, that is, force. For example, kips-in/in = kips. The positive sign conventions are indicated in Figure 24.4. The moments are calculated by integrating the elementary stress couples through the thickness:

$$\begin{aligned} M_{xx} dy &= \int_{-h/2}^{h/2} -\sigma_{xx} z dy dz \Rightarrow M_{xx} = - \int_{-h/2}^{h/2} \sigma_{xx} z dz, \\ M_{yy} dx &= \int_{-h/2}^{h/2} -\sigma_{yy} z dx dz \Rightarrow M_{yy} = - \int_{-h/2}^{h/2} \sigma_{yy} z dz, \\ M_{xy} dy &= \int_{-h/2}^{h/2} -\sigma_{xy} z dy dz \Rightarrow M_{xy} = - \int_{-h/2}^{h/2} \sigma_{xy} z dz, \\ M_{yx} dx &= \int_{-h/2}^{h/2} -\sigma_{yx} z dx dz \Rightarrow M_{yx} = - \int_{-h/2}^{h/2} \sigma_{yx} z dz. \end{aligned} \quad (24.6)$$

It will be shown later that rotational moment equilibrium implies $M_{xy} = M_{yx}$. Consequently only M_{xx} , M_{yy} and M_{xy} need to be calculated. Inserting (24.5) into (24.6) and integrating, one obtains the moment-curvature relation

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \frac{h^3}{12} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}. \quad (24.7)$$

The $D_{ij} = E_{ij}h^3/12$ for $i, j = 1, 2, 3$ are called the *plate rigidity coefficients*. They have dimension of force $\times$ length. For an isotropic material of elastic modulus E and Poisson's ratio ν , (24.7) specializes to

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1 + \nu) \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}. \quad (24.8)$$

Here $D = \frac{1}{12}Eh^3/(1 - \nu^2)$ is called the *isotropic plate rigidity*.

If the bending moments M_{xx} , M_{yy} and M_{xy} are given, the maximum values of the corresponding in-plane stress components can be recovered from

$$\sigma_{xx}^{max,min} = \pm \frac{6M_{xx}}{h^2}, \quad \sigma_{yy}^{max,min} = \pm \frac{6M_{yy}}{h^2}, \quad \sigma_{xy}^{max,min} = \pm \frac{6M_{xy}}{h^2} = \sigma_{yx}^{max,min}. \quad (24.9)$$

These max/min values occur on the plate surfaces as illustrated in Figure 24.4. Formulas (24.9) are useful for stress design.

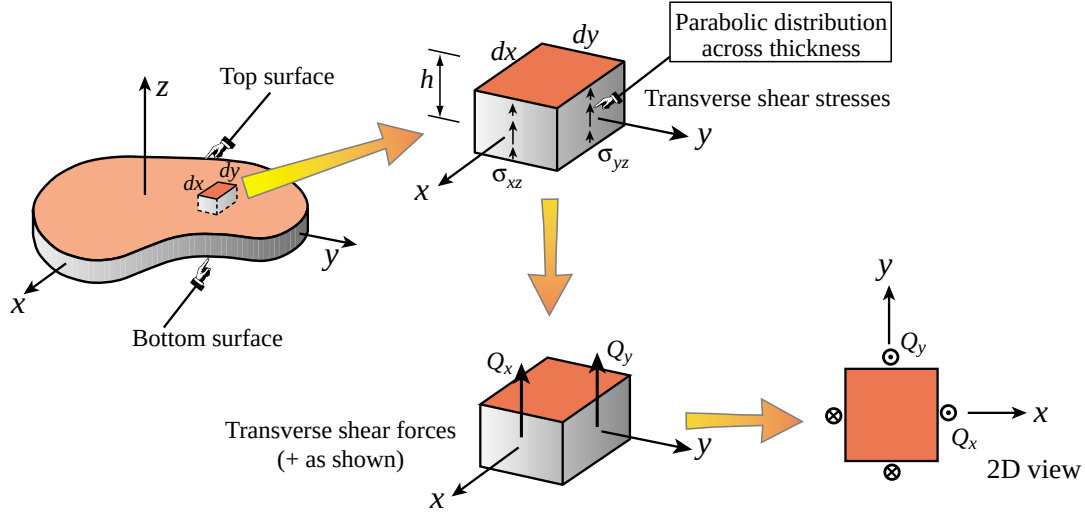


FIGURE 24.5. Transverse shear forces and stresses in a Kirchhoff plate, illustrating sign conventions.

Remark 24.3. For non-homogeneous plates, which are fabricated with different materials, the essential steps are the same but the integration over the plate thickness may be significantly more laborious. (For common structures such as reinforced concrete slabs or laminated composites, the integration process is explained in specialized books.) If the plate fabrication is symmetric about the midsurface, inextensional bending is possible, and the end result is a moment-curvature relation such as (24.8). If the wall fabrication is not symmetric, however, coupling occurs between membrane and bending effects even if the plate is only laterally loaded. The Kirchhoff and the plane stress model need to be linked to account for those effects. This coupling is examined further in chapters dealing with shell models.

§24.3.3. Transverse Shear Forces and Stresses

The equilibrium equations derived in §24.3.4 require the presence of transverse shear forces. Their components in the $\{x, y\}$ system are called Q_x and Q_y . These are defined as shown in Figure 24.5. These are *forces per unit of length*, with physical dimensions such as N/cm or kips/in.

Associated with these shear forces are transverse shear stresses σ_{xz} and σ_{yz} . For a homogeneous plate and using an equilibrium argument similar to Euler-Bernoulli beams, the stresses may be shown to vary parabolically over the thickness, as illustrated in Figure 24.5:

$$\sigma_{xz} = \sigma_{xz}^{max} \left(1 - \frac{4z^2}{h^2}\right), \quad \sigma_{yz} = \sigma_{yz}^{max} \left(1 - \frac{4z^2}{h^2}\right). \quad (24.10)$$

in which the peak values σ_{xz}^{max} and σ_{yz}^{max} , which occur on the midsurface $z = 0$, are only function of x and y . Integrating over the thickness gives

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz} dz = \frac{2}{3} \sigma_{xz}^{max} h, \quad Q_y = \int_{-h/2}^{h/2} \sigma_{yz} dz = \frac{2}{3} \sigma_{yz}^{max} h, \quad (24.11)$$

If Q_x and Q_y are given,

$$\sigma_{xz}^{max} = \frac{3}{2} \frac{Q_x}{h}, \quad \sigma_{yz}^{max} = \frac{3}{2} \frac{Q_y}{h}. \quad (24.12)$$

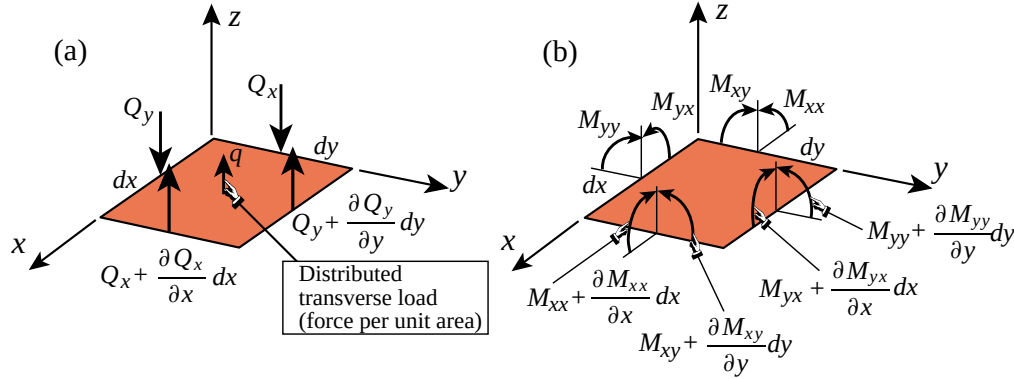


FIGURE 24.6. Differential midsurface elements used to derive the equilibrium equations of a Kirchhoff plate.

As in the case of Bernoulli-Euler beams, predictions such as (24.12) must come entirely from equilibrium analysis. The Kirchhoff plate model *ignores the transverse shear energy* and in fact predicts $\sigma_{xz} = \sigma_{yz} = 0$ from the kinematic equations (24.3). Cf. Remark 24.2. In practical terms this means that stresses (24.12) should be significantly smaller than (24.9). If they are not, the Kirchhoff model is not appropriate, and one should move to the Reissner-Mindlin model, which accounts for transverse shear energy.

§24.3.4. Equilibrium Equations

To derive the interior equilibrium equations we consider differential midsurface elements $dx \times dy$ aligned with the $\{x, y\}$ axes as illustrated in Figure 24.6. Consideration of force equilibrium along the z direction in Figure 24.6(a) yields the shear equilibrium equation

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q. \quad (24.13)$$

where q is the applied lateral force per unit of area. Force equilibrium along the x and y axes is automatically satisfied and does not give additional equilibrium equations.

Consideration of moment equilibrium about y and x in Figure 24.6(b) yields two moment differential equations:

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -Q_x, \quad \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = -Q_y. \quad (24.14)$$

Moment equilibrium about z gives⁷

$$M_{xy} = M_{yx}. \quad (24.15)$$

The four equilibrium equations (24.13)–(24.15) relate six fields: M_{xx} , M_{xy} , M_{yx} , M_{yy} , Q_x and Q_y . Hence plate problems are statically indeterminate.

⁷ Timoshenko's book (cited in §24.5) unfortunately defines M_{yx} so that $M_{xy} = -M_{yx}$. The present convention is far more common in recent books and agrees with the shear reciprocity $\sigma_{xy} = \sigma_{yx}$ of elasticity theory.

Table 24.1 Kirchhoff Plate Field Equations in Matrix and Indicial Form

<i>Field eqn</i>	<i>Matrix form</i>	<i>Indicial form</i>	<i>Equation name for plate problem</i>
KE	$\boldsymbol{\kappa} = \mathbf{P} w$	$\kappa_{\alpha\beta} = w_{,\alpha\beta}$	Kinematic equation
CE	$\mathbf{M} = \mathbf{D} \boldsymbol{\kappa}$	$M_{\alpha\beta} = D_{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}$	Moment-curvature equation
BE	$\mathbf{P}^T \mathbf{M} = q$	$M_{\alpha\beta,\alpha\beta} = q$	Internal equilibrium equation

Here $\mathbf{P}^T = [\partial^2/\partial x^2 \quad \partial^2/\partial y^2 \quad 2\partial^2/\partial x\partial y] = [\partial^2/\partial x_1\partial x_1 \quad \partial^2/\partial x_2\partial x_2 \quad 2\partial^2/\partial x_1\partial x_2]$,
 $\mathbf{M}^T = [M_{xx} \quad M_{yy} \quad M_{xy}] = [M_{11} \quad M_{22} \quad M_{12}]$,
 $\boldsymbol{\kappa}^T = [\kappa_{xx} \quad \kappa_{yy} \quad 2\kappa_{xy}] = [\kappa_{11} \quad \kappa_{22} \quad 2\kappa_{12}]$.
 Greek indices, such as α , run over 1,2 only.

Eliminating Q_x and Q_y from (24.13) and (24.14), and setting $M_{yx} = M_{xy}$, gives the following moment equilibrium equation in terms of the load:

$$\boxed{\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = q.} \quad (24.16)$$

For cylindrical bending in the x direction we recover the Bernoulli-Euler beam equilibrium equation $M'' = q$, where $M \equiv M_{xx}$ and q are understood as given per unit of y width.

§24.3.5. Indicial and Matrix Forms

The foregoing field equations have been given in full Cartesian form. To facilitate reading the existing literature and the working out the finite element formulation, both matrix and indicial forms are displayed in Table 24.1.

§24.3.6. Skew Cuts

The preceding relations were obtained by assuming that differential elements were aligned with x and y . Consider now the case of stress resultants acting on a plane section *skewed* with respect to the x and y axes. This is illustrated in Figure 24.7, which defines notation and sign conventions. The exterior normal $\mathbf{n}$ to this cut forms an angle ϕ with x , positive counterclockwise about z . The tangential direction $\mathbf{s}$ forms an angle $90^\circ + \phi$ with x .

The bending moments and transverse shear forces defined in Figure 24.7 are related to the Cartesian components by

$$\begin{aligned} Q_n &= Q_x c_\phi + Q_y s_\phi, \\ M_{nn} &= M_{xx} c_\phi^2 + M_{yy} s_\phi^2 + 2M_{xy} s_\phi c_\phi, \\ M_{ns} &= (M_{yy} - M_{xx}) s_\phi c_\phi + M_{xy} (c_\phi^2 - s_\phi^2), \end{aligned} \quad (24.17)$$

in which $c_\phi = \cos \phi$ and $s_\phi = \sin \phi$.

These relations can be obtained directly by considering the equilibrium of the midsurface triangles shown in the bottom of Figure 24.7. Alternatively they can be derived by transforming the wall stresses σ_{xx} , σ_{xy} , σ_{yx} , σ_{xz} and σ_{yz} to an oblique cut, and then integrating through the thickness.

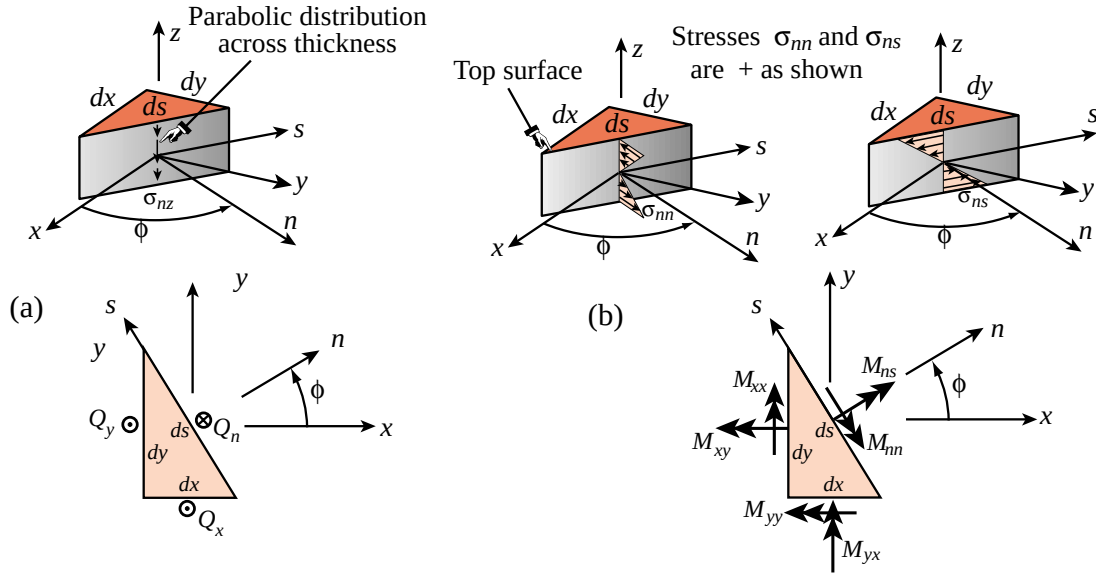


FIGURE 24.7. Skew plane sections in Kirchhoff plate.

§24.3.7. The Strong Form Diagram

Figure 24.8 shows the Strong Form diagram for the field equations of the Kirchhoff plate. The boundary conditions are omitted since those are treated in the next Chapter.

§24.4. The Biharmonic Equation

Consider a *homogeneous isotropic* plate of *constant rigidity* D . Elimination of the bending moments and curvatures from the field equations yields the famous equation for thin plates, first derived by Lagrange⁸

$$D\nabla^4 w = D\nabla^2 \nabla^2 w = q, \quad (24.18)$$

in which

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \quad (24.19)$$

is the biharmonic operator. Thus, under the foregoing constitutive and fabrication assumptions the plate deflection $w(x, y)$ satisfies a non-homogeneous biharmonic equation. If the lateral load q vanishes, as happens with plates loaded only on their boundaries, the deflection w satisfies the homogeneous biharmonic equation.

The biharmonic equation is the analog of the Bernoulli-Euler beam equation for uniform bending rigidity EI : $EI w^{IV} = q$, where $w^{IV} = d^4 w / dx^4$. Equation (24.18) was the basis of much of the early work in plates, both analytical and numerical (the latter by either finite difference or Galerkin methods). After the advent of finite elements it is largely a mathematical curiosity.

⁸ Never published; found posthumously in his Notes (1813).

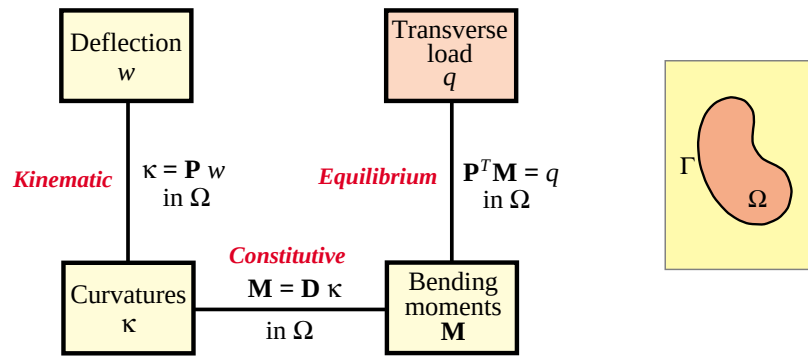


FIGURE 24.8. Strong Form diagram of the field equations of the Kirchhoff plate model. Boundary condition links omitted.

Notes and Bibliography

There is not shortage of material on plates. The problem is one of selection. Some of the best known English textbooks and monographs are listed below. Only books examined by the writer are commented upon. Others are simply listed. Emphasis is on technology oriented books of interest to engineers.

S. Ambartsumyan, *Theory of Anisotropic Plates: Strength, Stability, and Vibrations*, Hemisphere Pubs. Corp., 1991

C. R. Calladine, *Engineering Plasticity*, Pergamon Press, 1969. A textbook containing a brief treatment of plastic behavior of homogeneous (metal) plates. Many exercises, well written.

L. H. Donnell, *Beams, Plates and Shells*, McGraw-Hill, 1976. Unlike Timoshenko, largely obsolete.

D. J. Gorman, *Vibration Analysis of Plates by the Superposition Method*, World Scientific Pub. Co, 1999

R. Hussein, *Composite Panels and Plates: Analysis and Design*, Technomic, Lancaster, PA, 1986.

S. G. Letnitskii, *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day, 1963. Translated from Russian. See comments for next one.

S. G. Letnitskii, *Anisotropic Plates*, Gordon and Breach, 1968. Translated from Russian. One of the earliest monographs devoted to the title subject. Focus is on analytical solutions of tractable problems of anisotropic plates.⁹

K. M. Liew et al. (eds.), *Vibration of Mindlin Plates: Programming the P-Version Ritz Method*, Elsevier Science Ltd, 1998.

F. F. Ling (ed.), *Vibrations of Elastic Plates : Linear and Nonlinear Dynamical Modeling of Sandwiches, Laminated Composites, and Piezoelectric Layers*, Springer-Verlag, New York, 1995.

A. E. H. Love, *Theory of Elasticity*, Cambridge, 4th ed 1927, still reprinted as a Dover book. A classical (late XIX century) treatment of elasticity by a renowned applied mathematician contains several sections on thin plates. Material and notation is outdated. Don't expect any physical insight; Love has no interest in engineering applications. Useful mostly for tracing historical references to specific problems.

P. G. Lowe, *Basic Principles of Plate Theory*, Surrey Univ. Press, Blackie Group, London, 1982. An elementary treatment of plate mechanics that covers a lot of ground, including plastic and optimal design, in 175 pages. Appropriate for supplementary reading in a senior level course.

⁹ The advent of finite elements has made the distinction between isotropic and anisotropic plates unimportant from a computational viewpoint.

L. S. D. Morley, *Skew Plates and Structures*, Pergamon Press, 1963. This tiny monograph was motivated by the increasing importance of skew shapes in high speed aircraft after WWII. Compact and well written, still worth consulting for FEM benchmarks.

J. N. Reddy, *Mechanics of Laminated Composite Plates*, CRC Press, Boca Raton, FL, 1997. Good coverage of methods for composite wall constructions.

W. Soedel, *Vibrations of Plates and Shells*, Marcel Dekker, 1993.

R. Szilard, *Theory and Analysis of Plates*, Prentice-Hall, 1974. A massive work (710 pages) similar in level to Timoshenko's but covering more ground: numerical methods, dynamics, vibration and stability analysis. Although intended as a graduate textbook it contains no exercises, which detracts from its usefulness. The treatment of numerical methods is obsolete, with too many pages devoted to finite difference methods in vogue before 1960.

S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, 2nd edition, 1959. A classical reference book. Not a textbook (it lacks exercises) but case-by-case driven. It is quintessential Timoshenko: a minimum of generalities and lots of examples. The title is a bit misleading: only shells of special geometries are dealt with, in 3 chapters out of 16. As for plates, the emphasis is on linear static problems using the Kirchhoff model. There are only two chapters treating moderately large deflections, one with the von Karman model. There is no treatment of buckling or vibration, which are covered in other Timoshenko books.¹⁰ The reader will find no composite wall constructions, only one chapter on orthotropic plates (called "anisotropic" in the book) and a brief description of the Reissner-Mindlin model, which had been published in between the 1st and 2nd editions. Despite these shortcomings it is still *the* reference book to have for solution of specific plate problems. The physical insight is unmatched. Old-fashioned full notation is used: no matrices, vectors or tensors. Strangely, this long-windedness fits nicely with the deliberate pace of engineering before computers, where understanding of the physics was more important than numbers. The vast compendium of solutions comes handy when checking FEM codes, and for preliminary design work. (Unfortunately sign conventions are at odds with presently preferred ones, which can be a source of errors.) Exhaustive reference source for all work before 1959.

J. K. Whitney, *Structural Analysis of Laminated Anisotropic Plates*, Technomic Pubs Co, Lancaster, PA, 1987. Expanded version of an 1970 book by Ashton and Whitney. Concentrates on layered composite and sandwich wall constructions of interest in aircraft structures.

R. H. Wood, *Plastic Analysis of Slabs and Plates*, Thames and Hudson, 1961. Focuses on plastic analysis of reinforced concrete plates by yield line methods. Long portions are devoted to the author's research and supporting experimental work. Oriented to civil engineers.

The well known handbook by Young and Roark contains chapters collecting specific solutions of plate statics and vibration, with references as to sources.

For general books on composite materials and their use in aerospace, civil, marine, and mechanical engineering, a web search on new and used titles in <http://www3.addall.com> is recommended.

¹⁰ The companion volumes *Vibrations of Engineering Structures* and *Theory of Elastic Stability*.

Homework Exercises for Chapter 24**Kirchhoff Plates: Field Equations**

EXERCISE 24.1 [A:15] The biharmonic operator (24.19) can be written symbolically as $\nabla^4 = \mathbf{P}^T \mathbf{W} \mathbf{P}$, where $\mathbf{W}$ is a diagonal weighting matrix and $\mathbf{P}$ is defined in Table 24.1. Find $\mathbf{W}$.

EXERCISE 24.2 [A:15] Using the matrix form of the Kirchhoff plate equations given in Table 24.1, eliminate the intermediate variables κ and $\mathbf{M}$ to show that the generalized form of the biharmonic plate equation (24.18) is $\mathbf{P}^T \mathbf{D} \mathbf{P} w = q$. This form is valid for anisotropic plates of variable thickness.

EXERCISE 24.3 [A:15] Rewrite $\mathbf{P}^T \mathbf{D} \mathbf{P} w = q$ in indicial notation, assuming that $\mathbf{D}$ is constant over the plate.

EXERCISE 24.4 [A:20] Rewrite $\mathbf{P}^T \mathbf{D} \mathbf{P} w = q$ in glorious full form in Cartesian coordinates, assuming that $\mathbf{D}$ is constant over the plate.

EXERCISE 24.5 [A:20] Express the rotations θ_n and θ_s about the normal and tangential directions, respectively, in terms of θ_x , θ_y and the angle ϕ defined in Figure 24.7.

25

Kirchhoff Plates: BCs and Variational Forms

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§25.1. Introduction

In this Chapter we continue the discussion of the governing equations of Kirchhoff plates with the consideration of boundary conditions (BCs) and variational forms of the plate equations.

When plates and shells are spatially discretized by FEM the proper modeling of boundary conditions can be a tough subject. Two factors contribute. First, displacement derivatives in the form of rotations are now involved in the kinematic boundary conditions. Second, the correlation between physical support conditions and mathematical B.C.s can be tenuous. Some mathematical BC used in practice are nearly impossible to reproduce in the laboratory, let alone on an actual structure.

§25.2. Boundary Conditions for Kirchhoff Plate

One of the mathematical difficulties associated with the Kirchhoff model is the “Poisson paradox”:

- The plate deflection satisfies a fourth order partial differential equation (PDE). For the isotropic homogeneous plate this is the biharmonic equation $\Delta^2 w = q/D$.
- A fourth order PDE can only have *two* boundary conditions at each boundary point.
- But *three* conjugate quantities: normal moment, twist moment and transverse shear appear naturally at a boundary point.

The reduction from three to two requires variational methods. This was done first by Kirchhoff, thus achieving mathematical closure. But it is not necessary to look at a complete functional such as the TPE. The procedure can be explained directly through virtual work principles.

§25.2.1. Conjugate Quantities

Consider a Kirchhoff plate of general shape as in Figure 25.1(a). Assume that the boundary Γ is smooth, that is, contains no corners. Under those assumptions the exterior normal $\mathbf{n}$ and tangential direction $\mathbf{s}$ at each boundary point B are unique, and form a system of local Cartesian axes.

The kinematic quantities referred to these local axes are

$$w, \quad \frac{\partial w}{\partial n} = -\theta_s, \quad \frac{\partial w}{\partial s} = \theta_n, \quad (25.1)$$

where θ_n and θ_s denote the rotations of the midsurface at B about axes n and t , respectively, see Figure 25.1(b). The *work conjugate* static quantities, shown in Figure 25.1(c), are

$$Q_n, \quad M_{nn}, \quad M_{ns}, \quad (25.2)$$

respectively. By ‘conjugate’ it is meant that the boundary work can be expressed as the line integral

$$W_B = \int_{\Gamma} \left(Q_n w + M_{ns} \frac{\partial w}{\partial s} + M_{nn} \frac{\partial w}{\partial n} \right) ds = \int_{\Gamma} \left(Q_n w + M_{ns} \theta_n - M_{nn} \theta_s \right) ds \quad (25.3)$$

where $ds \equiv d\Gamma$ denotes the differential boundary arclength. This integral appears naturally in the process of forming the energy functionals of the plate. Given the foregoing configuration of W_B , it appears at first sight as if *three* boundary conditions can be assigned at each boundary point, taken from the conjugate sets (25.1) and (25.2). For example:

$$\begin{array}{ll} \text{Simply supported edge} & w = 0 \quad \theta_n = 0 \quad M_{nn} = 0, \\ \text{Free edge} & Q_n = 0 \quad M_{ns} = 0 \quad M_{nn} = 0. \end{array} \quad (25.4)$$

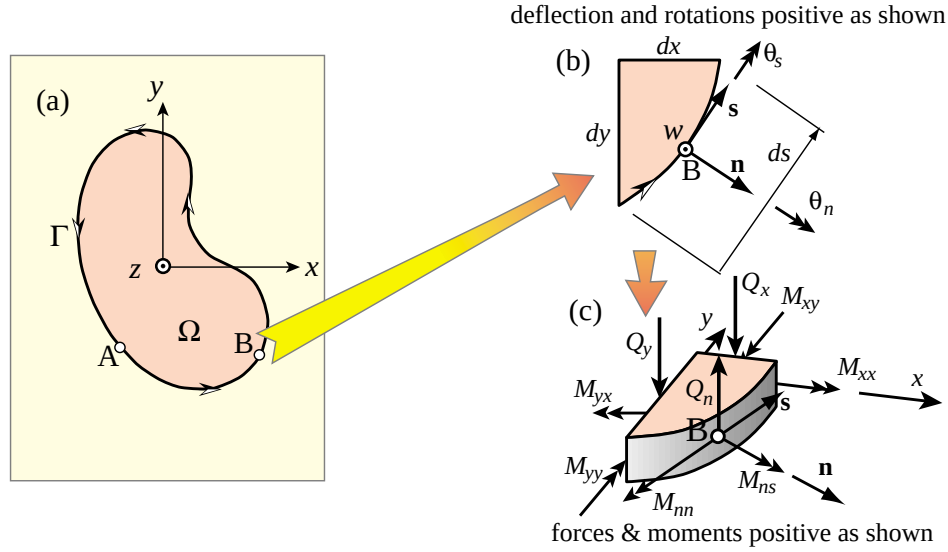


FIGURE 25.1. BCs at a *smooth* boundary point B of a Kirchhoff plate: $\mathbf{n}$ = external normal, $\mathbf{s}$ = tangential direction. (a) boundary traversed in the counterclockwise sense (looking down from $+z$) leaving the plate proper on the left; (b) edge kinematic quantities w , θ_s and θ_n ; (c) conjugate force and moment quantities Q_n , M_{nn} and M_{ns} on the boundary face.

The boundary conditions for a free edge were indeed expressed by Poisson in this form.¹ As noted previously, this is inconsistent with the order of the governing PDE. Kirchhoff showed² that three conditions are too many and in fact only two are independent.

§25.2.2. The Modified Shear

The reduction to two independent conjugate pairs may be demonstrated through integration of (25.3) by parts with respect to s , along a segment AB of the boundary Γ :

$$W_B|_A^B = \int_A^B \left[\left(Q_n - \frac{\partial M_{ns}}{\partial s} \right) w + M_{nn} \frac{\partial w}{\partial n} \right] dt + M_{ns} w|_A^B. \quad (25.5)$$

Introducing the *modified shear*³

$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s}, \quad (25.6)$$

we may rewrite (25.5) as

$$W_B|_A^B = \int_A^B \left(V_n w + M_{nn} \frac{\partial w}{\partial n} \right) dt + M_{ns} w|_A^B. \quad (25.7)$$

This transformation reduces the conjugate quantities to two work pairs:

$$V_n, w \quad \text{and} \quad M_{nn}, \frac{\partial w}{\partial n} = -\theta_s. \quad (25.8)$$

¹ See for example I. Todhunter and K. Pierson, *History of Theory of Elasticity*, Vol. I.

² G. Kirchhoff, publications cited in §24.2.

³ Also called *Kirchhoff equivalent force*, a translation of Kirchhoffische Ersatzkräfte.

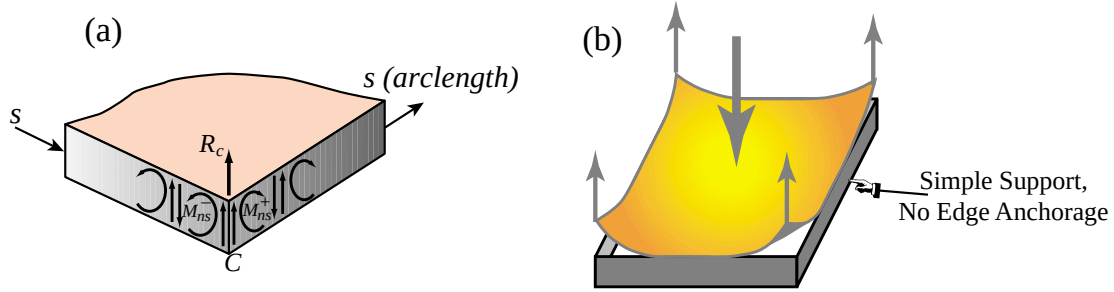


FIGURE 25.2. Effect of modified shear at a plate corner: (a) force-pairs do not cancel, producing a corner force R_c ; (b) physical manifestation of modified shear as corner lifting forces.

§25.2.3. Corner Forces

The last term in (25.7) deserves analysis. First consider a plate with *smooth* boundary as in Figure 25.1(a). Assuming that M_{ns} is continuous over Γ , and we go completely “around” the plate so that $A \equiv B$,

$$M_{ns} w|_A^B = 0. \quad (25.9)$$

Next consider the case of a plate with a corner C as in Figure 25.2(a). At C the twisting moment jumps from, say, M_{ns}^- to M_{ns}^+ . The transverse displacement w must be continuous (if fact, C^1 continuous). Place A and B to each side of C , so that $A \rightarrow C$ from the minus side while $C \leftarrow B$ from the plus side. Then

$$M_{ns}|_A^B = R_c w = (M_{ns}^+ - M_{ns}^-)w, \quad \text{with} \quad R_c = M_{ns}^+ - M_{ns}^-. \quad (25.10)$$

This jump in the twisting moment is called the *corner force* R_c , as shown in Figure 25.2(a). Note that R_c has the physical dimension of force, because the twisting moment M_{ns} is a moment (force times length) per unit length.

Remark 25.1. The physical interpretation of modified shears and of corner forces is well covered in Timoshenko and Woinowsky-Krieger.⁴ Suffices to say that if a plate corner is constrained not to move laterally, a concentrated force R_c called the corner reaction, appears. If the corner is not held down the reaction cannot be transmitted to the supports and the plate will have a tendency to move away from the support. This is the source of the well known “corner lifting” phenomenon that may be observed on a laterally loaded square plate with simply supported edges that do not prevent lifting. See Figure 25.2(b). Notice that if the boundary is smooth, as in a circular plate, the phenomenon will not be observed.

This effect does not appear if the edges meeting at C are free or clamped, because if so the twist moment M_{ns} on both sides of the corner point are zero.

§25.2.4. Common Boundary Conditions

Below we state *homogeneous* boundary conditions frequently encountered in Kirchhoff plates as selected combinations of the conjugate quantities (25.8). Some BCs are illustrated in the structures depicted in Figure 25.3.

⁴ *Theory of Plates and Shells* monograph cited in previous Chapter

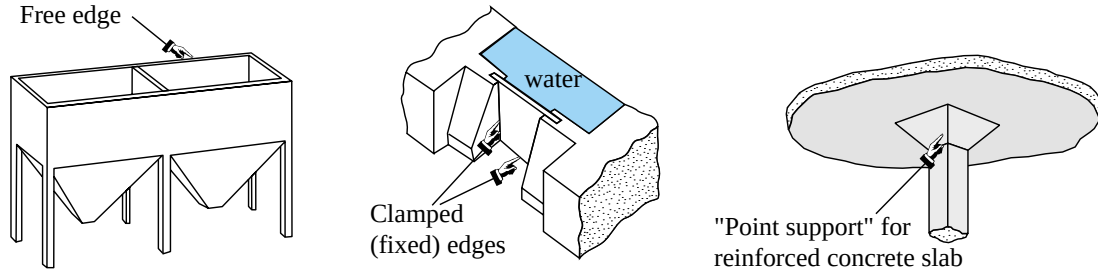


FIGURE 25.3. Boundary condition examples.

Clamped or Fixed Edge (with s along edge):

$$w = 0, \quad \frac{\partial w}{\partial n} = -\theta_s = 0. \quad (25.11)$$

Simply Supported Edge (with s along edge):

$$w = 0, \quad M_{nn} = 0. \quad (25.12)$$

Free Edge (with s along edge):

$$V_n = 0, \quad M_{nn} = 0. \quad (25.13)$$

Symmetry Line (with s along line):

$$V_n = 0, \quad \frac{\partial w}{\partial n} = -\theta_s = 0. \quad (25.14)$$

Point Support:

$$w = 0 \quad (25.15)$$

Non-homogeneous boundary conditions of force type involving prescribed normal moment $\hat{M}_{nn}$ or prescribed modified shear $\hat{V}_n$, are also quite common in practice. Non-homogeneous B.C. involving prescribed nonzero transverse displacements or rotations are less common.

§25.2.5. Strong Form Diagram

The Strong Form diagram of the governing equations, including boundary conditions, for the Kirchhoff plate model is shown in Figure 25.4.

We are now ready to present several energy functionals of the Kirchhoff plate that have been used in the construction of finite elements.

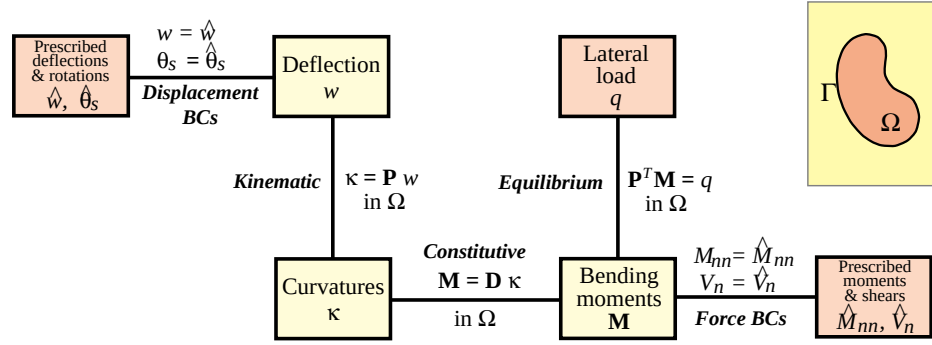


FIGURE 25.4. The Strong Form diagram for the Kirchhoff plate, including boundary conditions.

§25.3. The Total Potential Energy Principle

The only master field is the transverse displacement w . The departure Weak Form is shown in Figure 25.5. The weak links are the internal equilibrium equations and the force boundary conditions.

§25.3.1. The TPE Functional

Starting from the Weak Form flowcharted in Figure 25.5, and proceeding as in previous chapters one finally arrives at the Total Potential Energy (TPE) functional. Processing of the boundary terms is laborious. Timoshenko and Woinowsky-Krieger's book takes 6 pages in getting to the final destination.⁵ Here we just state the final result. The TPE functional with the conventional forcing potential is

$$\Pi_{\text{TPE}}[w] = U_{\text{TPE}}[w] - W_{\text{TPE}}[w] \quad (25.16)$$

The internal energy is

$$U_{\text{TPE}}[w] = \frac{1}{2} \int_{\Omega} (\mathbf{M}^w)^T \boldsymbol{\kappa}^w d\Omega = \frac{1}{2} \int_{\Omega} (\boldsymbol{\kappa}^w)^T \mathbf{D} \boldsymbol{\kappa}^w d\Omega = \frac{1}{2} \int_{\Omega} (w \mathbf{P}^T) \mathbf{D} (\mathbf{P} w) d\Omega, \quad (25.17)$$

where $\mathbf{P} = [\partial^2/\partial x^2 \quad \partial^2/\partial y^2 \quad 2\partial^2/\partial x \partial y]^T$ is the curvature-displacement operator. The groupings $\mathbf{P}w$ in the last of (25.17) emphasize that $\mathbf{P}$ is to be applied to w to form the slave curvatures $\boldsymbol{\kappa}^w$.

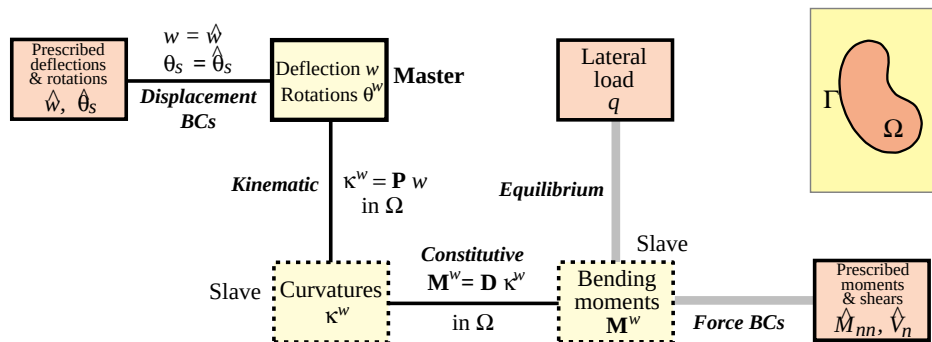


FIGURE 25.5. The Weak Form departure point to derive the TPE variational principle for a Kirchhoff plate.

⁵ Part of the length is due to use of full notation.

The external work $W_{\text{TPE}}[w]$ is more complicated than in plane stress and solids. It is best presented as the sum of three components. These are due to apply lateral loads, applied edge moments and transverse shears, and to corner loads, respectively:

$$W_{\text{TPE}}[w] = W_q[w] + W_B[w] + W_C[w], \quad (25.18)$$

The first two terms apply to all plate geometries and are

$$W_q[w] = \int_{\Omega} q w \, d\Omega, \quad W_B[w] = \int_{\Gamma_{VM}} (\hat{V}_n w - \hat{M}_{nn} \theta_s^w) \, d\Gamma. \quad (25.19)$$

The last term: W_C arises if the plate has $j = 1, 2, \dots, n_c$ corners at which the displacement w_j is *not* prescribed. If so,

$$W_C = \sum_{j=1}^{n_c} R_{cj} w_j = \sum_{j=1}^{n_c} (\hat{M}_{ns}^+ - \hat{M}_{ns}^-) w. \quad (25.20)$$

If the transverse displacement of a corner is prescribed, the contribution of that corner to the functional vanishes because the displacement variation is zero.

Remark 25.2. If neither the corner displacement nor the twist moments M_{ns}^- and M_{ns}^+ at the corner are known, the problem becomes nonlinear because the extent over which the plate lifts from supports is not known *a priori*. This problem becomes one of contact type and will require an iterative method to be solved.

§25.3.2. Finite Element Conditions

The variational index of the TPE functional is $m = 2$ because second derivatives of w (the curvatures) appear in U_{TPE} . The classical-Ritz convergence conditions for finite elements derived using this principle are:

Completeness. The assumed w over each element should reproduce exactly all $\{x, y\}$ polynomials of order ≤ 2 .

Continuity. The assumed w should be C^2 continuous inside the element and C^1 interelement-continuous.

Stability. Elements should be rank sufficient and the Jacobian determinant everywhere positive.

The C^1 interelement continuity condition is the tough one to crack. It is not easy to satisfy using standard polynomial assumptions. These difficulties have motivated, since the early 1960s, the development of various techniques to alleviate (or totally get rid of) that continuity requirement.

§25.4. The Hellinger-Reissner Principle

The Weak Form useful as departure point for the HR principle is shown in Figure 25.6. Both the transverse displacement w and the bending moment field $\mathbf{M}$ are chosen as master fields. The weak links are the internal equilibrium equations,

The HR functional with the conventional forcing potential is

$$\Pi_{\text{HR}}[w, \mathbf{M}] = U_{\text{HR}}[w, \mathbf{M}] - W_{\text{HR}}[w, \mathbf{M}]. \quad (25.21)$$

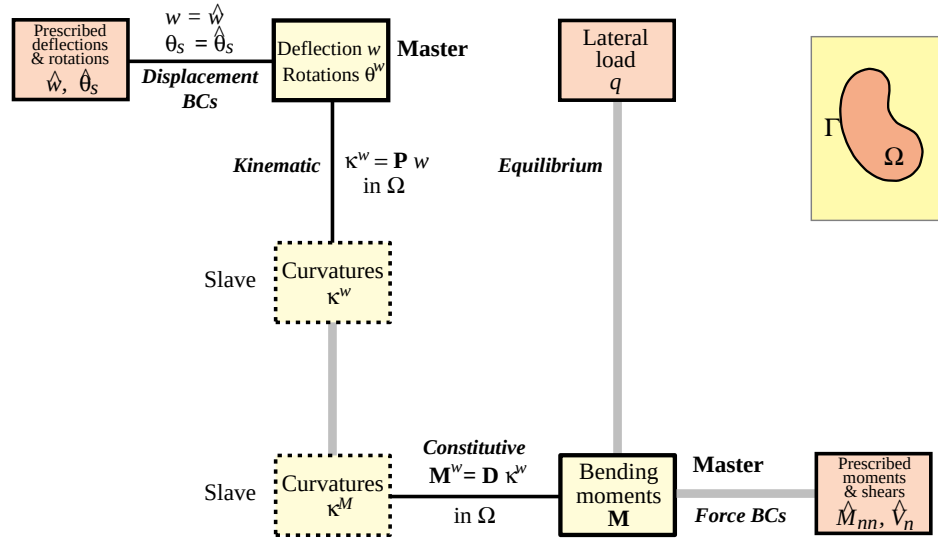


FIGURE 25.6. The Weak Form departure point to derive the Hellinger-Reissner (HR) variational principle for a Kirchhoff plate.

The internal energy is

$$U_{\text{HR}}[w, \mathbf{M}] = \int_{\Omega} (\mathbf{M}^T \boldsymbol{\kappa}^w - \frac{1}{2} \mathbf{M}^T \mathbf{D}^{-1} \mathbf{M}) d\Omega = \int_{\Omega} (\mathbf{M}^T \boldsymbol{\kappa}^w - \mathcal{U}^*) d\Omega. \quad (25.22)$$

Here $\mathcal{U}^* = \frac{1}{2} \mathbf{M}^T \mathbf{D}^{-1} \mathbf{M}$ is the complementary energy density (per unit of plate area) written in terms of the bending moments. Integration of this over Ω gives the total complementary energy U^* .

The external work W_{HR} is the same as for the TPE principle treated in the previous section.

§25.4.1. Finite Element Conditions

The variational indices of the HR functional are $m_w = 2$ for the transverse deflection and $m_M = 0$ for the bending moments. Consequently the completeness and continuity conditions for w are the same as for the TPE, and nothing is gained by going to the more complicated functional.

It is possible to *balance* the variational indices so that $m_w = m_M = 1$ by integrating the previous form by parts once. The resulting principle was exploited by Herrmann⁶ to construct a plate element with linearly varying w , M_{xx} , M_{yy} , M_{xy} . This element, however, was disappointing in accuracy. Furthermore enforcing moment continuity can be physically wrong. Progress in the construction of elements of this type was achieved later using hybrid principles. This advance will not be covered here since the advent of the Post-1980 formulations (FF, ANDES, etc) have taken care of the problem.

⁶ L. R. Herrmann, A bending analysis for plates, in *Proceedings 1st Conference on Matrix Methods in Structural Mechanics*, AFFDL-TR-66-80, Air Force Institute of Technology, Dayton, Ohio, pp. 577-604, 1966.

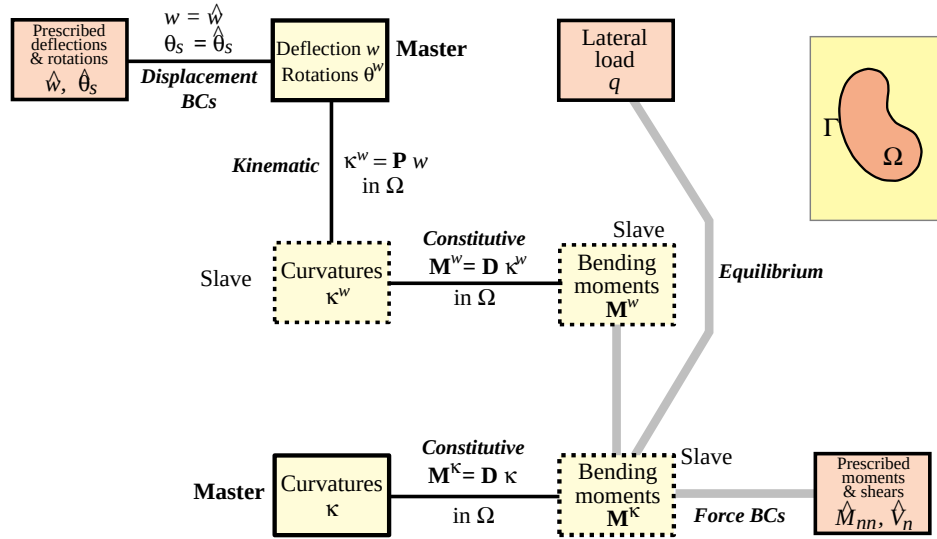


FIGURE 25.7. The Weak Form departure point to derive the Fraeijs de Veubeke curvature-displacement variational principle for a Kirchhoff plate.

§25.5. The Curvature-Displacement Veubeke Principle

This kind of principle (for elastic solids) was introduced by Fraeijs de Veubeke, and functionals will be accordingly identified by a FdV subscript.

§25.5.1. The Fraeijs de Veubeke Functional

In this case both the transverse displacement w and the curvatures field κ are chosen as master fields. The departure Weak Form is shown in Figure 25.7.

$$\boxed{\Pi_{\text{FdV}}[w, \kappa] = U_{\text{FdV}}[w, \kappa] - W_{\text{FdV}}[w, \kappa].} \quad (25.23)$$

The internal functional is

$$\Pi_{\text{FdV}}[w, \kappa] = \int_{\Omega} (\kappa^T \mathbf{M}^w - \frac{1}{2} \kappa^T \mathbf{D} \kappa) d\Omega. \quad (25.24)$$

The external work is the same as for TPE.

§25.5.2. Finite Element Conditions

The variational indices of the Π_{FdV} functional is $m_w = 2$ for the transverse displacement and $m_\kappa = 0$ for the curvatures. Consequently the completeness and continuity conditions for w are the same as for the TPE, as nothing is gained by going to the more complicated functional. To get a practical scheme that reduces the continuity order one can either integrate by parts the internal energy, or proceed to hybrid principles. Since the latter are implicitly used in the Post-1984 advanced FEM formulations, the procedure will not be covered here.

26

Thin Plate Elements: Overview

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This Chapter presents an overview of finite element models for thin plates using the Kirchhoff Plate bending (KPB) model. The derivation of shape functions for the triangle geometry is covered in the next chapter.

§26.1. An Overview of KTP FE Models

The plate domain Ω is subdivided into finite elements in the usual way, as illustrated in Figure 26.1. The most widely used geometries are triangles and quadrilaterals with straight sides. Curved side KPB elements are rare. They are more widely seen in shear-endowed C^0 models derived by the degenerate solid approach.

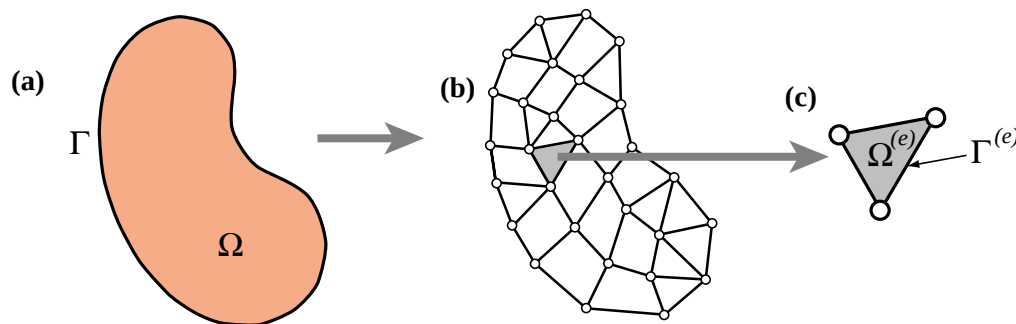


Figure 26.1. A thin plate subdivided into finite elements.

§26.1.1. Triangles

This and following Chapter will focus on KPB *triangular* elements only. These triangles will invariably have straight sides. Their geometry is defined by the position of the three corners as pictured in Figure 26.2(a). The positive sense of traversal of the boundary is shown in Figure 26.2. This sense defines three side directions: s_1 , s_2 and s_3 , which are aligned with the sides opposite corners 1, 2 and 3, respectively. The *external* normal directions n_1 , n_2 and n_3 shown there are oriented at -90° from s_1 , s_2 and s_3 .¹

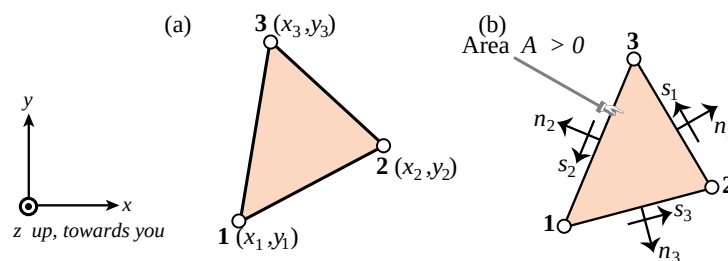


Figure 26.2. Triangular geometry and side-normal directions.

¹ This means that $\{n_i, s_i, z\}$ for $i = 1, 2, 3$ form three right-handed RCC systems, one for each side.

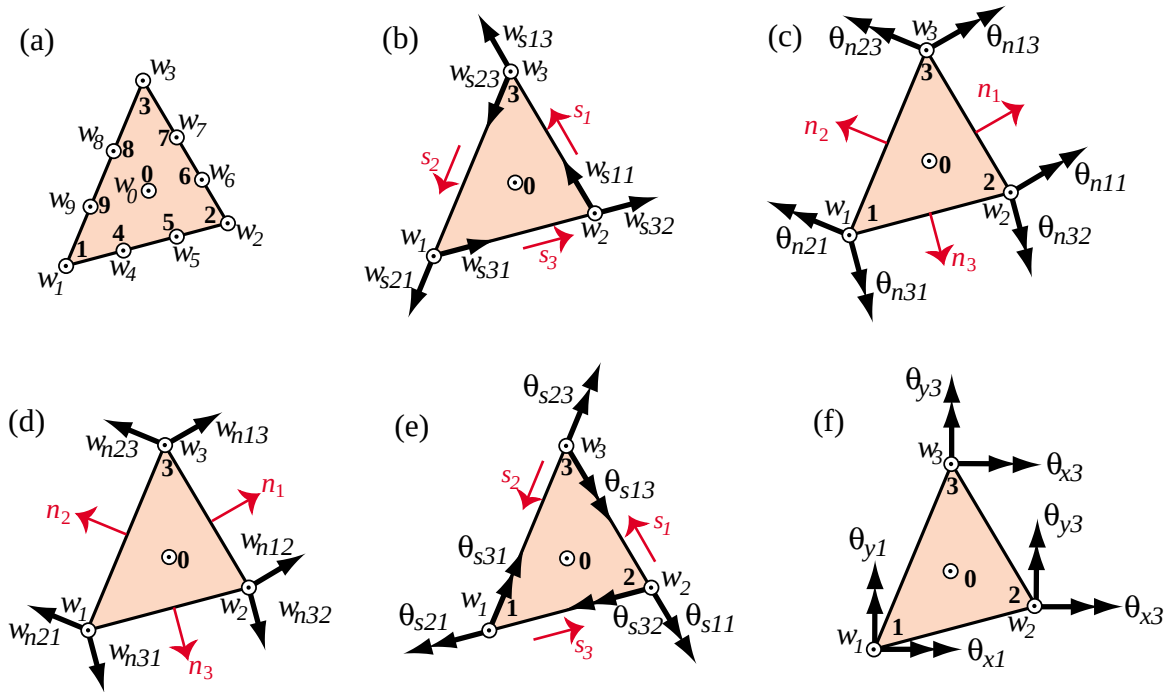


Figure 26.3. Several 10-DOF configurations for expressing the complete cubic interpolation of the lateral deflection w over a KPB triangle. Normal to sides and tangential directions identified by red arrows to avoid confusion with DOFs shown in black.

The area of the triangle, denoted by A , is a signed quantity given by

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1). \quad (26.1)$$

We require that $A > 0$.

§26.1.2. A Potpourri of Freedom Configurations

In KPB elements treated by assumed transverse displacements, the minimum polynomial expansion of w to achieve at least partly the compatibility requirements, is cubic. A complete cubic has 10 terms and consequently can accommodate 10 element degrees of freedom (DOFs). Figure 26.3 shows several 10-DOF configurations from which the cubic interpolation over the complete triangle can be written as an interpolation formula, with shape functions expressed in terms of the geometry data and the triangular coordinates. These interpolation formulas are studied in the next Chapter.

Because a complete polynomial is invariant with respect to a change in basis, all of the configurations depicted in Figure 26.3 are *equivalent* in providing the *same* interpolation over the triangle. They differ, however, when connecting to adjacent elements. Only configuration (f) is practical for connecting elements over arbitrary meshes using the Direct Stiffness Method (DSM). The other configurations are valuable in intermediate derivations, or for various theoretical studies.

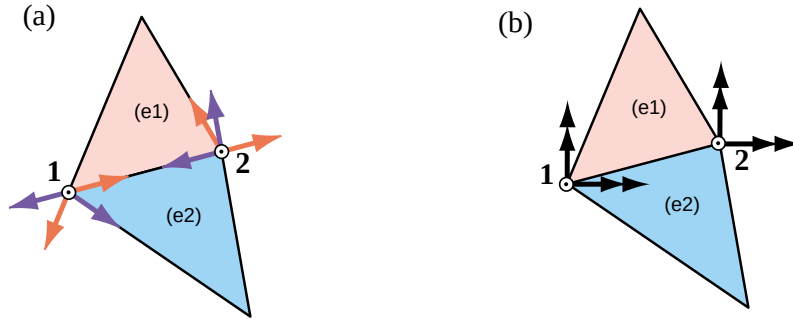


Figure 26.4. Connecting KPB elements.

The 10-node configuration (a) specifies the cubic by the 10 values w_i , $i = 1, \dots, 10$, of the deflection at corners, thirdpoints of sides, and centroid. This is a useful starting point because the associated shape functions can be constructed directly using the technique explained in Chapter 17 of IFEM. The resulting plate element is useless, however, because it does not enforce interelement C^1 continuity at any boundary point.

From (a) one can pass to any of (b) through (d), the choice being primarily a matter of taste or objectives. Configurations (b) and (d) use the six corner partial derivatives of w along the side directions or the normal to the sides, respectively. The notation is $w_{sij} = (\partial w / \partial s_i)_j$ and $w_{nij} = (\partial w / \partial n_i)_j$, where i is the side index and j the corner index. (Sides are identified by the number of the opposite corner.) For example, $w_{s21} = (\partial w / \partial s_2)_1$. These partials are briefly called *side slopes* and *normal slopes*, respectively, on account of their physical meaning.

According to the fundamental kinematic assumption of the KPB model, a w slope along a midsurface direction is equivalent for small deflections to a *midsurface rotation* about a line perpendicular to and forming a -90° angle with that direction. Rotations about the s_i and n_i directions are called *side rotations* and *normal rotations*, respectively, for brevity.²

For example at corner 1, normal rotation θ_{n21} equals side slope w_{s21} . Replacing the six side-slope DOF w_{sij} of Figure 26.3(b) by the normal rotations θ_{nij} produces configuration (c). Similarly, replacing the six normal-slope DOF w_{nij} of Figure 26.3(d) by the side rotations θ_{sij} produces configuration (e). Note that the positive sense of the θ_{sij} , viewed as vectors, is opposite that of s_i ; this is a consequence of the sign conventions and positive-rotation rule.

§26.1.3. Connectors

If corner slopes along two noncoincident directions are given, the slope along any other corner direction is known. The same is true for corner rotations. It follows that for any of the configurations of Figure 26.3(b) through (e), *the deflection and tangent plane at the 3 corners are known*. However that information cannot be readily communicated to adjacent elements.

The difficulties are illustrated with Figure 26.4(a), which shows two adjacent triangles: red element (e1) and blue element (e2), possessing the DOF configuration of Figure 26.3(b). The deflections w_1 and w_2 match without problems because direction z is shared. But the color-coded side slopes

² Note that in passing from slopes to equivalent rotations, the qualifiers “side” and “normal” exchange.

do not match.³ For more elements meeting at a corner the result is chaotic.

To make the element suitable for implementation in a DSM-based program, it is necessary to transform slope or rotational DOFs to *global directions*. The obvious choices are the midsurface axes $\{x, y\}$. Most FEM codes use rotations instead of slopes since that simplifies connection of different element types (e.g., shells to beams) in three dimensions. Choosing corner rotations θ_{xi} and θ_{yi} as DOF we are led to the configuration of Figure 26.3(f). As illustrated in Figure 26.4(b), the connection problem is solved and the elements are now suitable for the DSM.

§26.2. Convergence Conditions

The foregoing exposition has centered on displacement assumed elements where w is the master field. Element stiffness equations are obtained through the Total Potential Energy (TPE) variational principle presented in the previous Chapter. The completeness and continuity requirements are summarized in §23.3 on the basis of a variational index $m_w = 2$. These are now studied in more detail for cubic triangles.

§26.2.1. Completeness

The TPE variational index $m_w = 2$ requires that all w -polynomials of order 0, 1 and 2 in $\{x, y\}$ be exactly represented over each element. Constant and linear polynomials represent rigid body motions, whereas quadratic polynomials represent constant curvature states.

Now if w is interpolated by a complete cubic, the ten terms $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$ are automatically present for any freedom configuration. This appears to be more than enough. Nothing to worry about, right? Wrong. Preservation of such terms over each triangle is guaranteed only if full C^1 continuity is verified. But, as discussed below, attaining C^1 continuity is difficult. To get it one while sticking to cubics one must make substantial changes in the construction of w . Since those changes do not necessarily preserve completeness, that requirement appears as an *a posteriori* constraint. Alternatively, to make life easier C^1 continuity may be abandoned except at corners. If so completeness may again be lost, for example by a seemingly harmless static condensation of w_0 . Again this has to be kept as a constraint.

The conclusion is that *completeness cannot be taken for granted* in displacement-assumed KPB elements. Gone is the “IFEM easy ride” of isoparametric elements for variational index 1.

§26.2.2. Continuity Games

To explain what C^1 interelement continuity entails, it is convenient to break this condition into two levels:⁴

C^0 Continuity. The element is C^0 compatible if w over any side is completely specified by DOFs on that side.

C^1 Continuity. The element is C^1 compatible if it is C^0 compatible, *and* the normal slope $\partial w / \partial n$ over any side is completely specified by DOFs on that side.

³ Positive slopes along the common side point in opposite directions.

⁴ It is tacitly assumed that the condition is satisfied inside the element.

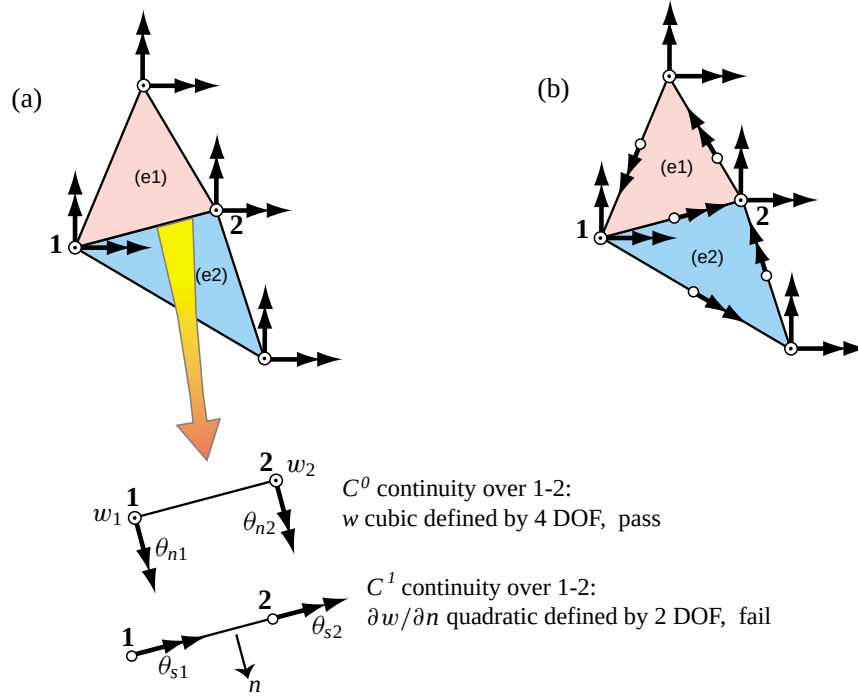


Figure 26.5. Checking interelement continuity of KPB triangles along common side 1-2.

The first level: C^0 continuity, is straightforward. In the IFEM course, which for 2D problems deals with variational index $m = 1$ only, the condition is easily achieved with the isoparametric formulation. The second level is far more difficult. Attaining it is the exception rather than the rule, and elements that make it are not necessarily the best performing ones. Nevertheless it is worth studying since so many theory advances in FEM: hybrid principles, the patch test, etc., came as a result of research in C^1 plate elements.

To appreciate the difficulties in attaining C^1 continuity consider two cubic triangles with the freedom configuration of Figure 26.3(f) connected as shown in Figure 26.5(a). At corners 1 and 2 the rotational freedoms are rotated to align with common side 1-2 and its normal as shown underneath Figure 26.5(a). Over side 1-2 the deflection w varies cubically. This variation is defined by four DOF on that side: w_1 , w_2 , θ_{n1} and θ_{n2} . Consequently C^0 continuity holds. Over side 1-2 the normal slope $\partial w / \partial n$ varies quadratically since it comes from differentiating w once. A quadratic is defined by three values; but there are only two DOF that can control the normal slope: θ_{s1} and θ_{s2} . Consequently C^1 continuity is violated between corner points.

To control a quadratic variation of $\partial w / \partial n = \theta_s$ an additional DOF on the side is needed. The simplest implementation of this idea is illustrated in Figure 26.5(b): add a side rotation DOF at the midpoint. But this increases the total number of DOF of the triangle to at least 12: 9 at the corners and 3 at the midpoints.⁵ Because a cubic has only 10 independent terms, terms from a quartic polynomial are needed if we want to keep just a polynomial expansion over the full triangle. But that raises the side variation orders of w and $\partial w / \partial n$ to 4 and 3, respectively, and again we are short. Limitation Theorem III given below state that it is impossible to “catch up” under these conditions.

⁵ The number climbs to 13 if the centroid deflection w_0 is kept as a DOF.

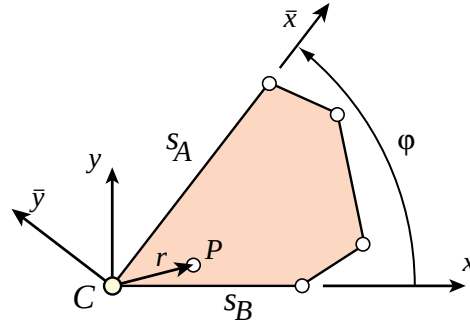


Figure 26.6. A corner C of a polygonal KPB element.

§26.3. *Kinematic Limitation Principles

This section examines *kinematic limitation principles* that place constraints on the construction of KPB displacement-assumed elements. The principles are useful in ruling out once and for all the easy road to constructing such elements, and in explaining why researchers turned their attention elsewhere.

Limitation principles 1 and 2 are valid for an arbitrary element polygonal shape as illustrated in Figure 26.6, that has only corner DOF on its boundary.⁶

Select a corner C bounded by sides s_A and s_B , which subtend angle φ . We use the abbreviations $s_\varphi = \sin \varphi$ and $c_\varphi = \cos \varphi$. Select a rectangular Cartesian coordinate (RCC) system: $\{x, y\}$ with origin at C and x along side s_A . Another RCC system $\{\bar{x}, \bar{y}\}$ is placed with $\bar{x}$ along side s_B . The systems are related by $\{\bar{x} = xc_\varphi + ys_\varphi, \bar{y} = -xs_\varphi + yc_\varphi\}$ and $\{x = \bar{x}c_\varphi - \bar{y}s_\varphi, y = \bar{x}s_\varphi + \bar{y}c_\varphi\}$

We focus on limitations related to assuming that w has *continuous second derivatives* at C. That is, the following Taylor expansion holds at a point $P(x, y)$ at a distance r from C:

$$w = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + O(r^3) \quad (26.2)$$

We need the following results derivable from (26.2). The lateral deflections over s_A and s_B are

$$\begin{aligned} w_A &= a_0 + a_1 x + a_3 x^2 + O(r^3), \\ w_B &= a_0 + (a_1 c_\varphi + a_2 s_\varphi) \bar{x} + (a_3 c_\varphi^2 + a_4 s_\varphi c_\varphi + a_5 s_\varphi^2) \bar{x}^2 + O(r^3). \end{aligned} \quad (26.3)$$

The along-the-side slopes over s_A and s_B are obtained by evaluating $\partial w / \partial x$ at $y = 0$ and $\partial w / \partial \bar{x}$ at $\bar{y} = 0$:

$$\begin{aligned} w_{sA} &= a_1 + 2a_3 x + O(r^2), \\ w_{sB} &= a_1 c_\varphi + a_2 s_\varphi + 2(a_3 c_\varphi^2 + a_4 s_\varphi c_\varphi + a_5 s_\varphi^2) \bar{x} + O(r^2). \end{aligned} \quad (26.4)$$

The normal slopes over s_A and s_B are obtained by evaluating $-w_y = -\partial w / \partial y = -a_2 - a_4 x - 2a_5 y + O(r^2)$ at $y = 0$, and $w_{\bar{y}} = \partial w / \partial \bar{y} = -(a_1 + 2a_3 x + a_4 y) s_\varphi + (a_2 + a_4 x + 2a_5 y) c_\varphi + O(r^2)$ at $\bar{y} = 0$. This gives

$$\begin{aligned} w_{nA} &= -a_2 - a_4 x + O(r^2), \\ w_{nB} &= -a_1 s_\varphi + a_2 c_\varphi + (a_4(c_\varphi^2 - s_\varphi^2) - 2(a_3 - a_5)s_\varphi c_\varphi) \bar{x} + O(r^2) \end{aligned} \quad (26.5)$$

Assume that the element satisfies the following four assumptions.

⁶ The presence of internal DOFs is not excluded.

- (I) The Taylor series (26.2) at C is valid; thus the deflection w has second derivatives at C .
- (II) Three nodal values are chosen at C : $w_C = a_0$, $\theta_{xC} = (\partial w / \partial y)_C = a_1$ and $\theta_{yC} = -(\partial w / \partial x)_C = -a_2$. This is the standard choice for plate elements.
- (III) Completeness is satisfied in that the six states $w = \{1, x, y, x^2, xy, y^2\}$ are exactly representable over the element.
- (IV) The variation of the normal slope $\partial w / \partial n$ along the element sides is linear.

§26.3.1. Limitation Theorem I

A KPB element cannot satisfy (I), (II), (III) and (IV) simultaneously.

Proof. Choose three set of corner DOF at C to satisfy:

$$\begin{aligned}
 \text{Set 1:} \quad w_C &= 1, \quad \left(\frac{\partial w}{\partial n_A} \right)_C = 0, \quad \left(\frac{\partial w}{\partial n_B} \right)_C = 0, \\
 \text{Set 2:} \quad w_C &= 0, \quad \left(\frac{\partial w}{\partial n_A} \right)_C = 1, \quad \left(\frac{\partial w}{\partial n_B} \right)_C = 0, \\
 \text{Set 3:} \quad w_C &= 0, \quad \left(\frac{\partial w}{\partial n_A} \right)_C = 0, \quad \left(\frac{\partial w}{\partial n_B} \right)_C = 1.
 \end{aligned} \tag{26.6}$$

while all other DOF are set to zero.

Set 1 imposes $a_0 = 1$ and $a_1 = a_2 = 0$. Both normal slopes at C are zero, and so are at other corners. Because of the linear variation assumption (IV), $w_{nA} = \partial w / \partial n_A \equiv 0$ and $w_{nB} = \partial w / \partial n_B \equiv 0$. Expressions (26.5) require $a_4 = 0$ and $a_3 = a_5$.

Set 2 imposes $a_0 = 0$, $a_1 = 1$ and $a_2 = c_\varphi / s_\varphi$. Now $w_{nB} \equiv 0$. This requires $a_4 = 0$ and $a_3 = a_5$, as above. Replacing gives $w_{nA} = 1$, which contradicts (IV).

Set 3 imposes $a_0 = a_1 = 0$, and $a_2 = 1$. Now $w_{nA} = 0$ identically, which forces $a_4 = 0$.

An arbitrary set of values for the DOFs at C can be written as a linear combination of (26.6). But any such combination requires $a_4 = 0$, making the twist term vanish identically in (26.1). Thus the assumption (IV) of completeness cannot be satisfied.

Oddly enough the proof needs no assumption about how w varies along the sides; that is, C^0 compatibility. Just the assumption that the normal slope varies linearly is enough to kill completeness.

This theorem says that to get a C^1 compatible element while retaining assumptions (I), (II) and (III) the normal slope variation cannot be linear. Such conforming elements can be constructed, for example, using product of cubic Hermitian functions along side directions with suitable damping factors along the other directions. But this approach runs into serious trouble as shown by the next limitation principle.

§26.3.2. Limitation Theorem II

Any C^1 -compatible, non rectangular KPB element that satisfies conditions (I) and (II) cannot represent exactly all constant curvature states.

Proof. If the element is exactly in a constant curvature state, the deflection w must be quadratic in $\{x, y\}$. Hence the normal slope variation must be linear. But according to Limitation Theorem I the element cannot represent the constant twist state.

This theorem shows that (I), (II) and (III) are incompatible. A more detailed study shows that for a C^1 compatible rectangular element with sides aligned with $\{x, y\}$ only the twist state is lost but that x^2 and y^2 can be exactly represented. For non-rectangular geometries all constant curvature states are lost.

If one insists in C^1 continuity there are two ways out:

Abandon (I): Keep a single polynomial over the element but admit higher order derivatives as corner degrees of freedom.

Abandon (II): Permit discontinuous second derivatives at corners through the use of non-polynomial assumptions, or macroelements.

Both techniques have been tried with success. The use of second derivatives as DOFs, if (I) is abandoned, is forced by the next limitation principle.

§26.3.3. Limitation Theorem III

Suppose that a simple complete polynomial expansion of order $n \geq 3$ is assumed for w over a triangle. At each corner i the deflection w_i , the slopes w_{xi} , w_{yi} and all midsurface derivatives up to order $m \geq 1$ are taken as degrees of freedom. Then C^1 continuity requires $m \geq 2$ and $n \geq 5$.

Proof. Proven in the writer's thesis.⁷

Here is an informal summary of the proof. The total number of DOFs for a complete polynomial is $P_n = (n+1)(n+2)/2 = F_n + B_n$. Of these $F_n = \min((n+1)(n+2)/2, 6n-9)$ are called *fundamental freedoms* in the sense that they affect interelement compatibility. The $B_n = \max(0, (n-5)(n-4)/2)$ are called *bubble freedoms*, which have zero value and normal slopes along the three sides. Bubbles occur only if $n \geq 6$.

Over each side the variation of w is of order n and that of $w_n = \partial w / \partial n$ of order $n-1$. This requires $n+1$ and n control DOF on the side, respectively. The number of corner freedoms is $N_c = (m+1)(m+2)/2$, which provides $2(m+1)$ and $2m$ controls on w and w_n , respectively. Within the side (e.g. at a midpoint) one need to add $N_{ws} = n+1-2(m+1) \geq 0$ control DOFs for C^0 continuity in w and $N_{wns} = n-2m \geq 0$ control DOFs for C^1 continuity in w_n . The grand total of boundary DOFs is $N_b = 3(N_c + N_{ws} + N_{wns})$. This has to be equal to the number of fundamental freedoms F_n . Here is a tabulation for various values of n and m . N/A means that the interpolation order is not applicable as being too low as it gives $N_{ws} < 0$ and/or $N_{wns} < 0$.

	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
	$F_n = 10$	$F_n = 15$	$F_n = 21$	$F_n = 27$	$F_n = 33$
$m = 1$	$N_b = 12$	$N_b = 18$	$N_b = 24$	$N_b = 30$	$N_b = 36$
$m = 2$	N/A	N/A	$N_b = 21$	$N_b = 27$	$N_b = 33$
$m = 3$	N/A	N/A	N/A	N/A	$N_b = 33$

The first interesting solution is boxed. It corresponds to a complete quintic polynomial $n = 5$ with 21 DOFs, all fundamental. Six degrees of freedom are required at each corner: w_i , w_{xi} , w_{yi} , w_{xxi} , w_{xyi} , w_{yyi} , $i = 1, 2, 3$, plus one normal slope (or side rotation) at each midpoint. The resulting element, called CCT-21, was presented in the thesis cited in footnote 7. The element, however, is too complex for use in standard FEM codes because it does not mix easily with other elements such as beams. So it has only been used in special purpose codes for plates only.

⁷ C. A. Felippa, Refined finite element analysis of linear and nonlinear two-dimensional structures, *Ph.D. Dissertation*, Department of Civil Engineering, University of California at Berkeley, 1966.

§26.4. Early Work

By the late 1950s the success of the Finite Element Method with membrane problems (for example, for wing covers and fuselage panels) led to high hopes in its application to plate bending and shell problems. The first results were published by 1960. But until 1965 only rectangular models gave satisfactory results. The construction of successful triangular elements to model plates and shells of arbitrary geometry proved more difficult than expected. Early failures, however, led to a more complete understanding of the theoretical basis of FEM and motivated several advances taken for granted today.

The major source of difficulties in plates is due to stricter continuity requirements. The objective of attaining normal slope interelement compatibility posed serious problems, documented in the form of Limitation Theorems in §26.3. By 1963 researchers were looking around “escape ways” to bypass those problems. It was recognized that completeness, in the form of exact representation of rigid body and constant curvature modes, was fundamental for convergence to the analytical solution, a criterion first enunciated by Melosh.⁸ The effect of compatibility violations was more difficult to understand until the patch test came along.

§26.4.1. Rectangular Elements

The first successful rectangular plate bending element was developed by Adini and Clough⁹. This element has 12 degrees of freedom (DOF). It used a complete third order polynomial expansion in x and y , aligned with the rectangle sides, plus two additional x^3y and xy^3 terms. The element satisfies completeness as well as transverse deflection continuity but normal slope continuity is only maintained at the four corner points. The same element results from another expansion proposed by Melosh (1963 reference cited), which erroneously states that the element satisfies C^1 continuity. The error was noted in subsequent discussion.¹⁰ In 1961 Melosh had proposed¹¹ had proposed a rectangular plate element constructed with beam-like edge functions damped linearly toward the opposite side, plus a uniform twisting mode. Again C^0 continuity was achieved but not C^1 except at corners.

Both of the foregoing elements displayed good convergence characteristics when used for rectangular plates. However the search for a compatible displacement field was underway to try to achieve monotonic convergence. A fully compatible 12-DOF rectangular element was apparently first developed by Papenfuss in an obscure reference.¹² The element appears to have been rediscovered several times. The simplest derivation can be carried out with products of Hermite cubic polynomials, as noted below. Unfortunately the uniform twist state is not included in the expansion and consequently the element fails the completeness requirement, converging monotonically to a zero twist-curvature solution: the right answer for the wrong problem.

In a brief but important paper, Irons and Draper¹³ stressed the importance of completeness for uniform strain modes (constant curvature modes in the case of plate bending). They proved that it is impossible to construct any polygonal-shape plate element with only 3 DOFs per corner and continuous corner curvatures that can simultaneously maintain normal slope conformity and inclusion of the uniform twist mode. This negative

⁸ R. J. Melosh, Bases for the derivation of stiffness matrices for solid continua, *AIAA J.*, bf 1, 1631–1637, 1963.

⁹ A. Adini and R. W. Clough, Analysis of plate bending by the finite element method, NSF report for Grant G-7337, Dept. of Civil Engineering, University of California, Berkeley, 1960. Also A. Adini, Analysis of shell structures by the finite element method, Ph.D. Dissertation, Department of Civil Engineering, University of California, Berkeley, 1961.

¹⁰ J. L. Tocher and K. K. Kapur, Comment on Melosh’s paper, *AIAA J.*, **3**, 1215–1216, 1965.

¹¹ R. J. Melosh, A stiffness matrix for the analysis of thin plates in bending, *J. Aero. Sci.*, **28**, 34–42, 1961.

¹² S. W. Papenfuss, Lateral plate deflection by stiffness methods and application to a marquee, M. S. Thesis, Department of Civil Engineering, University of Washington, Seattle, WA, 1959.

¹³ B. M. Irons and K. Draper, Inadequacy of nodal connections in a stiffness solution for plate bending, *AIAA J.*, **3**, 965–966, 1965.

result, presented in §26.3 as Limitation Theorem II, effectively closed the door to the construction of the analog of isoparametric elements in plate bending.

The construction of fully compatible polynomial expansions of various orders for rectangular shapes was solved by Bogner et al in 1965¹⁴ through Hermitian interpolation functions. In their paper they rederived Papenfuss' element, but in an Addendum¹⁵ they recognised the lack of the twist mode and an additional degree of freedom: the twist curvature, was added at each corner. The 16-DOF element is complete and compatible, and produced excellent results. More refined rectangular elements with 36 DOFs have been also developed using fifth order Hermite polynomials.

§26.4.2. Triangular Elements

Flat triangular plate elements have a wider range of application than rectangular elements since they naturally conform to the analysis of plates and shells of arbitrary geometry for small and large deflections. But as previously noted the development of adequate kinematic expansions was not an easy problem and has kept researchers busy for decades.

The success of incompatible rectangular elements is due to the fact that the assumed polynomial expansions for w can be considered as “natural” deformation modes, after a trivial reduction to nondimensional form. They are intrinsically related to the geometry of the element because the local system is chosen along two preferred directions. Lack of C^1 continuity between corners disappears in the limit of a mesh refinement.

Early attempts to construct triangular elements tried to mimic that scheme, using a RCC system arbitrarily oriented with respect to the element. This led to an unpleasant lack of invariance whenever an *incomplete* polynomial was selected, since kinematic constraints were artificially imposed. Furthermore the role of completeness was not understood. Thus the first suggested expansion for a triangular element with 9 DOFs¹⁶

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 x^3 + \alpha_7 x^2 y + \alpha_8 x y^2 + \alpha_9 y^3 \quad (26.7)$$

in which the xy term is missing, violates compatibility, completeness and invariance requirements. The element converges, but to the wrong solution with zero twist curvature.

Tocher in his thesis cited above tried two variants of the cubic expansion:

- 1 Combining the two cubic terms: $x^2 y + x y^2$.
- 2 Using a complete 10-term cubic polynomial The first choice satisfies completeness but violates compatibility and invariance. The second assumption satisfies completeness and invariance but violates compatibility and poses the problem: what to do with the extra DOF? Tocher decided to eliminate it by a generalized inversion process, which unfortunately leads to discarding a fundamental degree of freedom. This led to an extremely flexible (and non convergent) element. The elimination technique of Bazeley et. al. discussed in Chapter 25 was more successful and produced an element which is still in use today.

The first fully compatible 9-DOF cubic triangle was finally constructed by the macroelement technique.¹⁷ The triangle was divided into three subtriangles, over each of which a cubic expansion with linear variation along

¹⁴ F. K. Bogner, R. L. Fox and L. A. Schmidt Jr., The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas, *Proc. Conf. on Matrix Methods in Structural Mechanics*, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, pp. 397–444, 1966.

¹⁵ Addendum to aforementioned paper, 411–413 in *AFFDL TR 66-80*.

¹⁶ J. L. Tocher, Analysis of plate bending using triangular elements, *Ph. D. Dissertation*, Dept. of Civil Engineering, University of California, Berkeley, California, 1963.

¹⁷ R. W. Clough and J. L. Tocher, Finite element stiffness matrices for analysis of plate bending, *Proc. Conf. on Matrix Methods in Structural Mechanics*, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, 515–545, 1966.

the exterior side was assumed. A similar element with quadratic slope variation and 12 DOF was constructed by the writer.¹⁸ The original derivations, carried out in x, y coordinates were considerably simplified later by using triangular coordinates.

The 1965 paper by Bazeley et al.¹⁹ was an important milestone. In it three plate bending triangles were developed. Two compatible elements were developed using rational functions. Experiments showed them to be quite stiff and have no interest today. An incompatible element called the BCIZ triangle since was obtained by eliminating the 10th DOF from a complete cubic in such a way that completeness was maintained. This element is incompatible. Numerical experiments showed that it converged for some mesh patterns but not for others. This puzzling behavior led to the invention of the patch test.²⁰ The patch test was further developed by Irons and coworkers in the 1970s.²¹ A mathematical version is presented in the Strang-Fix monograph.²²

§26.4.3. Quadrilateral Elements

Arbitrary quadrilaterals can be constructed by assembling several triangles, and eliminating internal DOFs, if any by static condensation. This represents an efficient procedure to take into account that the four corners need not be on a plane. The article by Clough and Felippa cited above presents the first quadrilateral element constructed this way. That element was included in the open-source SAP family of FEM codes and used for shell analysis since 1968.

A direct construction of an arbitrary quadrilateral with 16 DOFs was presented by de Veubeke.²³ The quadrilateral is formed by a macroassembly of four triangles by the two diagonals, which are selected as a skew Cartesian coordinate system to develop the finite element fields.

§26.5. More Recent Work

The fully conforming elements developed in the mid 1960s proved “safe” for FEM program users in that convergence could be guaranteed. Performance was another matter. Triangular elements proved to be excessively stiff, particularly for high aspect ratios. A significant improvement in performance was achieved by Razzaque²⁴ who replaced the shape function curvatures with least-square-fitted smooth functions. This technique was later shown to be equivalent to the stress-hybrid formulation.

The first application of mixed functionals to finite elements was actually to the plate bending problem. Herrmann²⁵ developed a mixed triangular model in which transverse displacements and bending moments are

¹⁸ R. W. Clough and C. A. Felippa, A refined quadrilateral element for analysis of plate bending, Proc. 2nd Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, in *AFFDL TR 69-23*, 1969

¹⁹ G. P. Bazeley, Y. K. Cheung, B. M. Irons and O. C. Zienkiewicz, Triangular elements in plate bending — conforming and nonconforming solutions, Proc. Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, pp. 547–576, 1966.

²⁰ Addendum to Bazeley et. al. paper cited above, pp. 573–576 in *AFFDL TR 66-80*.

²¹ B. M. Irons and A. Razzaque, Experiences with the patch test for convergence of finite elements, in *Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, ed. by K. Aziz, Academic Press, New York, 1972.

B. M. Irons and S. Ahmad, *Techniques of Finite Elements*, Ellis Horwood Ltd, Chichester, England, 1980.

²² G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

²³ B. Fraeijs de Veubeke, A conforming finite element for plate bending, *Int. J. Solids Struct.*, **4**, 95–108, 1968

²⁴ A. Razzaque, Program for triangular bending elements with derivative smoothing, *Int. J. Numer. Meth. Engrg.*, **6**, 333–343, 1973.

²⁵ L. R. Herrmann, A bending analysis for plates, in *Proceedings 1st Conference on Matrix Methods in Structural Mechanics*, AFFDL-TR-66-80, Air Force Institute of Technology, Dayton, Ohio, 577–604, 1966.

selected as master variables. A linear variation was assumed for both variables. This work was based on the HR variational principle and included the transversal shear energy. The element did not perform well in practice.

Successful plate bending elements have also been constructed by Pian's assumed-stress hybrid method²⁶ The 9-dof triangles in this class are normally derived by assuming cubic deflection and linear slope variations along the element sides, and a linear variation of the internal moment field. Efficient formulations of such elements have been published.²⁷ Hybrid elements generally give better moment accuracy than conforming displacement elements. The derivation of these elements, however, is more involved in that it depends on finding equilibrium moments fields within the element, which is not a straightforward matter if the moments vary within the element or large deflections are considered.

Much of the recent research on displacement-assumed models has focused on relaxing or abandoning the assumptions of Kirchhoff thin-plate theory. Relaxing these assumptions has produced elements based on the so-called discrete Kirchhoff theory.²⁸ In this method the primary expansion is made for the plate rotations. The rotations are linked to the nodal freedoms by introduction of thin-plate normality conditions at selected boundary points, and then interpolating displacements and rotations along the boundary. The initial applications of this method appear unduly complicated. A clear and relatively simple account is given by Batoz, Bathe and Ho.²⁹ The most successful of these elements to date is the DKT (Discrete Kirchhoff Triangle), an explicit formulation of which has been presented by Batoz.³⁰

A more drastic step consists of abandoning the Kirchhoff theory in favor of the Reissner-Mindlin theory of moderately thick plates. The continuity requirements for the displacement assumption are lowered to C^0 (hence the name " C^0 bending elements"), but the transverse shear becomes an integral part of the formulation.

Historically the first fully conforming triangular plate elements were not Clough-Tocher's but C^0 elements called "facet" elements that were derived in the late 1950s, although an account of their formulation was not published until 1965.³¹ Facet elements, however, suffer from severe numerical problems for thin-plate and obtuse-angle conditions. The approach was revived later by Argyris *et al.*³² within the context of degenerated "brick" elements.

Successful quadrilateral C^0 elements have been developed by Hughes, Taylor and Kanolkulchai,³³ Pugh,

²⁶ T. H. H. Pian, Derivation of element stiffness matrices by assumed stress distributions, *AIAA J.*, **2**, 1333–1336, 1964. T. H. H. Pian and P. Tong, Basis of finite element methods for solid continua, *Int. J. Numer. Meth. Engrg.*, **1**, 3–29, 1969.

²⁷ O. C. Zienkiewicz, *The Finite Element Method in Engineering Science*, McGraw-Hill, New York, 3rd edn., 1977. D. J. Allman, Triangular finite elements for plate bending with constant and linearly varying bending moments, *Proc. IUTAM Conf. on High Speed Computing of Elastic Structures*, Liège, Belgium, 105–136, 1970.

²⁸ J. Stricklin, W. Haisler, P. Tisdale and R. Gunderson, A rapidly converging triangular plate bending element, *AIAA J.*, **7**, 180–181, 1969. Also G. Dhett, An efficient triangular shell element, *AIAA J.*, **8**, No. 11, 2100–2102, 1970.

²⁹ J. L. Batoz, K.-J. Bathe and Lee-Wing Ho, A study of three-node triangular plate bending elements, *Int. J. Numer. Meth. Engrg.*, **15**, 1771–1812, 1980.

³⁰ J. L. Batoz, An explicit formulation for an efficient triangular plate-bending element, *Int. J. Numer. Meth. Engrg.*, **18**, 1077–1089, 1982.

³¹ R. J. Melosh, A flat triangular shell element stiffness matrix, Proc. Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, 503–509, 1966.

³² J. H. Argyris, P. C. Dunne, G. A. Malejannakis and E. Schelkle, A simple triangular facet shell element with applications to linear and nonlinear equilibrium and elastic stability problems, *Comp. Meths. Appl. Mech. Engrg.*, **11**, 215–247, 1977.

³³ T. J. R. Hughes, R. Taylor and W. Kanolkulchai, A simple and efficient finite element for plate bending, *Int. J. Numer. Meth. Engrg.*, **11**, 1529–1543, 1977.

Hinton and Zienkiewicz,³⁴ MacNeal,³⁵ Crisfield,³⁶ Tessler and Hughes,³⁷ Dvorkin and Bathe,³⁸ and Park and Stanley.³⁹

Triangular elements in this class have been presented by Belytschko, Stolarski and Carpenter.⁴⁰ The construction of robust C^0 bending elements is delicate, as they are susceptible to ‘shear locking’ effects in the thin-plate regime if fully integrated, and to kinematic deficiencies (spurious modes) if they are not. When the proper care is exercised good results have been reported for quadrilateral elements and, more recently, for triangular elements.⁴¹

A different path has been taken by Bergan and coworkers, who retained the classical Kirchhoff formulation but in conjunction with the use of highly nonconforming (C^{-1}) shape functions. They have shown that interelement continuity is not an obstacle to convergence provided the shape functions satisfy certain energy and force orthogonality conditions⁴² or the stiffness matrix is constructed using the *free formulation*⁴³ rather than the standard potential energy formulation. A characteristic feature of these formulations is the careful separation between basic and higher order assumed displacement functions or “modes”. Results for triangular bending elements derived through this approach have reported satisfactory performance.⁴⁴ One of these elements, which is based on force-orthogonal higher order functions, was rated in 1983 as the best performer in its class.⁴⁵

³⁴ E. D. Pugh, E. Hinton and O. C. Zienkiewicz, A study of quadrilateral plate bending elements with reduced integration, *Int. J. Numer. Meth. Engrg.*, **12**, 1059–1078, 1978.

³⁵ R. H. MacNeal, A simple quadrilateral shell element, *Computers & Structures*, **8**, 175–183, 1978.

R. H. MacNeal, Derivation of stiffness matrices by assumed strain distributions, *Nucl. Engrg. Design*, **70**, 3–12, 1982.

³⁶ M. A. Crisfield, A four-noded thin plate bending element using shear constraints – a modified version of Lyons’ element, *Comp. Meths. Appl. Mech. Engrg.*, **39**, 93–120, 1983.

³⁷ A. Tessler and T. J. R. Hughes, A three-node Mindlin plate element with improved transverse shear, *Comp. Meths. Appl. Mech. Engrg.*, **50**, 71–101, 1985.

³⁸ E. N. Dvorkin and K. J. Bathe, A continuum mechanics based four-node shell element for general nonlinear analysis, *Engrg. Comp.*, **1**, 77–88, 1984.

³⁹ G. M. Stanley, Continuum-based shell elements, *Ph. D. Dissertation*, Department of Mechanical Engineering, Stanford University, 1985.

K. C. Park and G. M. Stanley, A Curved C^0 shell element based on assumed natural-coordinate strains, *J. Appl. Mech.*, **108**, 278–286, 1986.

⁴⁰ T. Belytschko, H. Stolarski and N. Carpenter, A C^0 triangular plate element with one-point quadrature, *Int. J. Numer. Meth. Engrg.*, **20**, 787–802, 1984.

⁴¹ A. Tessler and T. J. R. Hughes, A three-node Mindlin plate element with improved transverse shear, *Comp. Meths. Appl. Mech. Engrg.*, **50**, 71–101, 1985.

⁴² P. G. Bergan, Finite elements based on energy-orthogonal functions, *Int. J. Numer. Meth. Engrg.*, **11**, 1529–1543, 1977.

⁴³ P. G. Bergan and M. K. Nygård, Finite elements with increased freedom in choosing shape functions, *Int. J. Numer. Meth. Engrg.*, **20**, 643–664, 1984.

⁴⁴ P. G. Bergan and L. Hanssen, A new approach for deriving “good” finite elements, MAFELAP II Conference, Brunel University, 1975, in *The Mathematics of Finite Elements and Applications – Vol. II*, ed. by J. R. Whiteman, Academic Press, London, 1976. L. Hanssen, T. G. Syvertsen and P. G. Bergan, Stiffness derivation based on element convergence requirements, MAFELAP III Conference, Brunel University, 1978, in *The Mathematics of Finite Elements and Applications – Vol III*, ed. by J. R. Whiteman, Academic Press, London, 1979. P. G. Bergan and M. K. Nygård, Nonlinear shell analysis using free formulation finite elements, *Proc. Europe-US Symposium on Finite Element Methods for Nonlinear Problems*, Springer-Verlag, 1986.

⁴⁵ B. M. Irons, Putative high-performance plate bending element, Letter to Editor, *Int. J. Numer. Meth. Engrg.*, **19**, 310, 1983.

27

Triangular Plate Displacement Elements

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The purpose of this Chapter is to explain the construction of displacement-based triangular Kirchhoff plate bending elements through the development of the necessary interpolation formulas. The resulting element is complete but does not satisfy full normal-slope conformity.

§27.1. Triangular Element Properties

We recall the following properties of a straight-sided triangular element, taken from Chapter 15 of IFEM.

§27.1.1. Geometry

The geometry of the 3-node triangle shown in Figure 27.1(a) is specified by the location of its three corner nodes on the $\{x, y\}$ plane. The nodes are labeled 1, 2, 3 while traversing the sides in *counterclockwise* fashion. The location of the corners is defined by their coordinates:

$$x_i, y_i, \quad i = 1, 2, 3 \quad (27.1)$$

The area of the triangle is denoted by A and is given by

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1). \quad (27.2)$$

It is important to realize that the area given by formula (27.2) is a *signed* quantity. It is positive if the corners are numbered in counterclockwise order as shown in Figure 27.1(b). This convention is followed in the sequel.

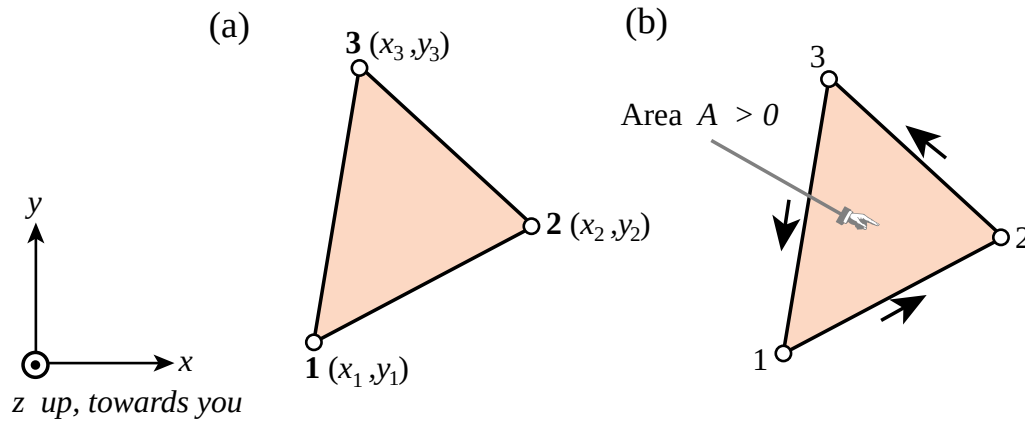


Figure 27.1. Geometry of straight-sided triangular element.

§27.1.2. Triangular Coordinates

Points of the triangle may also be located in terms of a *parametric* coordinate system:

$$\zeta_1, \zeta_2, \zeta_3. \quad (27.3)$$

In the literature these three parameters receive many names. In the sequel the name *triangular coordinates* will be used to stress its close association with this particular geometry.

§27.1.3. Properties of Triangular Coordinates

Equations

$$\zeta_i = \text{constant} \quad (27.4)$$

represent a set of straight lines parallel to the side opposite to the i^{th} corner. See Figure 27.2. The equation of sides 12, 23 and 31 are $\zeta_1 = 0$, $\zeta_2 = 0$ and $\zeta_3 = 0$, respectively. The three corners have coordinates $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The three midpoints of the sides have coordinates $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$, the centroid $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and so on. The coordinates are not independent because their sum is unity:

$$\zeta_1 + \zeta_2 + \zeta_3 = 1. \quad (27.5)$$

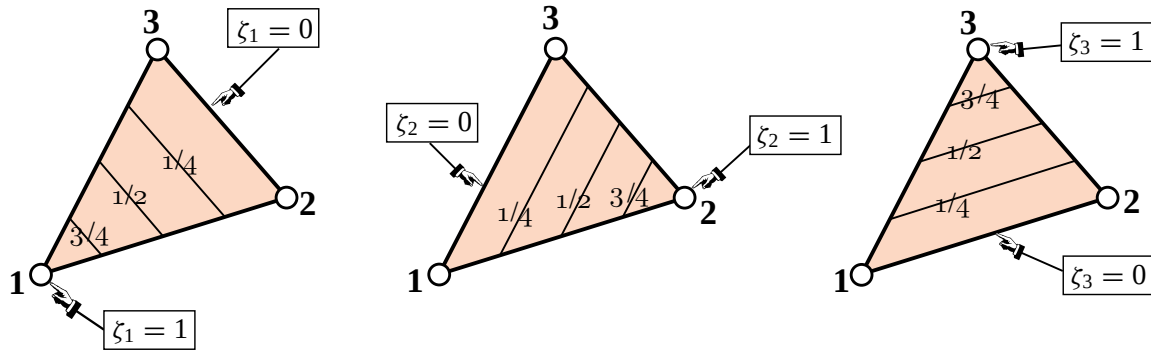


Figure 27.2. Triangular coordinates.

§27.1.4. Linear Interpolation

Consider a function $w(x, y)$ that varies *linearly* over the triangle domain. In terms of Cartesian coordinates it may be expressed as

$$w(x, y) = a_0 + a_1x + a_2y, \quad (27.6)$$

where a_0 , a_1 and a_2 are coefficients to be determined from three conditions. In finite element work such conditions are often the *nodal values* taken by w at the corners:

$$w_1, w_2, w_3 \quad (27.7)$$

The expression in triangular coordinates makes direct use of these three values:

$$w(\zeta_1, \zeta_2, \zeta_3) = w_1\zeta_1 + w_2\zeta_2 + w_3\zeta_3 = [w_1 \quad w_2 \quad w_3] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = [\zeta_1 \quad \zeta_2 \quad \zeta_3] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}. \quad (27.8)$$

Expression (27.8) is called a *linear interpolant* for w . See Figure 27.3(a).

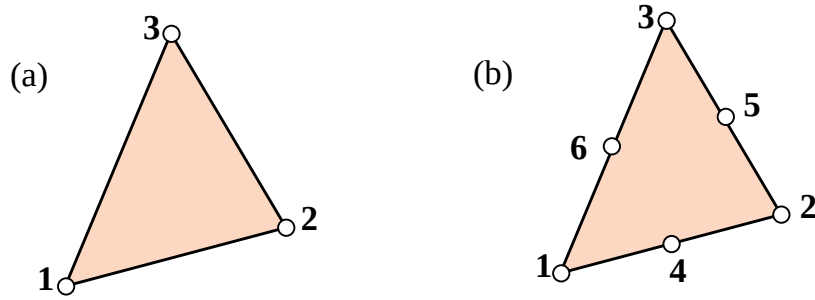


Figure 27.3. Nodal configurations for: (a) linear interpolation of w by three values w_i , at corners $i = 1, 2, 3$; (b) quadratic interpolation of w by six values w_i at corners $i = 1, 2, 3$ and midpoints $i = 4, 5, 6$.

§27.1.5. Coordinate Transformations

Quantities which are closely linked with the element geometry are naturally expressed in triangular coordinates. On the other hand, quantities such as displacements, strains and stresses are often expressed in the Cartesian system x, y . We therefore need transformation equations through which we can pass from one coordinate system to the other.

Cartesian and triangular coordinates are linked by the relation

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}. \quad (27.9)$$

The first equation says that the sum of the three coordinates is one. The second and third express x and y linearly as homogeneous forms in the triangular coordinates. These simply apply the linear interpolant formula (27.8) to the Cartesian coordinates: $x = x_1\zeta_1 + x_2\zeta_2 + x_3\zeta_3$ and $y = y_1\zeta_1 + y_2\zeta_2 + y_3\zeta_3$.

Inversion of (27.9) yields

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}. \quad (27.10)$$

Here $x_{jk} = x_j - x_k$, $y_{jk} = y_j - y_k$, A is the triangle area given by (27.2) and A_{jk} denotes the area subtended by corners j, k and the origin of the x - y system. If this origin is taken at the centroid of the triangle, $A_{23} = A_{31} = A_{12} = A/3$.

§27.1.6. Partial Derivatives

From equations (27.9) and (27.10) we immediately obtain the following relations between partial derivatives:

$$\frac{\partial x}{\partial \zeta_i} = x_i, \quad \frac{\partial y}{\partial \zeta_i} = y_i, \quad (27.11)$$

$$2A \frac{\partial \zeta_i}{\partial x} = y_{jk}, \quad 2A \frac{\partial \zeta_i}{\partial y} = x_{kj}. \quad (27.12)$$

In (27.12) j and k denote the *cyclic permutations* of i . For example, if $i = 2$, then $j = 3$ and $k = 1$. The derivatives of a function $w(\zeta_1, \zeta_2, \zeta_3)$ with respect to x or y follow immediately from (27.12) and application of the chain rule:

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{2A} \left(\frac{\partial w}{\partial \zeta_1} y_{23} + \frac{\partial w}{\partial \zeta_2} y_{31} + \frac{\partial w}{\partial \zeta_3} y_{12} \right) \\ \frac{\partial w}{\partial y} &= \frac{1}{2A} \left(\frac{\partial w}{\partial \zeta_1} x_{32} + \frac{\partial w}{\partial \zeta_2} x_{13} + \frac{\partial w}{\partial \zeta_3} x_{21} \right) \end{aligned} \quad (27.13)$$

which in matrix form is

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial \zeta_1} \\ \frac{\partial w}{\partial \zeta_2} \\ \frac{\partial w}{\partial \zeta_3} \end{bmatrix}. \quad (27.14)$$

§27.1.7. Six Node Quadratic Interpolation

Consider next the six-node triangle shown in Figure 27.3(b). This element still has straight sides, and nodes 4, 5 and 6 are located at the midpoint of the sides. In Chapter 16 of the IFEM course it was shown a function $w(x, y)$ that varies quadratically over the element and takes on the node values w_i , $i = 1, 2, 3, 4, 5, 6$, can be interpolated in terms of the triangular coordinates by the formula

$$w = [w_1 \ w_2 \ w_3 \ w_4 \ w_5 \ w_6] \begin{bmatrix} N_1^q \\ N_2^q \\ N_3^q \\ N_4^q \\ N_5^q \\ N_6^q \end{bmatrix}. \quad (27.15)$$

where the N_i^q are the quadratic shape functions

$$\begin{aligned} N_1^q &= \zeta_1(2\zeta_1 - 1) & N_2^q &= \zeta_2(2\zeta_2 - 1) & N_3^q &= \zeta_3(2\zeta_3 - 1) \\ N_4^q &= 4\zeta_1\zeta_2 & N_5^q &= 4\zeta_2\zeta_3 & N_6^q &= 4\zeta_3\zeta_1 \end{aligned} \quad (27.16)$$

The geometry is still defined by (27.9) because we are not permitting nodes 4,5,6 to be away from the midpoint positions. Similarly, the partial derivative expressions of §27.1.6 remain valid.

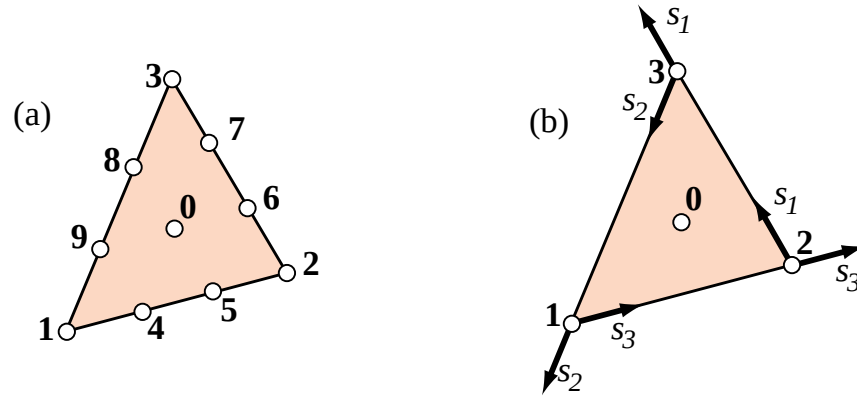


Figure 27.4. Nodal configurations for two forms of cubic interpolation of $w(x, y)$ over triangle: (a) interpolation by ten values w_i at corners $i = 1, 2, 3$, thirdpoints $i = 4, 5, 6, 7, 8, 9$ and centroid $i = 0$; (b) interpolation by four w_i values at corners $i = 1, 2, 3$ and centroid $i = 0$, plus six side slopes $(\partial w / \partial s_2)_1, (\partial w / \partial s_3)_1, (\partial w / \partial s_3)_2, (\partial w / \partial s_1)_2, (\partial w / \partial s_1)_3$ and $(\partial w / \partial s_2)_3$.

§27.2. Cubic Interpolants

Cubic interpolants for w are fundamental in the construction of the simplest Kirchhoff plate bending elements. The complete two-dimensional cubic polynomial has 10 terms. There are several choices for the selection of nodal values that determine that interpolation. Several important ones are examined next.

§27.2.1. Cubic Interpolation Choice 1: 10 Nodes

Next consider the ten-node triangle shown in Figure 27.4(a). This element still has straight sides. Nodes 4 through 9 are placed at the thirdpoints of the sides as indicated. The tenth node is placed at the centroid and is labeled 0.

In a Homework of the IFEM course it was shown that a function $w(x, y)$ that varies cubically over the element and takes on the node values $w_i, i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 0$, can be interpolated in terms of the triangular coordinates by the formula

$$w = [w_1 \ w_2 \ w_3 \ w_4 \ w_5 \ w_6 \ w_7 \ w_8 \ w_9 \ w_0] \begin{bmatrix} N_1^{c1} \\ N_2^{c1} \\ N_3^{c1} \\ N_4^{c1} \\ N_5^{c1} \\ N_6^{c1} \\ \vdots \\ N_0^{c1} \end{bmatrix}. \quad (27.17)$$

where the N_i^{c1} are the 10-node cubic shape functions

$$\begin{aligned}
 N_1^{c1} &= \frac{1}{2}\zeta_1(3\zeta_1 - 1)(3\zeta_1 - 1) & N_2^{c1} &= \frac{1}{2}\zeta_2(3\zeta_2 - 1)(3\zeta_2 - 1) & N_3^{c1} &= \frac{1}{2}\zeta_3(3\zeta_3 - 1)(3\zeta_3 - 1) \\
 N_4^{c1} &= \frac{9}{2}\zeta_1\zeta_2(3\zeta_1 - 1) & N_5^{c1} &= \frac{9}{2}\zeta_1\zeta_2(3\zeta_2 - 1) & N_6^{c1} &= \frac{9}{2}\zeta_2\zeta_3(3\zeta_2 - 1) \\
 N_7^{c1} &= \frac{9}{2}\zeta_2\zeta_3(3\zeta_3 - 1) & N_8^{c1} &= \frac{9}{2}\zeta_3\zeta_1(3\zeta_3 - 1) & N_9^{c1} &= \frac{9}{2}\zeta_3\zeta_1(3\zeta_1 - 1) \\
 N_0^{c1} &= 27\zeta_1\zeta_2\zeta_3
 \end{aligned} \tag{27.18}$$

The geometry is still defined by (27.8) because we are not permitting nodes 4 through 9 to be away from the thirdpoint positions as well as forcing 0 to be exactly at the centroid. Similarly, the partial derivative expressions of §27.1.6 remain valid.

§27.2.2. Cubic Interpolation Choice 2: 4 Nodes plus 6 Side Slopes

Consider now a variant of the cubic interpolation over the triangle as shown in Figure 27.4(b). This element still has straight sides. In addition to values at the corners 1,2,3 and the centroid 0, we specify the side-slope corner derivatives $w_{jk} = (\partial w / \partial s_j)_k$. There are six combinations: $w_{21} = (\partial w / \partial s_2)_1$, $w_{31} = (\partial w / \partial s_3)_1$, $w_{32} = (\partial w / \partial s_3)_2$, $w_{12} = (\partial w / \partial s_1)_2$, $w_{13} = (\partial w / \partial s_1)_3$ and $w_{23} = (\partial w / \partial s_2)_3$. The resulting interpolation is

$$w = [w_1 \ w_{21} \ w_{31} \ w_2 \ w_{32} \ w_{12} \ w_3 \ w_{13} \ w_{23} \ w_0] \begin{bmatrix} N_1^{c2} \\ N_2^{c2} \\ N_3^{c2} \\ N_4^{c2} \\ N_5^{c2} \\ N_6^{c2} \\ \vdots \\ N_0^{c2} \end{bmatrix}. \tag{27.19}$$

in which the N_i^{c2} are the shape functions

$$\begin{aligned}
 N_1^{c2} &= \zeta_1^2(\zeta_1 + 3\zeta_2 + 3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_2^{c2} &= -L_{31}(\zeta_3\zeta_1^2 - \zeta_1\zeta_2\zeta_3) \\
 N_3^{c2} &= L_{12}(\zeta_1\zeta_2^2 - \zeta_1\zeta_2\zeta_3) & N_4^{c2} &= \zeta_2^2(3\zeta_1 + \zeta_2 + 3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 \\
 N_5^{c2} &= -L_{12}(\zeta_1\zeta_2^2 - \zeta_1\zeta_2\zeta_3) & N_6^{c2} &= L_{23}(\zeta_2\zeta_3^2 - \zeta_1\zeta_2\zeta_3) \\
 N_7^{c2} &= \zeta_3^2(3\zeta_1 + 3\zeta_2 + \zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_8^{c2} &= -L_{23}(\zeta_2\zeta_3^2 - \zeta_1\zeta_2\zeta_3) \\
 N_9^{c2} &= L_{31}(\zeta_3\zeta_1^2 - \zeta_1\zeta_2\zeta_3) & N_0^{c2} &= 27\zeta_1\zeta_2\zeta_3
 \end{aligned} \tag{27.20}$$

and L_{jk} denotes the length of the side joining corners j and k . The function of the corrective terms $\zeta_1\zeta_2\zeta_3$ is to make the first nine functions (27.20) vanish at 0.

The $c1$ and $c2$ formulas are alledgely related by the transformation matrix

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_0 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 \\ 20 & 0 & 4L_{12} & 7 & -2L_{12} & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 2L_{12} & 20 & -4L_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 4L_{23} & 7 & -2L_{23} & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 2L_{23} & 20 & -4L_{23} & 0 & 0 \\ 7 & -2L_{31} & 0 & 0 & 0 & 0 & 20 & 0 & 4L_{31} & 0 \\ 20 & -4L_{31} & 0 & 0 & 0 & 0 & 7 & 0 & 2L_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 \end{bmatrix} \begin{bmatrix} w_1 \\ w_{21} \\ w_{31} \\ w_2 \\ w_{32} \\ w_{12} \\ w_3 \\ w_{13} \\ w_{23} \\ w_0 \end{bmatrix} \quad (27.21)$$

§27.2.3. Cubic Interpolation Choice 3: 4 Nodes plus 6 Cartesian Slopes

This choice is similar to the previous one but the corner derivatives are taken with respect to the directions x, y of the Cartesian reference system. The notation used for the slopes is $w_{x1} = (\partial w / \partial x)_1$, $w_{y1} = (\partial w / \partial y)_1$, $w_{x2} = (\partial w / \partial x)_2$, $w_{y2} = (\partial w / \partial y)_2$, $w_{x3} = (\partial w / \partial x)_3$ and $w_{y3} = (\partial w / \partial y)_3$. The resulting interpolation is

$$w = [w_1 \ w_{x1} \ w_{y1} \ w_2 \ w_{x2} \ w_{y2} \ w_3 \ w_{x3} \ w_{y3} \ w_0] \begin{bmatrix} N_1^{c3} \\ N_2^{c3} \\ N_3^{c3} \\ N_4^{c3} \\ N_5^{c3} \\ N_6^{c3} \\ \vdots \\ N_0^{c3} \end{bmatrix}. \quad (27.22)$$

in which the N_i^{c3} are the shape functions

$$\begin{aligned} N_1^{c3} &= \zeta_1^2(\zeta_1 + 3\zeta_2 + 3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_2^{c3} &= \zeta_1^2(x_{21}\zeta_2 - x_{13}\zeta_3) + (x_{13} - x_{21})\zeta_1\zeta_2\zeta_3 \\ N_3^{c3} &= \zeta_1^2(y_{21}\zeta_2 - y_{13}\zeta_3) + (y_{13} - y_{21})\zeta_1\zeta_2\zeta_3 & N_4^{c3} &= \zeta_2^2(3\zeta_1 + \zeta_2 + 3\zeta_3) - 7\zeta_1\zeta_2\zeta_3 \\ N_5^{c3} &= \zeta_2^2(x_{32}\zeta_3 - x_{21}\zeta_1) + (x_{21} - x_{32})\zeta_1\zeta_2\zeta_3 & N_6^{c3} &= \zeta_2^2(y_{32}\zeta_3 - y_{21}\zeta_1) + (y_{21} - y_{32})\zeta_1\zeta_2\zeta_3 \\ N_7^{c3} &= \zeta_3^2(3\zeta_1 + 3\zeta_2 + \zeta_3) - 7\zeta_1\zeta_2\zeta_3 & N_8^{c3} &= \zeta_3^2(x_{13}\zeta_1 - x_{32}\zeta_2) + (x_{32} - x_{13})\zeta_1\zeta_2\zeta_3 \\ N_9^{c3} &= \zeta_3^2(y_{13}\zeta_1 - y_{32}\zeta_2) + (y_{32} - y_{13})\zeta_1\zeta_2\zeta_3 & N_0^{c3} &= 27\zeta_1\zeta_2\zeta_3 \end{aligned} \quad (27.23)$$

in which $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$.

§27.2.4. Cubic Interpolation Choice 4: 4 Nodes plus 6 Cartesian Rotations

This interpolation is the same as the previous with some rearrangements. The rotational degrees of freedom at the corners are introduced using the relations $\theta_x = \partial w / \partial y$ and $\theta_y = -\partial w / \partial x$. The

resulting interpolation is

$$w = [w_1 \ \theta_{x1} \ \theta_{y1} \ w_2 \ \theta_{x2} \ \theta_{y2} \ w_3 \ \theta_{x3} \ \theta_{y3} \ w_0] \begin{bmatrix} N_1^{c4} \\ N_2^{c4} \\ N_3^{c4} \\ N_4^{c4} \\ N_5^{c4} \\ N_6^{c4} \\ \vdots \\ N_0^{c4} \end{bmatrix}. \quad (27.24)$$

in which the N_i^{c4} are the same shape functions of Choice 3 with some

$$\begin{array}{lll} N_1^{c4} = N_1^{c3} & N_2^{c4} = N_3^{c3} & N_3^{c4} = -N_2^{c3} \\ N_4^{c4} = N_4^{c3} & N_5^{c4} = N_6^{c3} & N_6^{c4} = -N_5^{c3} \\ N_7^{c4} = N_4^{c3} & N_8^{c4} = N_9^{c3} & N_9^{c4} = -N_8^{c3} \\ N_0^{c4} = N_4^{c3} & & \end{array} \quad (27.25)$$

§27.3. The BCIZ Plate Bending Element

One of the simplest Kirchhoff plate bending elements¹ was presented by Bazeley, Cheung, Irons and Zienkiewicz at the 1965 Wright-Patterson Conference.² This is called the “BCIZ element” after the authors initials.

This element can be derived from the cubic interpolation choice 4 (27.24). The technique appears a bit mysterious at first. Basically one can construct a 10×10 plate stiffness matrix. Freedom w_0 at the centroid can be statically condensed out and the resulting 9×9 stiffness used in finite element analysis. Unfortunately the static condensation destroys curvature completeness so the solutions will not generally converge. The idea behind the BCIZ element is that elimination of the centroidal DOF is done in such a way that completeness is maintained, using a kinematic constraint.

§27.3.1. The Kinematic Constraint

As explained above, we seek a kinematic constraint to eliminate w_0 in terms of the nine connector

¹ Historically the first triangular plate bending element that satisfied completeness and invariance. This element is also important as the motivation for development of the patch test.

² G. P. Bazeley, Y. K. Cheung, B. M. Irons and O. C. Zienkiewicz, Triangular Elements in Plate Bending – Conforming and Nonconforming Solutions, *Proceedings First Conference on Matrix Methods in Structural Mechanics*, AFFDL-TR-66-80, Air Force Institute of Technology, Dayton, Ohio, 547–576, 1966.

DOFs:

$$w_0 = [a_{w1} \ a_{\theta x1} \ a_{\theta y1} \ a_{w2} \ a_{\theta x2} \ a_{\theta y2} \ a_{w3} \ a_{\theta x3} \ a_{\theta y3}] \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \end{bmatrix} \quad (27.26)$$

Substituting this into any of the 4 choices of the previous section, w_0 is eliminated. For choice 4:

$$w = [w_1 \ \theta_{x1} \ \theta_{y1} \ w_2 \ \theta_{x2} \ \theta_{y2} \ w_3 \ \theta_{x3} \ \theta_{y3}] \begin{bmatrix} N_1^{c4} + a_{w1}N_0 \\ N_2^{c4} + a_{\theta x1}N_0 \\ N_3^{c4} + a_{\theta y1}N_0 \\ \vdots \\ N_9^{c4} + a_{\theta y3}N_0 \end{bmatrix}. \quad (27.27)$$

in which $N_0 = 27\zeta_1\zeta_2\zeta_3$ is the same for all choices. To determine the a 's in (27.27) we impose the condition that *all constant curvature states must be exactly represented*. This is a completeness condition. Constant curvature states are associated with quadratic variations of w , which are defined by the 6-node interpolation (27.16). The procedure is as follows:

- (i) Use (27.16) to express w_0 in terms of w_1 through w_6 by setting $\zeta_1 = \zeta_2 = \zeta_3 = 1/3$.
- (ii) Take the cubic interpolation choice 4 (27.24) and express the value of w_4 , w_5 and w_6 in terms of the nine connection DOF w_1 through θ_{y3} by setting ζ_1 , ζ_2 and ζ_3 to appropriate midpoint coordinates.
- (iii) Eliminate w_4 , w_5 and w_6 from (i) and (ii) to get (27.26) and hence (27.27). The resulting formula is called the BCIZ interpolation. The associated N 's are the BCIZ shape functions.

It can be shown that the constraint is

$$\mathbf{q} = [1/3, (-2x_1 + x_2 + x_3)/18, (-2y_1 + y_2 + y_3)/18, \\ 1/3, (x_1 - 2x_2 + x_3)/18, (y_1 - 2y_2 + y_3)/18, \\ 1/3, (x_1 + x_2 - 2x_3)/18, (y_1 + y_2 - 2y_3)/18] \quad (27.28)$$

The derivation of the entries of (27.28) is the subject of a homework Exercise.

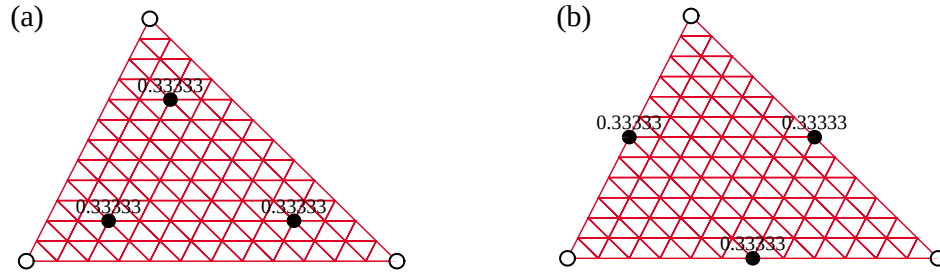


Figure 27.5. The two 3-point Gauss integration rules for triangles, which are useful in the computation of the plate element stiffness matrix $\mathbf{K}$. Numbers annotated near sample points are the weights.

§27.3.2. The Curvature Displacement Matrix

Once we have an interpolation formula for w , as in (27.27), the curvatures over the plate element can be evaluated by double differentiation of the shape functions with respect to x and y . The resulting relation can be expressed in matrix form as

$$\begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix} = \begin{bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial y \partial x \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} & B_{19} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} & B_{27} & B_{28} & B_{29} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} & B_{37} & B_{38} & B_{39} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \end{bmatrix} \quad (27.29)$$

or

$$\boldsymbol{\kappa} = \mathbf{B}\mathbf{u} \quad (27.30)$$

The entries B_{ij} are *linear* in the triangular coordinates ζ_1 , ζ_2 and ζ_3 .

The derivation of the entries of the $\mathbf{B}$ matrix is the subject of a homework Exercise.

§27.3.3. The Element Stiffness Matrix

For this displacement element, the stiffness matrix is given by the usual formula

$$\mathbf{K} = \int_{\Omega^{(e)}} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \quad (27.31)$$

where $\Omega^{(e)}$ is the element area, $\mathbf{D}$ is the plate rigidity matrix that relates moment to curvatures (see §9.3.2) and $\mathbf{B}$ is the curvature-displacement matrix defined in (27.30).

If $\mathbf{D}$ is constant over the element, as frequently assumed, the integrand of $\mathbf{K}$ is quadratic in the triangular coordinates. If so the integral can be done exactly using one of the two 3-point Gauss rule presented in Chapter 23 of IFEM. The two rules are reproduced below for convenience:

$$\frac{1}{A} \int_{\Omega^{(e)}} F(\zeta_1, \zeta_2, \zeta_3) d\Omega = \frac{1}{3} F\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right) + \frac{1}{3} F\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right) + \frac{1}{3} F\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right). \quad (27.32)$$

$$\frac{1}{A} \int_{\Omega^{(e)}} F(\zeta_1, \zeta_2, \zeta_3) d\Omega = \frac{1}{3} F\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{3} F\left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{3} F\left(\frac{1}{2}, 0, \frac{1}{2}\right). \quad (27.33)$$

The second one is called *the midpoint rule* because the three sample points are at the triangle midpoints. These rules are depicted in Figures 27.5(a) and (b), respectively. Both are exact up to quadratic polynomials in the triangular coordinates, which is what is needed for the plate stiffness matrix.

Homework Exercises for Chapter 27
Triangular Plate Displacement Elements

EXERCISE 27.1 [A:15] Check that the shape functions given for cubic interpolation choice 4 satisfy the conditions of being 1 for the associated DOF, and zero for all others. For example N_2 , which is associated with θ_{x1} , must satisfy

$$N_2(1, 0, 0) = 0, \quad \frac{\partial N_2}{\partial y}(1, 0, 0) = 1 = \theta_{x1}, \quad -\frac{\partial N_2}{\partial x}(1, 0, 0) = 0 = \theta_{y1}, \quad \text{etc.} \quad (\text{E27.1})$$

Note: It is sufficient to verify N_1 , N_2 , and N_3 because the next six shape functions are obtained by cyclic permutation of subscripts. N_0 is easy to check since its corner value and corner slopes are zero.

EXERCISE 27.2 [A:15] Using the chain rule and the results of §27.1.6, show that the following formula can be used to compute the plate curvatures from an interpolation formula in w :

$$\begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{23}^2 & x_{32}^2 & 2x_{32}y_{23} \\ y_{31}^2 & x_{13}^2 & 2x_{13}y_{31} \\ y_{12}^2 & x_{21}^2 & 2x_{21}y_{12} \\ 2y_{23}y_{31} & 2x_{32}x_{13} & 2(x_{32}y_{31} + x_{13}y_{23}) \\ 2y_{31}y_{12} & 2x_{13}x_{21} & 2(x_{13}y_{12} + x_{21}y_{31}) \\ 2y_{12}y_{23} & 2x_{21}x_{32} & 2(x_{21}y_{23} + x_{32}y_{12}) \end{bmatrix}^T \begin{bmatrix} \frac{\partial^2 w}{\partial \xi_1^2} \\ \frac{\partial^2 w}{\partial \xi_2^2} \\ \frac{\partial^2 w}{\partial \xi_3^2} \\ \frac{\partial^2 w}{\partial \xi_1 \partial \xi_2} \\ \frac{\partial^2 w}{\partial \xi_2 \partial \xi_3} \\ \frac{\partial^2 w}{\partial \xi_3 \partial \xi_1} \end{bmatrix}, \quad (\text{E27.2})$$

EXERCISE 27.3 [A:15] Derive the expression for the a 's in (27.27) following the procedure outline in that section, and derive the modified shape functions appearing in (27.28).

EXERCISE 27.4 [A:25] Program the BCIZ element starting from the interpolation formula provided by the previous exercise, the expression (E27.2) to form the curvature-displacement matrix $\mathbf{B}$ and one of the 3-point Gauss rules given in §27.3.3 to evaluate $\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega$. Assume that $\mathbf{D} = (h^3/12)\mathbf{E}$ in which both the thickness h and the 3×3 stress-strain matrix $\mathbf{E}$ are constant over the element, and supplied through arguments.

EXERCISE 27.5 [A:15] Evaluate $\mathbf{K}^{(e)}$ of the BCIZ element for a triangle with corners at $(0, 0)$, $(2, 1)$ and $(1, 3)$, fabricated with isotropic material of $E = 120$ and $\nu = 0$, and constant thickness $h = 1$. As a check, compute the 9 eigenvalues; three of them should be zero and six positive.

28

Finite Element Templates for Plate Bending

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RECENT ADVANCES IN FINITE ELEMENT TEMPLATES

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Abstract: A finite element template is a parametrized algebraic form that reduces to specific finite elements by setting numerical values to the free parameters. Following an outline of high performance elements, templates for Kirchhoff Plate-Bending Triangles (KPT) with 3 nodes and 9 degrees of freedom are studied. A 37-parameter template is constructed using the Assumed Natural Deviatoric Strain (ANDES) approach. Specialization of this template includes well known elements such as DKT and HCT. The question addressed here is: can these parameters be selected to produce high performance elements? The study is carried out by staged application of constraints on the free parameters. The first stage produces element families satisfying invariance and aspect ratio insensitivity conditions. Application of energy balance constraints produces specific elements. The performance of such elements in a preliminary set of benchmark tests is reported.

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§28.1. Introduction

The Finite Element Method (FEM) was first described in the presently dominant form by Turner et. al. [1]. It was baptized by Clough [2] at the beginning of an explosive growth period. The first applications book, by Zienkiewicz and Cheung [3] appeared seven years later. The first monograph on the mathematical foundations was written by Strang and Fix [4]. The opening sentence of this book already declared the FEM an “astonishing success.” And indeed the method had by then revolutionized computational structural mechanics and was in its way to impact non-structural applications.

The FEM was indeed the right idea at the right time. The key reinforcing factor was the expanding availability of digital computers. Lack of this enabling tool meant that earlier related proposals, notably that of Courant [5], had been forgotten. A second enabler was the heritage of classical structural mechanics and its reformulation in matrix form, which culminated in the elegant unification of Argyris and Kelsey [6]. A third influence was the victory of the Direct Stiffness Method (DSM) developed by

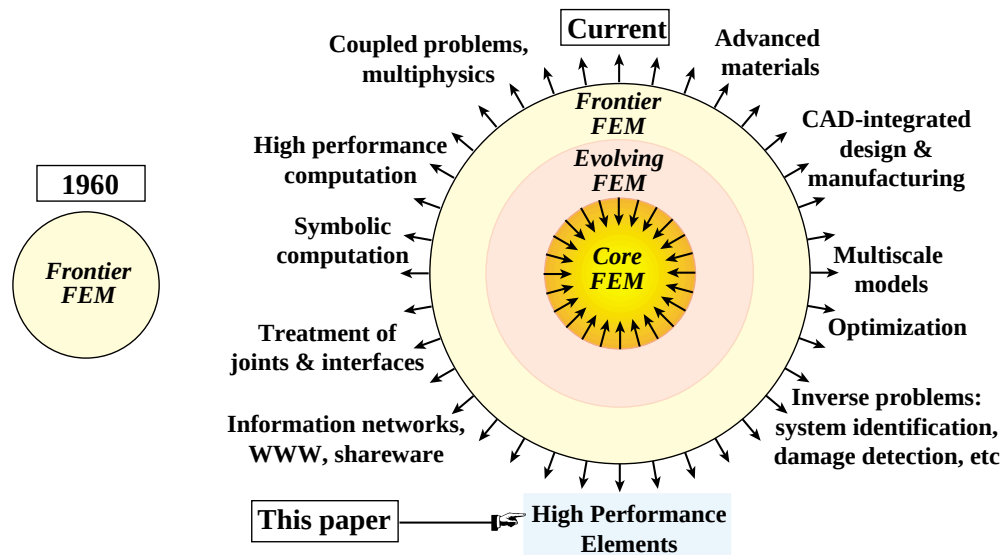


Figure 28.1. Evolution of the Finite Element Method.

Turner [7,8] over the venerable Force Method, a struggle recently chronicled by Felippa [9]. Victory was sealed by the adoption of the DSM in the earlier general-purpose FEM codes, notably NASTRAN, MARC and SAP. In the meantime the mathematical foundations were rapidly developed in the 1970s.

“Astonishing success”, however, carries its own dangers. By the early 1980s the FEM began to be regarded as “mature technology” by US funding agencies. By now that feeling has hardened to the point that it is virtually impossible to get significant research support for fundamental work in FEM. This viewpoint has been reinforced by major software developers, which proclaim their products as solutions to all user needs.

Is this perception correct? It certainly applies to the *core FEM*, or orthodox FEM. This is the material taught in textbooks and which is implemented in major software products. Core FEM follows what may be called the Ritz-Galerkin and Direct Stiffness Method canon. Beyond the core there is an *evolving FEM*. This is strongly rooted on the core but goes beyond textbooks. Finally there is a *frontier FEM*, which makes only partial or spotty use of core knowledge. See Figure 28.1.

By definition core FEM is mature. As time goes, it captures segments of the evolving FEM. For example, most of the topic of FEM mesh adaptivity can be classified as evolving, but will eventually become part of the core. Frontier FEM, on the other hand, can evolve unpredictably. Some components prosper, mature and eventually join the core, some survive but never become orthodox, while others wither and die.

Four brilliant contributions of Bruce Irons, all of which were frontier material when first published, can be cited as examples of the three outcomes. *Isoparametric elements* and *frontal solvers* rapidly became integral part of the core technology. The *patch test* has not become part of the core, but survives as a useful if controversial tool for element development and testing. The *semi-loof shell elements* quietly disappeared.

Topics that drive frontier FEM include multiphysics, multiscale models, symbolic and high performance computation, integrated design and manufacturing, advances in information technology, optimization, inverse problems, materials, treatment of joints and interfaces, and high performance elements. This paper deals with the last topic.

§28.2. High Performance Elements

An important area of frontier FEM is the construction of high performance (HP) finite elements. These were defined by Felippa and Militello [10] as “simple elements that deliver engineering accuracy with arbitrary coarse meshes.” Some of these terms require clarification.

Simple means the simplest geometry and freedom configuration that fits the problem and target accuracy consistent with human and computer resources. This can be summed up in one FEM modeling rule: *use the simplest element that will do the job.*

Engineering accuracy is that generally expected in most FEM applications in Aerospace, Civil and Mechanical Engineering. Typically this is 1% in displacements and 10% in strains, stresses and derived quantities. Some applications, notably in Aerospace, require higher precision in quantities such as natural frequencies, shape tolerances, or in long-time simulations.

Coarse mesh is one that suffices to capture the important physics in terms of geometry, material and load properties. It does not imply few elements. For example, a coarse mesh for a fighter aircraft undergoing maneuvers may require several million elements. For simple benchmark problems such as a uniformly loaded square plate, a mesh of 4 or 16 elements may be classified as coarse.

Finally, *arbitrary mesh* implies *low sensitivity to skewness and distortion*. This attribute is becoming important as push-button mesh generators gain importance, because generated meshes can be of low quality compared to those produced by an experienced analyst.

For practical reasons we are interested only in the construction of HP elements with *displacement* nodal degrees of freedom. Such elements are characterized by their stiffness equations, and thus can be plugged into any standard finite element program.

§28.2.1. Tools for Construction of HP Elements

The origins of HP finite elements may be traced to several investigators in the late 1960s and early 1970s. Notable early contributions are those of Clough, Irons, Taylor, Wilson and their coworkers. The construction techniques made use of incompatible shape functions, the patch test, reduced, selective and directional integration. These can be collectively categorized as unorthodox, and in fact were labeled as “variational crimes” at that time by Strang and Fix [4].

A more conventional development, pioneered by Pian, Tong and coworkers, made use of mixed and hybrid variational principles. They developed elements using stress or partial stress assumptions, but the end product were standard displacement elements. These techniques were further refined in the 1980s. A good expository summary is provided in the book by Zienkiewicz and Taylor [11].

New innovative approaches came into existence in the 1980s. The most notable have been the Free Formulation of Bergan and Nygård [12,13], and the Assumed Strain method pioneered by MacNeal [14]. The latter was further developed along different paths by Bathe and Dvorkin [15], Park and Stanley [16], and Simo and Hughes [17].

§28.2.2. Unification by Parametrized Variational Principles

The approach taken by the author started from collaborative work with Bergan in Free Formulation (FF) high performance elements. The results of this collaboration were a membrane triangle with drilling freedoms described in Bergan and Felippa [18] and a plate bending triangle presented by Felippa and Bergan [19]. It continued with exploratory work using the Assumed Natural Strain (ANS) method of Park and Stanley [16]. Eventually FF and ANS coalesced in a variant of ANS called Assumed

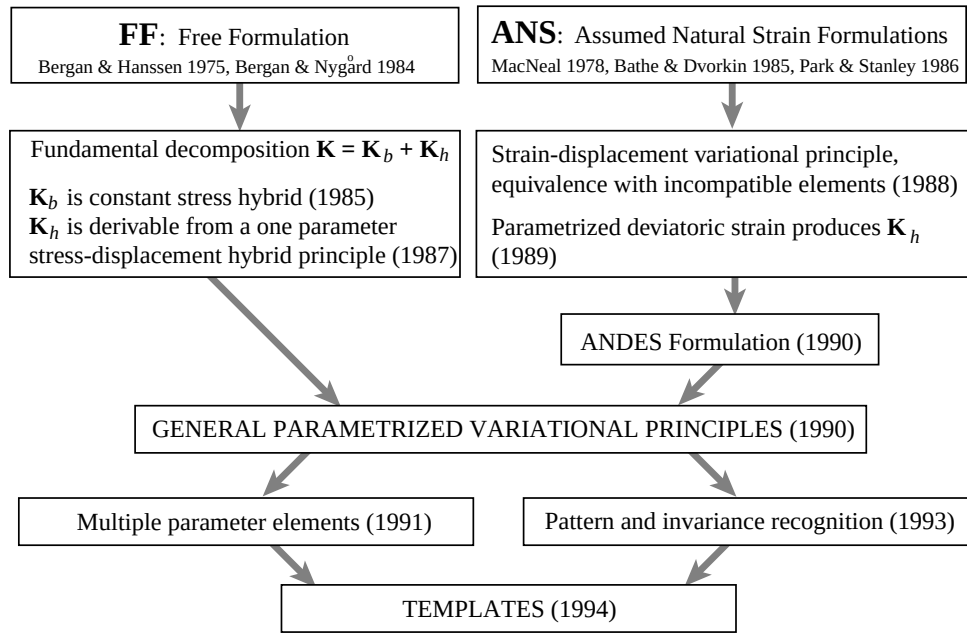


Figure 28.2. The road to templates.

Natural Deviatoric Strain, or ANDES. High performance elements based on the ANDES formulation are described by Militello and Felippa [20] and Felippa and Militello [21].

This unification work led naturally to a formulation of elasticity functionals containing free parameters. These were called parametrized variational principles, or PVPs in short. Setting the parameters to specific numerical values produced the classical functionals of elasticity such as Total Potential Energy, Hellinger-Reissner and Hu-Washizu. For linear elasticity, three free parameters in a three-field functional with independently varied displacements, strains and stresses are sufficient to embed all classical functionals. Two survey articles with references to the original papers are available [22,23].

One result from the PVP formulation is that, upon FEM discretization, free parameters appear at the element level. One thus naturally obtains *families* of elements. Setting the free parameters to numerical values produces specific elements. Although the PVP Euler-Lagrange equations are the same excepts for weights, the discrete solution produced by different elements are not. Thus an obvious question arises: which free parameters produce the best elements? It turns out that there is no clear answer to the question, because the best set of parameters depends on the element geometry. Hence the equivalent question: which is the best variational principle? makes no sense.

The PVP formulation led, however, to an unexpected discovery. The configuration of elements constructed according to PVPs and the usual assumptions on displacements, stresses and strains was observed to follow specific algebraic rules. *Such configurations could be parametrized directly without going through the source PVP.* This observation led to a general formulation of finite elements as *templates*.

The aforementioned developments are flowcharted in Figure 28.2.

§28.3. Finite Element Templates

A finite element template, or simply *template*, is an algebraic form that represents element-level stiffness equations, and which fulfills the following conditions:

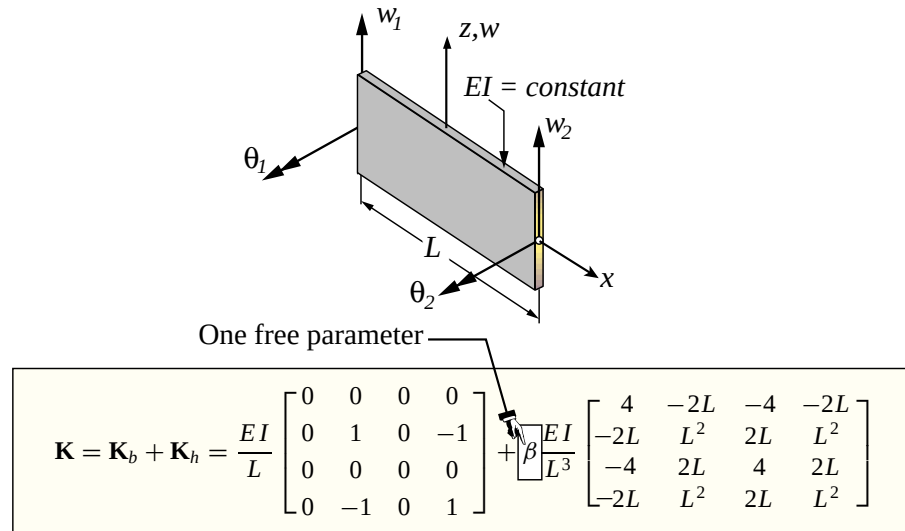


Figure 28.3. Template for Bernoulli-Euler prismatic plane beam.

(C) Consistency: the Individual Element Test (IET) form of the patch test, introduced by Bergan and Hanssen [24], is passed for any element geometry.

(S) Stability: the stiffness matrix satisfies correct rank and nonnegativity conditions.

(P) Parametrization: the element stiffness equations contain free parameters.

(I) Invariance: the element equations are observer invariant. In particular, they are independent of node numbering and choice of reference systems.

The first two conditions: (C) and (S), are imposed to ensure convergence. Property (P) permits performance optimization as well as tuning elements to specific needs. Property (I) helps predictability and benchmark testing.

Setting the free parameters to numeric values yields specific element instances.

§28.3.1. The Fundamental Decomposition

A stiffness matrix derived through the template approach has the fundamental decomposition

$$\mathbf{K} = \mathbf{K}_b(\alpha_i) + \mathbf{K}_h(\beta_j) \quad (28.1)$$

Here $\mathbf{K}_b$ and $\mathbf{K}_h$ are the basic and higher-order stiffness matrices, respectively. The basic stiffness matrix $\mathbf{K}_b$ is constructed for *consistency* and *mixability*, whereas the higher order stiffness $\mathbf{K}_h$ is constructed for *stability* (meaning rank sufficiency and nonnegativity) and *accuracy*. As further discussed below, the higher order stiffness $\mathbf{K}_h$ must be orthogonal to all rigid-body and constant-strain (curvature) modes.

In general both matrices contain free parameters. The number of parameters α_i in the basic stiffness is typically small for simple elements. For example, in the 3-node, 9-DOF KPT elements considered here there is only one basic parameter, called α . This number must be the *same for all elements in a mesh* to insure satisfaction of the IET [18].

On the other hand, the number of higher order parameters β_j can be in principle *infinite* if certain components of $\mathbf{K}_h$ can be represented as a polynomial series of element geometrical invariants. In practice, however, such series are truncated, leading to a finite number of β_j parameters. Although the

β_j may vary from element to element without impairing convergence, often the same parameters are retained for all elements.

As an illustration Figure 28.3 displays the template of a simple one-dimensional element: a 2-node, 4-DOF plane Bernoulli-Euler prismatic beam. This has only one free parameter: β , which scales the higher order stiffness. A simple calculation [22] shows that its optimal value is $\beta = 3$, which yields the well-known Hermitian beam stiffness. This is known as a *universal template* since it include all possible beam elements that satisfy the foregoing conditions.

§28.3.2. Constructing the Component Stiffness Matrices

The basic stiffness that satisfies condition (C) is the same for any formulation. It is simply a constant stress hybrid element [13,18]. For a specific element and freedom configuration, $\mathbf{K}_b$ can be constructed once and for all.

The formulation of the higher order stiffness $\mathbf{K}_h$ is not so clear-cut, as can be expected because of the larger number of free parameters. It can be done by a variety of techniques, which are summarized in a article by Felippa, Haugen and Militello [25]. Of these, one has proven exceedingly useful for the construction of templates: the ANDES formulation. ANDES stands for Assumed Natural DEviatoric Strains. It is based on assuming natural strains for the high order stiffness. For plate bending (as well as beams and shells) natural curvatures take the place of strains.

Second in usefulness is the Assumed Natural DEviatoric STRESSes or ANDESTRESS formulation, which for bending elements reduces to assuming deviatoric moments. This technique, which leads to stiffness templates that contain inverses of natural flexibilities, is not considered here.

§28.3.3. Basic Stiffness Properties

The following properties of the template stiffness equations are collected here for further use. They are discussed in more detail in the article by Felippa, Haugen and Militello [25]. Consider a test displacement field, which for thin plate bending will be a continuous transverse displacement mode $w(x, y)$. [In practical computations this will be a polynomial in x and y .] Evaluate this at the nodes to form the element node displacements $\mathbf{u}$. These can be decomposed into

$$\mathbf{u} = \mathbf{u}_b + \mathbf{u}_h = \mathbf{u}_r + \mathbf{u}_c + \mathbf{u}_h, \quad (28.2)$$

where $\mathbf{u}_r$, $\mathbf{u}_c$ and $\mathbf{u}_h$ are rigid body, constant strain and higher order components, respectively, of $\mathbf{u}$. The first two are collectively identified as the *basic* component $\mathbf{u}_b$. The matrices (28.1) must satisfy the stiffness orthogonality conditions

$$\mathbf{K}_b \mathbf{u}_r = \mathbf{0}, \quad \mathbf{K}_h \mathbf{u}_r = \mathbf{0}, \quad \mathbf{K}_h \mathbf{u}_c = \mathbf{0} \quad (28.3)$$

while $\mathbf{K}_b$ represents exactly the response to $\mathbf{u}_c$.

The strain energy taken up by the element under application of $\mathbf{u}$ is $U = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u}$. Decomposing $\mathbf{K}$ and $\mathbf{u}$ as per (28.1) and (28.2), respectively, and enforcing (28.3) yields

$$U = \frac{1}{2} (\mathbf{u}_b + \mathbf{u}_h)^T \mathbf{K}_b (\mathbf{u}_b + \mathbf{u}_h) + \frac{1}{2} \mathbf{u}_h^T \mathbf{K}_h \mathbf{u}_h = U_b + U_h \quad (28.4)$$

U_b and U_h are called the basic and higher order energy, respectively. Let U_{ex} be the exact energy taken up by the element as a continuum body subjected to the test displacement field. The element energy ratios are defined as

$$\rho = \frac{U}{U_{ex}} = \rho_b + \rho_h, \quad \rho_b = \frac{U_b}{U_{ex}}, \quad \rho_h = \frac{U_h}{U_{ex}}. \quad (28.5)$$

Here ρ_b and ρ_h are called the basic and higher order energy ratios, respectively. If $\mathbf{u}_h = \mathbf{0}$, $\rho = \rho_b = 1$ because the element must respond exactly to any basic mode by construction. For a general displacement mode in which $\mathbf{u}_h$ does not vanish, ρ_b is a function of the α_i whereas ρ_h is a function of the β_j .

§28.3.4. Constructing Optimal Elements

By making a template sufficiently general all published finite elements for a specific configuration can be generated. This includes those derivable by orthodox techniques (for example, shape functions) and those that are not. Furthermore, an infinite number of new elements arise. The same question previously posed for PVPs arises: Can one select the free parameters to produce an optimal element?

The answer is not yet known for general elements. The main unresolved difficulty is: which optimality conditions must be imposed at the local (element) level? While some of them are obvious, for example those requiring observer invariance, most of the others are not. The problem is that *a detailed connection between local and global optimality is not fully resolved by conventional FEM error analysis*. Such analysis can only provide convergence rates expressed as $C h^m$ in some error norm, where h is a characteristic mesh dimension and m is usually the same for all template instances. The key to high performance is the coefficient C , but this is problem dependent. Consequently, verification benchmarks are still inevitable.

As noted, conventional error analysis is of limited value because it only provides the exponent m , which is typically the *same* for all elements in a template. It follows that several template optimization constraints discussed later are heuristic. But even if the local-to-global connection were fully resolved, a second technical difficulty arises: the actual construction and optimization of templates poses formidable problems in symbolic matrix manipulation, because one has to carry along arbitrary geometries, materials and free parameters.

Until recently those manipulations were beyond the scope of computer algebra systems (CAS) for all but the simplest elements. As personal computers and workstations gain in CPU speed and storage, it is gradually becoming possible to process two-dimensional elements for plane stress and plate bending. Most three-dimensional and curved-shell elements, however, still lie beyond the power of present systems.

Practitioners of optimization are familiar with the dangers of excessive perfection. A system tuned to operate optimally for a narrow set of conditions often degrades rapidly under deviation from such conditions. The benchmarks of Section 8 show that a similar difficulty exists in the construction of optimal plate elements, and that expectations of an “element for all seasons” must be tempered.

§28.4. Templates for 3-Node KPT Elements

The application of the template approach is rendered specific by studying a particular configuration: a 3-node flat triangular element to model bending of Kirchhoff (thin) plates. The element has the conventional 3 degrees of freedom: one transverse displacement and three rotations at each corner, as illustrated in Figure 28.4.

For brevity this will be referred to as a Kirchhoff Plate Triangle, or KPT, in the sequel. The complete development of the template is given in the Appendix. In the following sections we summarize only the important results necessary for the application of local optimality constraints.

§28.4.1. Stiffness Decomposition

For the KPT elements under study the configuration of the stiffness matrices in (28.1) can be shown in more detail. Assuming that the 3×3 moment-curvature plate constitutive matrix $\mathbf{D}$ is constant over the

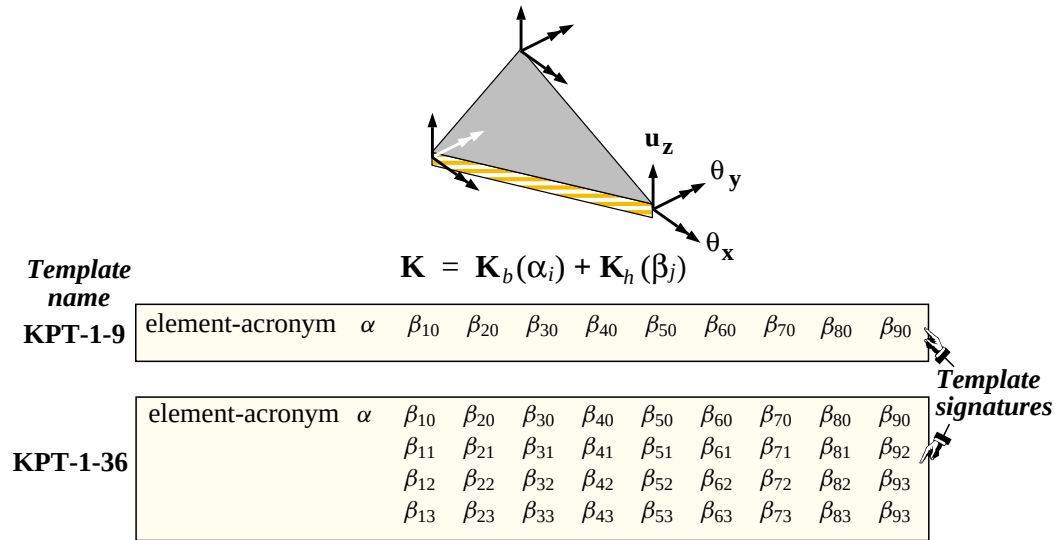


Figure 28.4. Template for Kirchhoff plate-bending triangles (KPT) studied here.

triangle, we have

$$\mathbf{K}_b = \frac{1}{A} \mathbf{L} \mathbf{D} \mathbf{L}^T, \quad \mathbf{K}_h = \frac{A}{3} [\mathbf{B}_{\chi 4}^T \mathbf{D}_{\chi} \mathbf{B}_{\chi 4} + \mathbf{B}_{\chi 5}^T \mathbf{D}_{\chi} \mathbf{B}_{\chi 5} + \mathbf{B}_{\chi 6}^T \mathbf{D}_{\chi} \mathbf{B}_{\chi 6}]. \quad (28.6)$$

Here A is the triangle area, $\mathbf{L}$ is the 9×3 force lumping matrix that transforms a constant internal moment field to node forces, $\mathbf{B}_{\chi m}$ are 3×9 matrices relating natural curvatures at triangle midpoints $m = 4, 5, 6$ to node displacements, and $\mathbf{D}_{\chi}$ is the plate constitutive matrix transformed to relate natural curvatures to natural moments. Parameter α appears in $\mathbf{L}$ whereas parameters β_j appear in $\mathbf{B}_{\chi m}$. Full expressions of these matrices are given in the Appendix.

§28.4.2. The KPT-1-36 and KPT-1-9 Templates

A useful KPT template is based on a 36-parameter representation of $\mathbf{K}_h$ in which the series noted above retains up to the *linear* terms in three triangle geometric invariants λ_1 , λ_2 and λ_3 , defined in the Appendix, which characterize the deviations from the equilateral-triangle shape. The template is said to be of *order one* in the λ s. It has a total of 37 free parameters: one α and 36 β s. Collectively this template is identified as KPT-1-36. Instances are displayed using the following tabular arrangement:

acronym	α	β_{10}	β_{20}	β_{30}	β_{40}	β_{50}	β_{60}	β_{70}	β_{80}	β_{90}	$/\beta_{sc}$
		β_{11}	β_{21}	β_{31}	β_{41}	β_{51}	β_{61}	β_{71}	β_{81}	β_{91}	
		β_{12}	β_{22}	β_{32}	β_{42}	β_{52}	β_{62}	β_{72}	β_{82}	β_{92}	
		β_{13}	β_{23}	β_{33}	β_{43}	β_{53}	β_{63}	β_{73}	β_{83}	β_{93}	

(28.7)

Here β_{sc} is a scaling factor by which all displayed β_{ij} must be divided; e.g. in the DKT element listed in Table 28.1 $\beta_{10} = -6/4 = -3/2$ and $\beta_{41} = 4/4 = 1$. If β_{sc} is omitted it is assumed to be one.

Setting the 37 parameters to numeric values yield specific elements, identified by the acronym displayed on the left. Some instances that are interesting on account of practical or historical reasons are collected in Table 28.1. Collectively these represents a tiny subset of the number of published KPT elements, which probably ranges in the hundreds, and is admittedly biased in favor of elements developed by

Table 28.1. Template Signatures of Some Existing KPT Elements

Acronym	α	β_{1j}	β_{2j}	β_{3j}	β_{4j}	β_{5j}	β_{6j}	β_{7j}	β_{8j}	β_{9j}	β_{sc}
ALR	0	−3	0	0	0	0	0	0	3	0	/2
		−6	0	0	0	0	0	0	0	0	
		0	0	0	0	0	0	0	0	0	
		0	0	0	0	0	0	0	−6	0	
AQR0	0	Same β s as AQR1									
AQRBE	$1/\sqrt{2}$	Same β s as AQR1									
AQR1	1	−3	0	0	0	0	0	0	3	0	/2
		−2	0	0	0	0	0	0	0	0	
		0	0	0	0	0	0	0	0	−4	
		0	0	0	0	0	0	0	−2	0	
AVG	0	−3	0	0	0	0	0	0	3	0	/2
BCIZ0	0	−3	1	1	0	0	−1	−1	3	0	/2
BCIZ1	1	−3	0	0	−1	1	0	0	3	0	/2
		0	0	0	2	0	0	2	0	0	
		0	0	−2	0	0	−2	0	0	0	
		0	2	0	0	2	0	0	0	0	
DKT	1	−6	1	1	−2	2	−1	−1	6	0	/4
		0	0	0	4	0	0	−2	0	0	
		0	0	2	0	0	2	0	0	0	
		0	−2	0	0	4	0	0	0	0	
FF0	0	−9	1	1	−2	2	−1	−1	9	0	/6
FF1	1	Same β s as FF0.									
HCT	1	−11	5	0	−2	2	0	−5	11	0	/4
		6	0	0	4	0	0	10	0	0	
		0	0	20	0	0	20	0	0	0	
		0	10	0	0	4	0	0	6	0	

the writer and colleagues. Table 28.2 identifies the acronyms of Table 28.1 correlated with original publications where appropriate.

An interesting subclass of (28.7) is that in which the bottom 3 rows vanish: $\beta_{11} = \beta_{12} = \dots \beta_{93} = 0$. This 10-parameter template is said to be of *order zero* because the invariants λ_1 , λ_2 and λ_3 do not appear in the higher order stiffness. It is identified as KPT-1-9. For brevity it will be written simply as

acronym	α	β_{10}	β_{20}	β_{30}	β_{40}	β_{50}	β_{60}	β_{70}	β_{80}	β_{90}	$/\beta_{sc}$

(28.8)

omitting the zero entries.

§28.4.3. Element Families

Specializations of (28.7) and (28.8) that still contain free parameters are called *element families*. In such a case the free parameters are usually written as arguments of the acronym. For example, Table 28.4 defines the ARI, or Aspect Ratio Insensitive, family derived in Section 6. ARI has seven free parameters identified as α , β_{10} , β_{20} , β_{30} , γ_0 , γ_1 and γ_2 . Consequently the template acronym is written $\text{ARI}(\alpha, \beta_{10}, \beta_{20}, \beta_{30}, \gamma_0, \gamma_1, \gamma_2)$.

A family whose only free parameter is α is called an α -family. Its instances are called α -variants. In some α families the β coefficients are fixed. For example in the $\text{AQR}(\alpha)$ and $\text{FF}(\alpha)$ families only α changes. Some practically important instances of those families are shown in Table 28.1. In other

Table 28.2. Element Identifiers Used in Table 28.1

Acronym	Description
ALR	Assumed Linear Rotation KPT element of Militello and Felippa [20].
AQR1	Assumed Quadratic Rotation KPT element of Militello and Felippa [20].
AQR0	α -variant of AQR1 with $\alpha = 0$.
AQRBE	α -variant of AQR1 with $\alpha = 1/\sqrt{2}$; of interest because it is BME.
AVG	Average curvature KPT element of Militello and Felippa [20].
BCIZ0	Nonconforming element of Bazeley, Cheung, Irons and Zienkiewicz [26] “sanitized” with $\alpha = 0$ as described by Felippa, Haugen and Militello [25]. Historically the first polynomial-based, complete, nonconforming KPT and the motivation for the original (multielement) patch test of Irons. See Section 4.4 for two clones of BCIZ0.
BCIZ1	Variant of above, in which the original BCIZ is sanitized with $\alpha = 1$.
DKT	Discrete Kirchhoff Triangle of Stricklin et al. [30] streamlined by Batoz [31]; see also Bathe, Batoz and Ho [32]
FF0	Free Formulation element of Felippa and Bergan [19].
FF1	α variant of FF0 with $\alpha = 1$.
HCT	Hsieh-Clough-Tocher element presented by Clough and Tocher [33] with curvature field collocated at the 3 midpoints. The original (macroelement) version was the first successful C^1 conforming KPT.

α -families the β s are functions of α . For example this happens in the BCIZ(α) family, two instances of which, obtained by setting $\alpha = 0$ and $\alpha = 1$, are shown on Table 28.1.

§28.4.4. Template Genetics: Signatures and Clones

An examination of Table 28.1 should convince the reader that template coefficients uniquely define an element once and for all, although the use of author-assigned acronyms is common in the FE literature. The parameter set can be likened to an “element genetic fingerprint” or “element DNA” that makes it a unique object. This set is called the element *signature*.

If signatures were randomly generated, the number of element instances would be of course huge: more precisely ∞^{37} for 37 parameters. But in practice elements are not fabricated at random. Attractors emerge. Some element derivation methods, notably those based on displacement shape functions, tend to “hit” certain signature patterns. The consequence is that the same element may be discovered separately by different authors, often using dissimilar derivation techniques. Such elements will be called *clones*. Cloning seems to be more prevalent among instances of the order-zero KPT-1-9 template (28.8). Some examples discovered in the course of this study are reported.

The first successful nonconforming triangular plate bending element was the original BCIZ [26]. This element, however, does not pass the Individual Element Test (IET), and in fact fails Irons’ original patch test for arbitrary mesh patterns. The source of the disease is the basic stiffness. The element can be

“sanitized” by removing the infected matrix as described by Felippa, Haugen and Militello [25]. This is replaced by a healthy $\mathbf{K}_b$ with, for example, $\alpha = 0$ or $\alpha = 1$. This transplant operation yields the elements called BCIZ0 and BCIZ1, respectively, in Table 28.1. [These are two instances of the BCIZ(α) family.] Note that BCIZ0 pertains to the KPT-1-9 template.

In the 3rd MAFELAP Conference, Hanssen, Bergan and Syversten [27] reported a nonconforming element which passed the IET and (for the time) was of competitive performance. Construction of its template signature revealed it to be a clone of BCIZ0. The plate bending part of the TRIC shell element [28] is also a clone of BCIZ0.

An energy orthogonal version of the HBS element was constructed by Nygård in his Ph.D. thesis [29]. Its signature turned out to agree with that of the FF0 element, constructed by Felippa and Bergan [19] with a different set of higher order shape functions.

Clones seem rarer in the realm of the full KPT-1-36 template because of its greater richness. The DKT [30–32] appears to be an exception. Although this popular element is usually constructed by assuming rotation fields, it coalesced with one of the ANDES elements derived by Militello and Felippa [20] by assuming natural deviatoric curvatures. At the time the coalescence was suspected from benchmarks, and later verified by direct examination of stiffness matrices. Using the template formulation such numerical tests can be bypassed, since it is sufficient to compare signatures.

§28.4.5. Parameter Constraints

To construct element families and in the limit, specific elements, constraints on the free parameters must be imposed. One key difficulty, already noted in Section 3.4, emerges. Constraints must be imposed at the *local* level of either an individual element or simple mesh units, but they should lead to high performance behavior at the *global* level. There is as yet no mathematical framework for establishing those connections.

Several constraint types have been used in this and previous work: (1) invariance, (2) skewness and aspect ratio insensitivity, (3) distortion insensitivity, (4) truncation error minimization, (5) energy balance, (6) energy orthogonality, (7) morphing, (8) mesh direction insensitivity. Whereas (1) and (2) have clear physical significance, the effect of the others has to be studied empirically on benchmark problems. Conditions that have produced satisfactory results are discussed below with reference to the KPT template. The reader should be cautioned, however, that these may not represent the final word inasmuch as templates are presently a frontier subject. For convenience the constraints can be divided into linear and nonlinear, the former being independent of constitutive properties.

§28.4.6. Staged Element Design

Taking an existing KPT element that passes the IET and finding its template signature is relatively straightforward with the help of a computer algebra program. Those listed in Table 28.1 were obtained using *Mathematica*. But in element design we are interested in the reverse process: starting from a general template such as KPT-1-36, to arrive at specific elements that display certain desirable characteristics. Experience shows that this is best done in two stages.

First, linear constraints on the free parameters are applied to generate *element families*. The dependence on the remaining free parameters is still linear.

Second, selected energy balance constraints are imposed. For linear elastic elements such constraints are quadratic in nature. Consequently there is no guarantee that real solutions exist. If they do, solutions typically produce families with few (usually 1 or 2) free parameters; in particular α families. Finally, setting the remaining parameters to specific values produces element instances.

Table 28.3 - Linear Constraints for KPT-1-36 Template

ARI2:	$\beta_{11} = (2\beta_{10} + \beta_{20} + 3\beta_{30} - 4\beta_{40})/3, \quad \beta_{22} = (8\beta_{20} - 4\beta_{30} + 2\beta_{40})/9$
ARI1:	$\beta_{33} = \beta_{20} - \beta_{30} - \beta_{22}, \quad \beta_{92} = 2\beta_{10} + \beta_{20} + 3\beta_{30} - 4\beta_{40} - \beta_{11}$
ARI1:	$\beta_{23} = -2\beta_{20} + \beta_{22}, \quad \beta_{32} = 2\beta_{30} + \beta_{33}, \quad \beta_{41} = -2\beta_{40} - \beta_{33}$
ARI1:	$\beta_{12} = \beta_{22}, \quad \beta_{93} = \beta_{33}, \quad \beta_{13} = -\beta_{33}, \quad \beta_{82} = \beta_{22}, \quad \beta_{43} = -\beta_{33}$
ARI0:	$\beta_{21} = \beta_{31} = \beta_{42} = \beta_{52} = \beta_{63} = \beta_{73} = 0$
ENO1:	$\beta_{32} = \beta_{23} + \beta_{41}, \quad \beta_{92} = 2\beta_{11} - \beta_{12} + \beta_{82}, \quad \beta_{13} = \beta_{93} - \beta_{82}, \quad \beta_{33} = \beta_{22} + \beta_{43}$
ENO0:	$\beta_{40} = -\beta_{20} - \beta_{30}$
OI1:	$\beta_{51} = \beta_{52} + \beta_{43} - \beta_{42}, \quad \beta_{53} = \beta_{52} - \beta_{42} + \beta_{41}, \quad \beta_{61} = \beta_{63} - \beta_{31} + \beta_{33},$
OI1:	$\beta_{62} = \beta_{63} + \beta_{32} - \beta_{31}, \quad \beta_{71} = \beta_{73} - \beta_{21} + \beta_{23}, \quad \beta_{72} = \beta_{73} - \beta_{21} + \beta_{22},$
OI1:	$\beta_{81} = \beta_{82} - \beta_{12} + \beta_{13}, \quad \beta_{83} = \beta_{82} - \beta_{12} + \beta_{11}, \quad \beta_{91} = \beta_{93}$
OI0:	$\beta_{50} = -\beta_{40}, \quad \beta_{80} = -\beta_{10}, \quad \beta_{60} = -\beta_{30}, \quad \beta_{70} = -\beta_{20}, \quad \beta_{90} = 0$

§28.5. Linear Constraints

Three types of linear constraints have been used to generate element families.

§28.5.1. Observer Invariance (OI) Constraints

These pertain to *observer invariance*. If the element geometry exhibits symmetries, those must be reflected in the stiffness equations. For example, if the triangle becomes equilateral or isocles, certain equality conditions between entries of the curvature-displacement matrices must hold. The resulting constraints are *linear* in the β s.

For the KPT-1-36 template one obtains the 14 constraints labeled as OI0 and OI1 in Table 28.3. The five OI0 constraints pertain to the order zero parameters and would be the only ones applicable to the KPT-1-9 template. They can be obtained by considering an equilateral triangle. The nine OI1 constraints link parameters of order one. This constraint set must be the first imposed and applies to any element.

§28.5.2. Aspect Ratio Insensitivity (ARI) Constraints

A second set of constraints can be found by requiring that the element be *aspect ratio insensitive*, or ARI for short, when subjected to arbitrary node displacements. A triangle that violates this requirement becomes infinitely stiff for certain geometries when a certain dimension aspect ratio r goes to infinite.

To express this mathematically, it is sufficient to consider the triangle configurations (A,B,C) depicted in Figure 28.5. In all cases L denotes a triangle dimension kept fixed while the aspect ratio r is increased. In configuration A, the angle ψ is kept fixed as $r \rightarrow \infty$. The opposite angles tend to zero and $\pi/2 - \psi$. The case $\psi = 90^\circ = \pi/2$ is particularly important as discussed later. In configurations (B) and (C) the ratio ξ is kept fixed as $r \rightarrow \infty$, and angles tend to π , $\pi/2$ or zero.

As higher order test displacements we select the four cubic modes $w_{30} = x^3$, $w_{21} = x^2y$, $w_{12} = xy^2$ and $w_{03} = y^3$. Any other cubic mode is a combination of those four. Construct the element energy ratios defined by Equation (28.5). The important dependence of those ratios on physical properties and free parameters is

$$\rho = \rho_b^m(r, \psi, \alpha) + \rho_h^m(r, \psi, \beta_j) \quad (28.9)$$

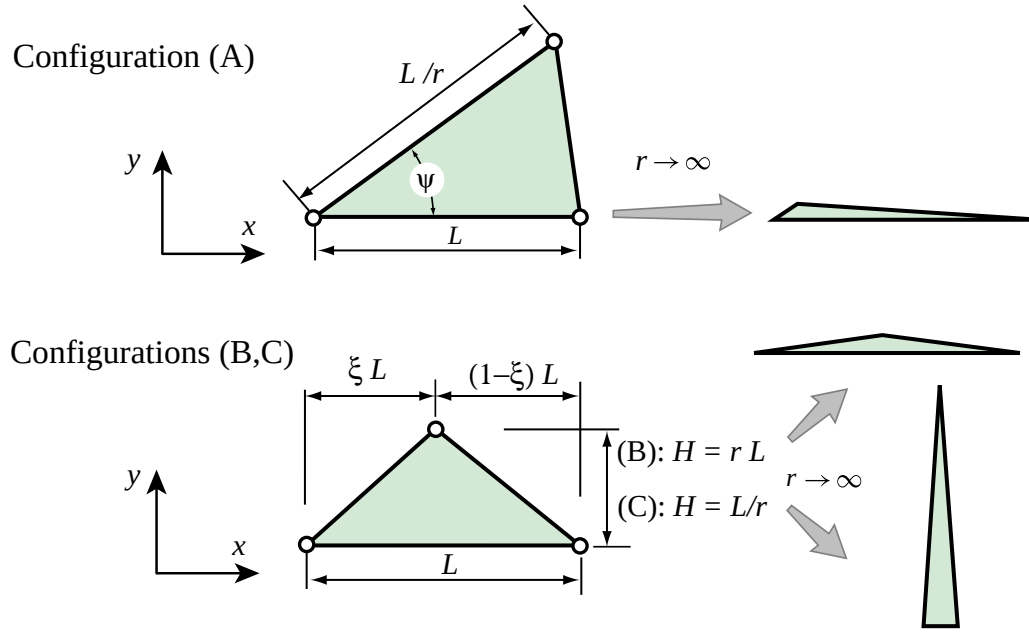


Figure 28.5. Triangle configurations for the study of ARI constraints.

where $m = 30, 21, 12, 03$ identifies modes x^3 , x^2y , xy^2 and y^3 , respectively. (The dependence on L and constitutive matrix $\mathbf{D}$ is innocuous for this study and omitted for simplicity). Take a particular configuration (A,B,C) and mode m , and let $r \rightarrow \infty$. If ρ remains nonzero and bounded the element is said to be *aspect ratio insensitive* for that combination. If $\rho \rightarrow \infty$ the element is said to experience *aspect ratio locking*, whereas if $\rho \rightarrow 0$ the element becomes infinitely flexible. If ρ remains nonzero and bounded for all modes and configurations the element is called *completely aspect ratio insensitive*. The question is whether free parameters can be chosen to attain this goal.

As posed an answer appears difficult because the ratios (28.9) are quadratic in the free parameters, rational in r , and transcendental in ψ . Fortunately the question can be reduced to looking at the dependence of the *curvature-displacement* matrices on r as $r \rightarrow \infty$. Entries of these matrices are linear functions of the free parameters. As $r \rightarrow \infty$ no entry must grow faster than r , because exact curvatures grow as $O(r)$. For example, if an entry grows as r^2 , setting its coefficient to zero provides a linear constraint from which the dependence on L and ψ is factored out. The material properties do not come in. Even with this substantial simplification the use of a CAS is mandatory to handle the elaborate symbolic algebra involved, which involves the Laurent expansion of all curvature matrices. The result of the investigation for the KPT-1-36 template can be summarized as follows.

- (1) The basic energy ratio ρ_b is bounded for all ψ and all m in configuration (A). In configurations (B) and (C) it is unbounded as $O(r)$ in two cases: $m = 30$ if $\alpha \neq 1$, and $m = 21$ for any α .
- (2) The higher order energy ratio ρ_h can be made bounded for all ψ and m by imposing the 18 linear constraints listed in Table 28.3 under the ARI label.
- (3) The foregoing bound on ρ_h is not possible for the KPT-1-9 template. Thus a signature of the form (28.8) is undesirable from a ARI standpoint.

Because of the basic stiffness shortcoming noted in (1), a completely ARI element cannot be constructed. However, the β s can be selected to guarantee that ρ_h is always bounded. The resulting constraints are listed in Table 28.3. They are grouped in three subsets labeled ARI0, ARI1 and ARI2.

Table 28.4. Three Element Families Derivable From KPT-1-36

$\text{ARI}(\alpha, \beta_{10}, \beta_{20}, \beta_{30}, \gamma_0, \gamma_1, \gamma_2)$	α	β_{10} β_{20} β_{30} β_{40} $-\beta_{40}$ $-\beta_{30}$ $-\beta_{20}$ $-\beta_{10}$ 0 β_{11} 0 0 β_{41} $-\beta_{61}$ β_{61} β_{71} $-\beta_{61}$ β_{61} β_{22} β_{22} β_{62} 0 0 β_{62} β_{22} β_{22} β_{92} $-\beta_{61}$ β_{71} β_{61} $-\beta_{61}$ β_{41} 0 0 β_{11} β_{61}
where $\beta_{40} = -\beta_{20} - \beta_{30} + \gamma_0$, $\beta_{11} = (2\beta_{10} + \beta_{20} + 3\beta_{30} - 4\beta_{40})/3 + \gamma_1$, $\beta_{22} = (8\beta_{20} - 4\beta_{30} + 2\beta_{40})/9 + \gamma_2$, $\beta_{61} = \beta_{20} - \beta_{22} - \beta_{30}$, $\beta_{41} = -\beta_{61} - 2\beta_{40}$, $\beta_{62} = \beta_{20} - \beta_{22} + \beta_{30}$, $\beta_{71} = -2\beta_{20} + \beta_{22}$ and $\beta_{92} = 2\beta_{10} - \beta_{11} + \beta_{20} + 3\beta_{30} - 4\beta_{40}$.		
$\text{ENO}(\alpha, \beta_{10}, \beta_{20}, \beta_{30}, \beta_{11}, \beta_{41}, \beta_{12}, \beta_{22}, \beta_{82}, \beta_{23}, \beta_{43}, \beta_{93})$	α	β_{10} β_{20} β_{30} β_{40} $-\beta_{40}$ $-\beta_{30}$ $-\beta_{20}$ $-\beta_{10}$ 0 β_{11} 0 0 β_{41} β_{43} β_{33} β_{23} β_{81} β_{93} β_{12} β_{22} β_{32} 0 0 β_{32} β_{22} β_{82} β_{92} β_{13} β_{23} β_{33} β_{43} β_{41} 0 0 β_{83} β_{93}
where $\beta_{40} = -\beta_{20} - \beta_{30}$, $\beta_{81} = -\beta_{12} + \beta_{93}$, $\beta_{32} = \beta_{23} + \beta_{41}$, $\beta_{92} = 2\beta_{11} - \beta_{12} + \beta_{82}$, $\beta_{13} = -\beta_{82} + \beta_{93}$, $\beta_{33} = \beta_{22} + \beta_{43}$ and $\beta_{83} = \beta_{11} - \beta_{12} + \beta_{82}$.		
$\text{ARIENO}(\alpha, \beta_{10}, \beta_{20}, \beta_{30})$		$\text{ARI}(\alpha, \beta_{10}, \beta_{20}, \beta_{30}, 0, 0, 0)$

Subset ARI0 requires $\beta_{21} = \beta_{31} = \beta_{42} = \beta_{52} = \beta_{63} = \beta_{73} = 0$. If any of these 6 parameters is nonzero, one or more entries of the deviatoric curvature displacement matrices grow as r^3 in the 3 configurations. This represents disastrous aspect ratio locking and renders any such element useless.

Once OI0, OI1 and ARI0 are imposed, subset ARI1 groups 10 constraints obtained by setting to zero $O(r^2)$ grow in configuration (A) for all HO modes. This insures that $\rho = \rho_b + \rho_h$ stays bounded in that configuration (because ρ_b stays bounded for any α). In Table 28.3 they are listed in reverse order of that found by the symbolic analysis program, which means that the most important ones appear at the end.

Once OI0, OI1 and ARI0 and ARI1 are imposed, the 2-constraint subset ARI2 guarantees that ρ_h is bounded for configurations (B) and (C). Since ρ_b is not necessarily bounded in this case, these conditions have minor practical importance.

Imposing OI0, OI1, ARI0 and ARI1 leaves $37 - 5 - 9 - 6 - 10 = 7$ free parameters, and produces the so-called ARI family defined in Table 28.4. Of the seven parameters, four are chosen to be the actual template parameters α , β_{10} , β_{20} , and β_{30} . Three auxiliary parameters, called γ_1 , γ_2 and γ_3 , are chosen to complete the generation of the ARI family as defined in Table 28.4.

This family is interesting in that it includes all existing high-performance elements, such as DKT and AQR1, as well as some new ones developed in the course of this study.

§28.5.3. Energy Orthogonality (ENO) Constraints

Energy orthogonality (ENO) means that the average value of deviatoric strains over the element is zero. Mathematically, $\mathbf{B}_{\chi 4} + \mathbf{B}_{\chi 5} + \mathbf{B}_{\chi 6} = \mathbf{0}$. This condition was a key part of the early developments of the Free Formulation by Bergan [12] and Bergan and Nygård [13], as well as of HP elements developed during the late 1980s and early 1990s.

The heuristic rationale behind ENO is to limit or preclude energy coupling between constant strain and higher order modes. A similar idea lurks behind the so-called “B-bar” formulation, which has a long and checkered history in modeling incompressibility and plastic flow.

For the KPT-1-36 element one may start with the OI0 and OI1 constraints as well as ARI0 (to preclude catastrophic locking), but ignoring ARI1 and ARI2. Then the ENO condition leads to the linear constraints labeled as ENO0 and ENO1 in Table 28.3. These group conditions on the order zero and one parameters, respectively. Imposing OI0, OI1, ARI0, ENO0 and ENO1 leads to the ENO family defined in Table 28.4, which has 12 free parameters. Elements ALR, FF0 and FF1 of Table 28.1 can be presented as instance of this family as shown later in a “genealogy” Table.

If instead one starts from the ARI family it can be verified that ENO is obtained if $\gamma_0 = \gamma_1 = \gamma_2 = 0$; that is, only three constraints are needed instead of five. [This is precisely the rationale for selecting those “ENO deviations” as free arguments]. Moreover, the order-one constraints $\gamma_1 = \gamma_2 = 0$ are precisely ARI2 in disguise.

Setting $\gamma_0 = \gamma_1 = \gamma_2 = 0$ in ARI yields a four-parameter family called ARIENO (pronounced like the French “Arienne”). The free parameters are α , β_{10} , β_{20} and β_{30} . Its template is defined in Table 28.4. As noted, this family incorporates all constraints listed in Table 28.3. (These add up to 37 relations, but there are 4 redundancies.)

As noted all HP elements found to date are ARI (ALR, FF0 and FF1 are not considered HP elements now). Most are ENO, but there are some that are not. One example is HCTS, a “smoothed HCT” element developed in this study. Hence circumstantial evidence suggests that ARI is more important than ENO. In the present investigation ENO is used as a guiding principle rather than an *a priori* constraint.

§28.6. Quadratic Constraints

As noted in the foregoing section, the ARI family — and its ARIENO subset — is a promising source of high performance KPT elements. But seven parameters remain to be set to meet additional conditions. These fall into the general class of *energy balance* constraints introduced in [18]. Unit energy ratios are imposed for specific mesh unit geometries, loadings and boundary conditions. Common feature of such constraints, for linear elements, are: they involve constitutive properties, and they are *quadratic* in the free parameters. Consequently real parameter solutions are not guaranteed. Even if they have real solutions, the resulting families may exist only for limited parameter ranges.

Numerous variants of the energy balance tests have been developed over the years. Because of space constraints only three variations under study are described below. All of them have immediate physical interpretation in terms of the design of custom elements. They have been applied assuming isotropic material with zero Poisson’s ratio.

§28.6.1. Morphing Constraints

This is a class of constraints that is presently being studied to ascertain whether enforcement would be generally beneficial to element performance. Consider the two-KPT-element rectangular mesh unit shown in Figure 28.6. The aspect ratio r is the ratio of the longest rectangle dimension L to the width $b = L/r$. The plate is fabricated of a homogeneous isotropic material with zero Poisson’s ratio and thickness t . Axis x is selected along the longitudinal direction. We study the two *morphing processes* depicted at the top of Figure 28.6. In both the aspect ratio r is made to increase, but with different objectives.

Plane Beam Limit. The width $b = L/r$ is decreased while keeping L and t fixed. The limit is the thin, Bernoulli-Euler plane beam member of rectangular cross section $t \times b$, $b \ll t$, shown at the end of the beam-morphing arrow in Figure 28.6. This member can carry exactly a linearly-varying bending moment $M(x)$ and a constant transverse shear V , although shear deformations are not considered.

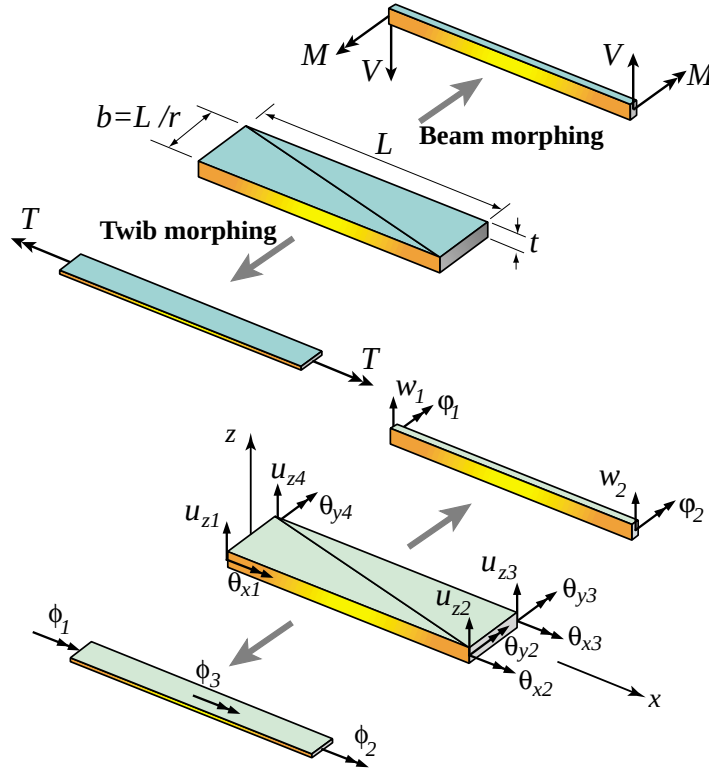


Figure 28.6. Morphing a rectangular plate mesh unit to beam and twib.

Twib Limit. Again the width $b = L/r$ is decreased by making r grow. The thickness t , however, is still considered small with respect to b . The limit is the twisted-ribbon member of cross section $t \times b$, $t \ll b$, shown on the left of Figure 28.6. This member, called a “twib” for brevity, can carry a longitudinal torque $T(x)$ that varies linearly in x .

Conditions called *morphing constraints* are now posed as follows.

- (1) Does the mesh unit approach the exact behavior of a Hermitian cubic beam as $r \rightarrow \infty$? If so, the plate element is said to be *beam morphing exact* or BE for short.
- (2) Does the mesh unit approach the exact beam behavior as both $r \rightarrow \infty$ and $r \rightarrow 0$? If so, the plate element is said to be *double beam morphing exact* or DBE for short.
- (3) Does the mesh unit approach the exact behavior of a twib under linearly varying torque? If so, the plate element is said to be *twib-morphing-exact* or TE.

To check these conditions the 12 plate-mesh-unit freedoms are congruentially transformed to the beam and twib freedoms shown at the bottom of Figure 28.6. The BE and TE conditions can be derived by symbolically expanding these transformed stiffness equations in Laurent series as $r \rightarrow \infty$. The DBE condition requires also a Taylor expansion as $r \rightarrow 0$. They can also be derived as asymptotic forms of energy ratios.

An element satisfying (1) and (3) is called BTE, and one satisfying (1), (2) and (3) is called DBTE. The practical interest for morphing conditions is the appearance of very high aspect ratios in modeling certain aerospace structures, such as the stiffened panel depicted in Figure 28.7.

§28.6.2. Mesh Direction Insensitivity Constraints

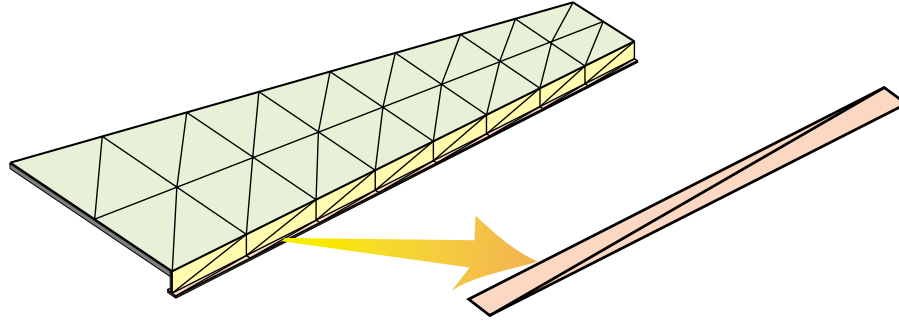


Figure 28.7. Stiffened panels modeled by facet shell elements are a common source of high aspect ratio elements.

Table 28.5. Template Signatures of Selected New KPT Elements

Acronym	α	β_{1j}	β_{2j}	β_{3j}	β_{4j}	β_{5j}	β_{6j}	β_{7j}	β_{8j}	β_{9j}	β_{sc}
BTE13	1/3	-9	2	-1	-1	1	1	-2	9	0	/6
		-5	0	0	1	-1	1	-2	-1	1	
		2	2	-1	0	0	-1	2	2	-10	
		-1	-2	1	-1	1	0	0	-5	1	
DBE00		ARIENO(0, -3/2, β_{20} , β_{30}), with $\beta_{20} = 3(\sqrt{6} - 2)/4$ and $\beta_{30} = -\beta_{20}$.									
DBE13		ARIENO(1/3, -3/2, β_{20} , β_{30}) with $\beta_{20} = -3/2 - (\sqrt{25 - \sqrt{609}})/12 + (25 + \sqrt{609})/4$ and $\beta_{30} = 3/2 - (\sqrt{25 - \sqrt{609}})/12 - (25 + \sqrt{609})/4$.									
DBE12		ARIENO(1/2, -3/2, β_{20} , β_{30}) with $\beta_{20} = (-12 - \sqrt{10 - 2\sqrt{21}} + 3\sqrt{10 + 2\sqrt{21}})/8$ and $\beta_{30} = (12 - \sqrt{10 - 2\sqrt{21}} - 3\sqrt{10 + 2\sqrt{21}})/8$.									
DBEN00	0	-3	0	0	0	0	0	0	3	0	/2
		2	0	0	0	0	0	0	0	0	
		0	0	0	0	0	0	0	0	-8	
		0	0	0	0	0	0	0	2	0	
DBEN13	1/3	-27	-1	-1	2	-2	1	1	27	0	/18
		$4\sqrt{61} - 22$	0	0	-4	0	0	2	0	0	
		0	0	-2	0	0	-2	0	0	$-4(\sqrt{61} + 11)$	
		0	2	0	0	-4	0	0	$4\sqrt{61} - 22$	0	
DBEN12	1/2	-12	-1	-1	2	-2	1	1	12	0	/8
		$4(\sqrt{6} - 3)$	0	0	-4	0	0	2	0	0	
		0	0	-2	0	0	-2	0	0	$-4(\sqrt{6} + 6)$	
		0	2	0	0	-4	0	0	$4(\sqrt{6} - 3)$	0	
DBTE13		ARIENO(1/3, -1.3100926, 0.40467862, -0.49926464)									
HCTS	1	-18	5	5	-1	1	-5	-5	18	0	/12
		8	0	0	2	0	0	-10	0	0	
		0	0	10	0	0	10	0	0	-20	
		0	-10	0	0	2	0	0	8	0	
MDIT1	1	-45	6	-6	0	0	6	-6	45	0	/30
		-34	0	0	-4	-4	4	-4	-4	4	
		8	8	-8	0	0	-8	8	8	-68	
		-4	-4	4	-4	-4	0	0	-34	4	

Table 28.6. Brief Description of New Elements Listed in Table 28.5

Name	Description
BTE13	Instance of the BTE(α) family for $\alpha = 1/3$. This is a subset of ARIENO which exhibits beam morphing and twib morphing exactness in the sense discussed in Section 6.1.
DBE00,DBE13, DBE12	Instances of the DBE(α) family for $\alpha = 0$, $\alpha = 1/3$ and $\alpha = 1/2$, respectively. DBE(α) is a subset of ARIENO which exhibits double-beam morphing exactness. The family exists (in the sense of having real solutions) for $\alpha \leq 1/\sqrt{2}$.
DBEN00,DBEN13, DBEN12	Instances of the DBEN(α) family for $\alpha = 0$, $\alpha = 1/3$ and $\alpha = 1/2$, respectively. Family exhibits double-beam morphing exactness but is not ENO. Exists for $\alpha \leq 1/\sqrt{2}$. DBEN00 coalesces with ALR.
DBTE13	An ARIENO instance with $\alpha = 1/3$ which exhibits double-beam and twib morphing exactness. It was found by minimizing a residual function. Only numerical values for the parameters are known. Such elements seem to exist only for a small α range.
HCTS	Derived from HCT, which is a too-stiff poor performer, by smoothing its curvature field. It displays very high performance if beam or twib exactness is not an issue. Member of ARI but not ARIENO.
MDIT1	Instance of the MDIT(α) family for $\alpha = 1$. MDI stands for Mesh-Direction Insensitive. Such elements display stiffness isotropy when assembled in a square mesh unit. Primarily of interest in high frequency dynamics as discussed in Section 7.2. Static performance also good to excellent for rectangular mesh units.

Table 28.7. Genealogy of Specific KPT Elements

Name	Source Family
ALR	ARI(0, $-3/2$, 0, 0, 0, -2 , 0)
AQR1	ARIENO(1, $-3/2$, 0, 0)
AQR0	ARIENO(0, $-3/2$, 0, 0)
AQRBE	ARIENO($1/\sqrt{2}$, $-3/2$, 0, 0)
AVG	ENO(0, $-3/2$, 0, 0, 0, 0, 0, 0, 0, 0, 0)
BCIZ0 (and clones), BCIZ1, HCT	Not ARI or ENO
BTE13	ARIENO($1/3$, $-3/2$, $1/3$, $-1/6$)
DBE00, DBE13, DBE12	See Table 28.5
DBEN00	ARI(0, $-3/2$, 0, 0, 0, 2, 0)
DBEN13	ARI(0, $-3/2$, $-1/18$, $-1/18$, 0, $2\sqrt{61}/9$, 0)
DBEN12	ARI(0, $-3/2$, $-1/8$, $-1/8$, 0, $\sqrt{3/2}$, 0)
DKT	ARIENO(1, $-3/2$, $1/4$, $1/4$)
FF0 (and clones)	ENO(0, $-3/2$, $1/6$, $1/6$, 0, 0, 0, 0, 0, 0, 0)
FF1	ENO(1, $-3/2$, $1/6$, $1/6$, 0, 0, 0, 0, 0, 0, 0)
HCTS	ARI(1, $-3/2$, $5/12$, $5/12$, $3/4$, 1, $-1/6$)
MDIT1	ARIENO(1, $-3/2$, $1/5$, $-1/5$)

This class of constraints is important in high frequency and wave propagation dynamics to minimize mesh induced anisotropy and dispersion. One important application of this kind of analysis is simulation of nondestructive testing of microelectronic devices and thin films by acoustic imaging.

Consider a square mesh unit fabricated of 4 overlapped triangles to try to minimize directionality from the start. Place Cartesian axes x and y at the mesh unit center. Apply higher order modes x_φ^3 and $x_\varphi^2 y_\varphi$, where x_φ forms an angle φ from x , and require that $\rho = 1$ for all φ . Starting from ARIENO one can construct α families which satisfy that constraint. The simplest one is the MDIT(α) family, which for $\alpha = 1$ yields the element MDIT1 defined in Table 28.5.

The correlation of this condition to flexural wave propagation and its extension to more general element lattices is under study.

§28.6.3. Distortion Minimization Constraints

An element is distortion insensitive when the solution hardly changes even when the mesh is significantly changed. The precise quantification of this definition in terms of energy ratios on standard benchmarks tests is presently under study. Preliminary conclusions suggest that sensitivity to distortion is primarily controlled by the basic stiffness parameter α whereas the higher order parameters β s only play a secondary role. For the KPT-1-36 template, $\alpha = 1$ appears to minimize the distortion sensitivity in ARIENO elements.

§28.7. New KPT Elements

Starting from the ARI family and applying various energy constraints, a set of new elements with various custom properties have been developed in this study. The most promising ones are listed in Tables 28.5 and chapdot6. Table 28.7 gives the genealogy — in the sense of “element family membership” — of all elements listed in Tables 28.1 and 28.6.

The performance of the old and new elements in a comprehensive set of plate bending benchmarks is still in progress. Preliminary results are reported in the next Section. It should be noted that the unification brought about by the template approach is facilitated for such comparisons, because all possible elements of a given type and node/freedom configuration can be implemented with a single program module. Furthermore the error-prone process of codifying somebody else’s published element is reduced as long as the signature is known.

§28.8. Benchmark Studies

Only a limited number of benchmark studies have been conducted to compare the new BTME elements with existing ones. One unfinished ingredient is the formulation of consistent node forces for distributed loads. These are difficult to construct because no transverse displacement shape functions are generally available, and the topic (as well as the formulation of consistent mass and geometric stiffness) will require further research. In all tests reported below the material is assumed isotropic with zero Poisson’s ratio.

§28.8.1. Simply Supported and Clamped Square Plates

The first test series involved the classical problem of a square plate with simply supported and clamped boundary conditions, subjected to either a uniform distributed load, or a concentrated central load. Only results for the latter are shown here. Meshes of $N \times N$ elements over one quadrant were used, with $N = 1, 2, 4, 8$ and 16. Mesh units are formed with four overlapped half-thickness elements to eliminate diagonal directionality effects.

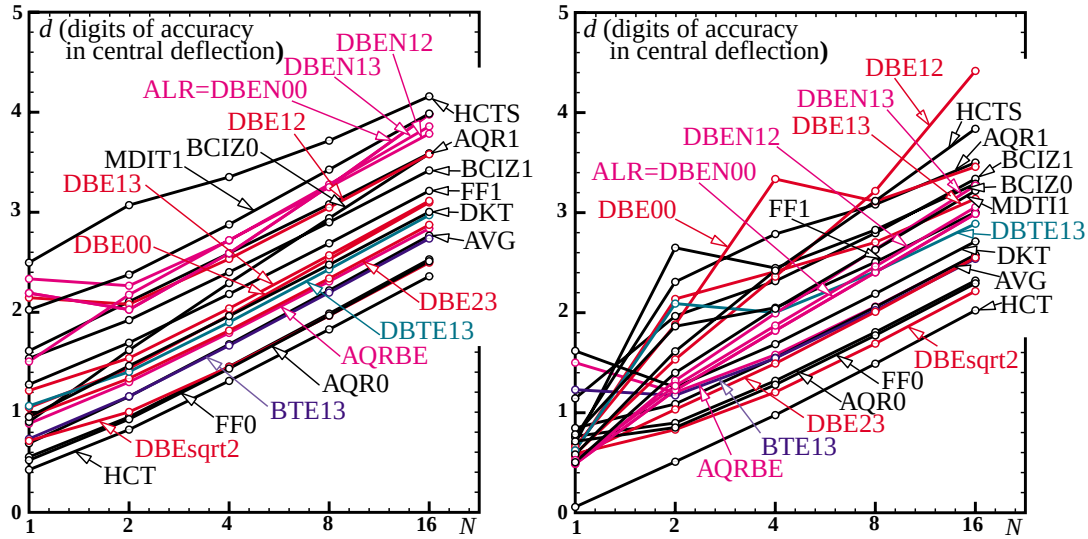


Figure 28.8. Convergence study of SS and clamped square plates under concentrated central load.

The accuracy of the computed central deflection w_C is reported as $d = -\log_{10} \epsilon_{rel}$ where ϵ_{rel} is the relative error $\epsilon_{rel} = |(w_C - w_{Cex})/w_{Cex}|$, w_{Cex} being the exact value that is known to at least five places; see [19]. Using d has the advantage of giving at a glance the number of correct digits. A shortcoming of d is that it ignores deflection error signs; this occasionally induces spurious “accuracy spikes” if computed results cross over the exact value before coming back. For this load case d also measures the energy accuracy.

Figure 28.8 displays log-log error plots in which d is shown as a function of $\log_2 N$. The ultimate slope of the log-log curve characterizes the asymptotic energy convergence rate. The plots indicate an asymptotic rate of $O(1/N^2)$ for all elements, meaning that ultimately a doubling of N gives $\log_{10} 4 \approx 0.6$ more digits. The same $O(h^2)$ convergence rate was obtained for energy, displacements, centroidal curvatures and centroidal moments, although only displacements are reported here.

Although the asymptotic convergence rate is the same for all elements in the template, big differences as regards error coefficients can be observed. These differences are visually dampened by the log scale. In both problems the difference between the worst element (HCT) and the best ones (HCTS, MDIT1, AQR1) spans roughly two orders of magnitude. The performance of the new BE and DBE elements depends significantly on the α parameter, with DBEN12, DBEN13 and DBE12 generally outperforming the others. The sanitized BCZ elements do well. DKT, FF0 and FF1 come in the middle of the pack. The results for distributed loads, not reported here, as well 2:1 rectangular plates largely corroborate these rankings, except for deteriorating performance of the BCZ elements.

§28.8.2. Uniformly Loaded Cantilever

Figure 28.9 reports the convergence study of a narrow cantilever plate subjected to uniform lateral load q . The aspect ratio is $r = 20$. Regular $N \times N$ meshes are used, with $N = 1, 2, 4$ and 8 . The consistent transformation of q to node forces has not been investigated. The analysis uses the consistent loads of FF0, for which the transverse displacement over the triangle is available. This shortcut is not believed to have a major effect on convergence once the mesh is sufficiently fine.

As expected the beam-exact elements do much better than the others, with an accuracy advantage of 3–4 orders of magnitude for $N \geq 4$. (The different superconvergence slopes have not been explained yet in

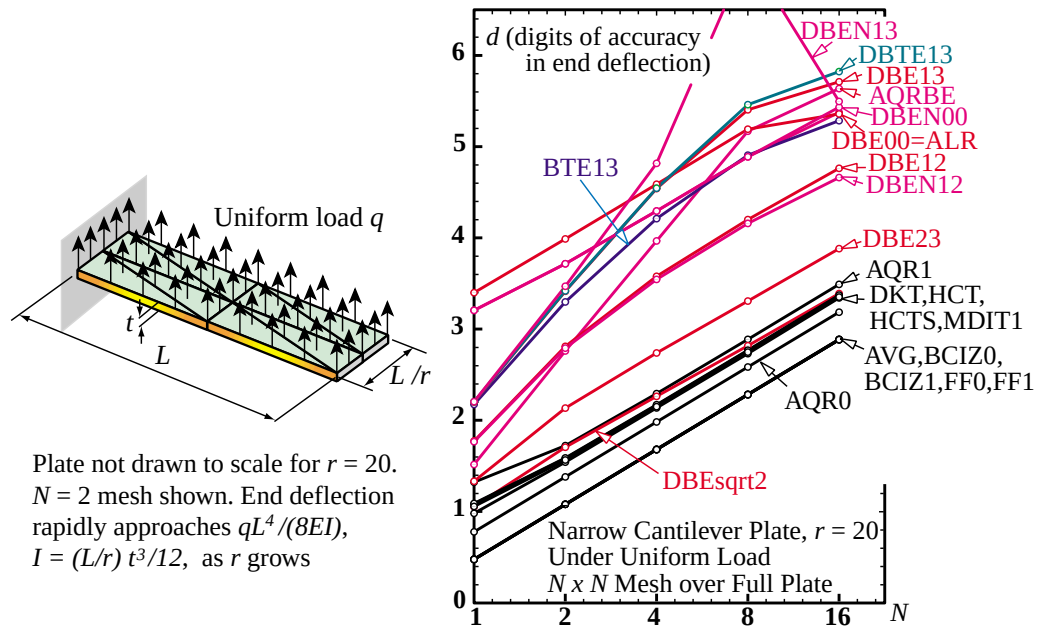


Figure 28.9. Convergence study on uniform loaded, narrow cantilever plate.

term of asymptotic expansions, pending a study of consistent load formulations.) The accuracy “spike” of DBE13 is a fluke caused by computed end deflections crossing over the exact answer before returning to it.

§28.8.3. Aspect Ratio Test of End Loaded Cantilever

Aspect ratio tests use a single mesh unit of length L and width $b = L/r$ subjected to simple end load systems. Only r is varied. Two kinds of tests, depicted in Figures 28.10 and 28.11, were carried out.

The analysis of the shear-end-loaded cantilever mesh unit, reported in Figure 28.10, simply confirms the asymptotic analysis of the beam morphing process. As expected, the BE and DBE elements designed to beam-morph exactly perform well, converging to the correct thin-cantilever answer $PL^3/(3EI)$ as $r \rightarrow \infty$. All other elements stay at the same deflection percentage beyond $r > 8$. For example, the five elements AVG, BCIZ0, BCIZ1, FF0 and FF1 yield 2/3 of the correct answer whereas DKT, MDIT1 yields 3/4 of it.

The end load system must include two fixed-end moments $\pm \alpha PLr/12$ to deliver the correct answers predicted by the $r \rightarrow \infty$ asymptotic analysis. Should the moments be removed, significant errors occur if $\alpha \neq 0$, and BME mesh units no longer converge to the correct answer as $r \rightarrow \infty$. Of course, in the case of a repeated mesh subdivision the effect of those moments would become progressively smaller.

§28.8.4. Aspect Ratio Test of Twisted Ribbon

Aspect ratio tests were also conducted on a twib subjected to a total applied end torque T using a single mesh unit fabricated of 4 overlaid triangles to avoid directional anisotropy. The consistent nodal force system shown in Figure 28.11 was used. Two boundary condition cases: (I) and (II), also defined in Figure 28.11, were considered.

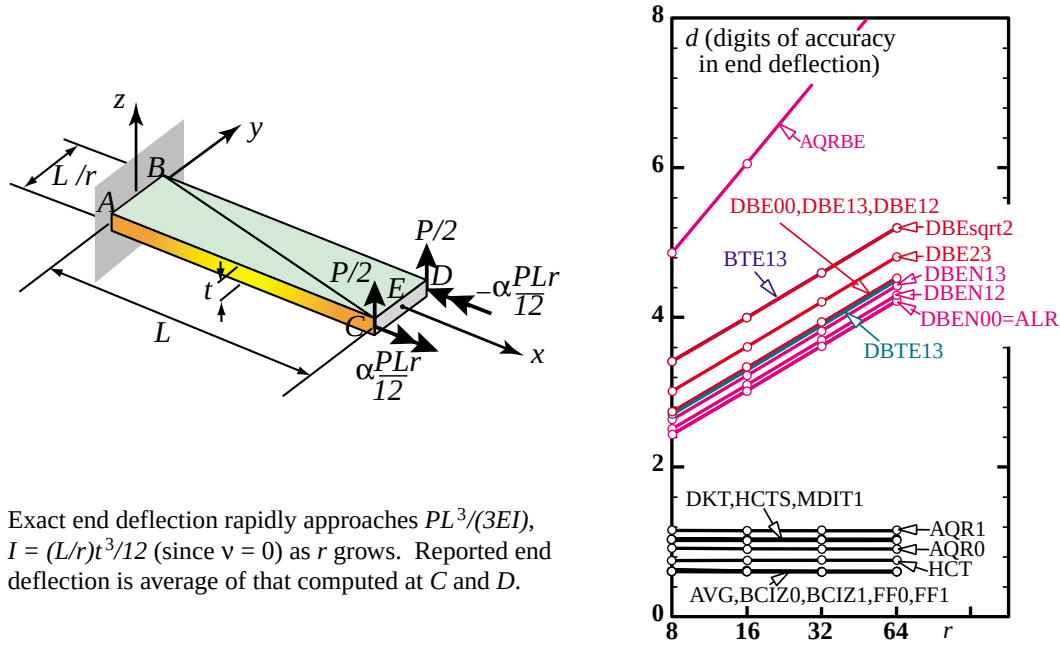


Figure 28.10. Aspect ratio study of end-loaded cantilever mesh unit.

Case (I) is essentially a constant twist-moment patch test. All elements in the KPT-1-36 template must pass the test, and indeed all elements gave the correct answer for any r . The total angle of twist must be $\phi_E = TL/GJ$, where $G = \frac{1}{2}E$ (because $\nu = 0$) and $J = \frac{1}{3}(L/r)t^3$. With the boundary conditions set as shown, the x rotations at both end nodes must be ϕ_E , the transverse deflections $\pm\phi_E(L/2r)$ and the y rotations $\pm\phi_E/(2r)$. Note, however, that if the applied load system shown in Figure 28.11 is not adhered to, the patch test is not passed. For example, the end moments cannot be left out unless $\alpha = 1$. This consistency result is believed to be new.

Case (II) is analogous to Robinson's twisted ribbon test [34], which is usually conducted with specific material and geometric properties. Working with symbolic algebraic computations there is no need to chose arbitrary numerical values. The end torque T was simply adjusted so that the pure twist solution gives $\phi_E = r$. The graph shown on the right of Figure 28.11 plots ϕ_E as function of r for specific elements in the range $r = 5$ to $r = 40$. Because the sign of the discrepancy is important, this plot does not use a log-log reduction. It is seen that most elements are too stiff, and that several of them, notably HCT, BCIZ and FF, exhibit severe aspect ratio locking. For the twib-exact (TE) elements, a symbolic analysis predicted that exact agreement with the pure twist solution for this boundary condition case can only be obtained if $\alpha = 1/3$. The computed response of BTE13 and DBTE13 verifies this prediction.

§28.8.5. The Score so Far

The benchmarks conducted so far, of which those reported in Section 8 constitute a small sample, suggest that the choice of an optimal "element for all seasons" will not be realized. For the classical rectangular plate benchmarks HCTS, MDTI1 and AQR1 consistently came at the top. The displacement-assumed HCT, BCZ and FF elements, which represent 1970-80 technology, did well for unit aspect ratios but deteriorated for narrower triangles. The new BE and DBE elements showed variable performance. The performance of DKT was mediocre to average. Skew plate tests, not reported here, were inconclusive.

For problems typified by uniaxial bending of long cantilevers, the BE and DBE elements outperformed all others as can be expected by construction. The displacement assumed elements did poorly. Distortion

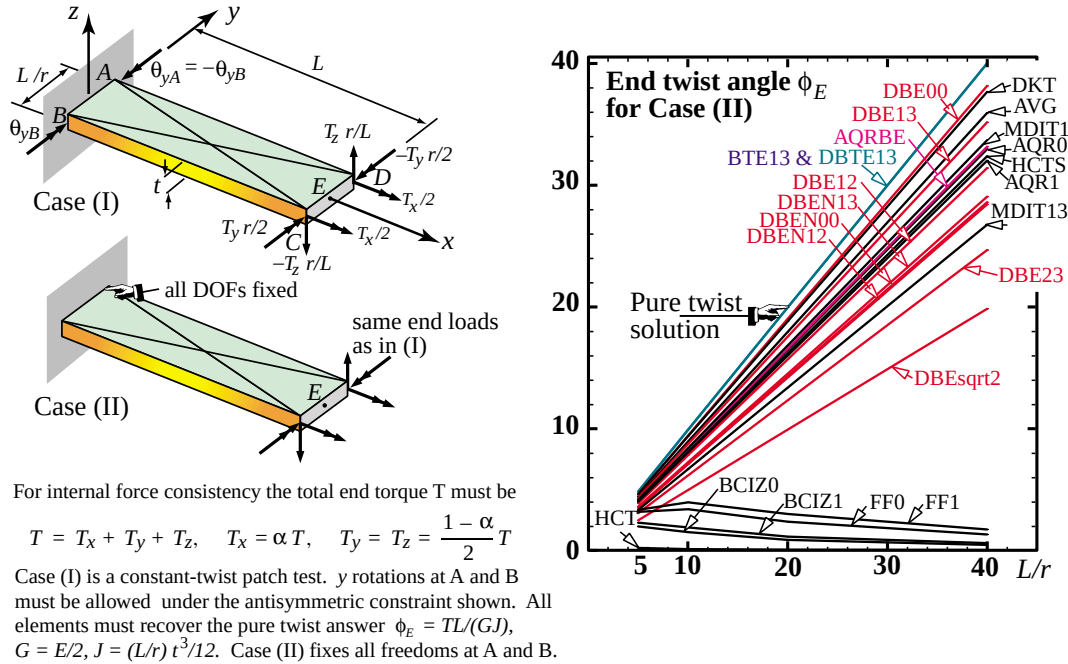


Figure 28.11. Twisted ribbon test.

benchmarks, not reported here, indicate that ARI elements with $\alpha = 1$ outperform elements with $\alpha \neq 1$, or non-ARI elements such as FF1 or HCT.

Based on the test set, MDIT1 and HCTS appear to the running neck and neck as the best all-purpose KPT elements, closely followed by AQR1. There are arguments for DBE13 as good choice for very high aspect ratio situations such as the stiffened panels illustrated in Figure 28.7. Unfortunately directly joining elements with different α s would violate the IET. Thus a mesh with, say, DBE13 in high aspect ratio regions and HCTS otherwise would be disallowed. Such combinations could be legalized by interface transition elements but that would represent a new research topic.

§28.9. Concluding Remarks

The usual finite element construction process, which involves *a priori* selection of a variational principle and shape functions, hinders the exploration of a wide range of admissible finite element models. As such it is ineffectual for the *design* of finite elements with desirable physical behavior. The application described here illustrates the effectiveness of templates to build custom elements. The template approach attempts to implement the hope long ago expressed by Bergan and Hanssen [24] in the Introduction of their MAFELAP II paper:

“An important observation is that each element is, in fact, only represented by the numbers in its stiffness matrix during the analysis of the assembled system. The origin of these stiffness coefficients is unimportant to this part of the solution process ... The present approach is in a sense the opposite of that normally used in that the starting point is a generally formulated convergence condition and from there the stiffness matrix is derived ... The patch test is particularly attractive [as such a condition] for the present investigation in that it is a direct test on the element stiffness matrix and requires no prior knowledge of interpolation functions, variational principles, etc.”

This statement sets out what may be called the *direct algebraic approach* to finite elements: the element stiffness is derived directly from consistency conditions — provided by the IET — plus stability and

accuracy considerations to determine algebraic redundancies if any. It has in fact many points in common with energy-based finite differences.

This ambitious goal has proven elusive because the direct algebraic construction of the stiffness matrix of most multidimensional elements becomes effectively a problem in constrained optimization. In the *symbolic form* necessitated by element design, such problem is much harder to tackle than the conventional element construction methods based on shape functions. Only with the advent and general availability of powerful computer algebra systems can the dream become a reality.

Acknowledgements

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Appendix A. Formulation of KPT-1-36 Template

This Appendix collects the formulas that fully define the KPT-1-36 element template.

A.1 Element Relations

The triangle geometry is defined by the corner coordinates in its (x, y) local system, which are (x_i, y_i) , $i = 1, 2, 3$. Coordinate differences are abbreviated as $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$. The signed triangle area A is given by $2A = x_{21}y_{31} - x_{31}y_{21} = x_{32}y_{12} - x_{12}y_{32} = x_{13}y_{23} - x_{23}y_{13}$ and we require that $A > 0$. The visible degrees of freedom of the element collected in $\mathbf{u}$ and the associated node forces collected in $\mathbf{f}$ are

$$\mathbf{u}^T = [u_{z1} \ \theta_{x1} \ \theta_{y1} \ u_{z2} \ \theta_{x2} \ \theta_{y2} \ u_{z3} \ \theta_{x3} \ \theta_{y3}]. \quad (28.10)$$

$$\mathbf{f}^T = [f_{z1} \ \mathcal{M}_{x1} \ \mathcal{M}_{y1} \ f_{z2} \ \mathcal{M}_{x2} \ \mathcal{M}_{y2} \ f_{z3} \ \mathcal{M}_{x3} \ \mathcal{M}_{y3}]. \quad (28.11)$$

The Cartesian components of the plate curvatures are κ_{xx} , κ_{yy} and $2\kappa_{xy} = \kappa_{xy} + \kappa_{yx}$, which are gathered in a 3-vector $\boldsymbol{\kappa}$. In the Kirchhoff model, curvatures and displacements are linked by

$$\kappa_{xx} = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_{yy} = \frac{\partial^2 w}{\partial y^2}, \quad 2\kappa_{xy} = 2 \frac{\partial^2 w}{\partial x \partial y}. \quad (28.12)$$

where $w = w(x, y) \equiv u_z$ is the plate transverse displacement. In the KPT elements considered here, however, the compatibility equations (28.12) must be understood in a weak sense because the assumed curvature field is not usually integrable. The internal moment field is defined by the Cartesian components m_{xx} , m_{yy} and m_{xy} , which are placed in a 3-vector $\mathbf{m}$. Curvatures and moments are linked by the constitutive relation

$$\mathbf{m} = \begin{bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix} = \mathbf{D}\boldsymbol{\kappa}. \quad (28.13)$$

where $\mathbf{D}$ results from integration through the thickness in the usual way. Three dimensionless side direction coordinates π_{21} , π_{32} are π_{13} are defined as going from 0 to 1 by marching along sides 12, 23 and 31, respectively. The side coordinate π_{ji} of a point not on a side is that of its projection on side ij . The second derivatives of $w \equiv u_z$ with respect to the dimensionless side directions will be called the *natural curvatures* and are denoted by $\chi_{ji} = \partial^2 w / \partial \pi_{ji}^2$. These curvatures have dimensions of displacement. They are related to the Cartesian plate curvatures by the matrix relation

$$\boldsymbol{\chi} = \begin{bmatrix} \chi_{21} \\ \chi_{32} \\ \chi_{13} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 w}{\partial \pi_{21}^2} \\ \frac{\partial^2 w}{\partial \pi_{32}^2} \\ \frac{\partial^2 w}{\partial \pi_{13}^2} \end{bmatrix} = \begin{bmatrix} x_{21}^2 & y_{21}^2 & x_{21}y_{21} \\ x_{32}^2 & y_{32}^2 & x_{32}y_{32} \\ x_{13}^2 & y_{13}^2 & x_{13}y_{13} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = \mathbf{T}^{-1}\boldsymbol{\kappa}, \quad (28.14)$$

the inverse of which is

$$\boldsymbol{\kappa} = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{23}y_{13} & y_{31}y_{21} & y_{12}y_{32} \\ x_{23}x_{13} & x_{31}x_{21} & x_{12}x_{32} \\ y_{23}x_{31} + x_{32}y_{13} & y_{31}x_{12} + x_{13}y_{21} & y_{12}x_{23} + x_{21}y_{32} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial \pi_{21}^2} \\ \frac{\partial^2 w}{\partial \pi_{32}^2} \\ \frac{\partial^2 w}{\partial \pi_{13}^2} \end{bmatrix} = \mathbf{T}\boldsymbol{\chi}. \quad (28.15)$$

The transformation equations (28.14) and (28.15) are assumed to hold even if $w(x, y)$ is only known in a weak sense.

A.2 The Basic Stiffness Template

Following Militello and Felippa [20] the α -parametrized basic stiffness is defined as the linear combination

$$\mathbf{K}_b = A^{-1} \mathbf{L} \mathbf{D} \mathbf{L}^T, \quad \mathbf{L} = (1 - \alpha) \mathbf{L}_l + \alpha \mathbf{L}_q \quad (28.16)$$

where $\mathbf{L}$ is a force-lumping matrix that maps an internal constant moment field to node forces. $\mathbf{L}_l$ and $\mathbf{L}_q$ are called the linear and quadratic versions, respectively, of $\mathbf{L}$:

$$\mathbf{L}_l^T = \begin{bmatrix} 0 & 0 & y_{32} & 0 & 0 & y_{13} & 0 & 0 & y_{21} \\ 0 & x_{32} & 0 & 0 & x_{13} & 0 & 0 & x_{21} & 0 \\ 0 & y_{23} & x_{23} & 0 & y_{31} & x_{31} & 0 & y_{12} & x_{12} \end{bmatrix} \quad (28.17)$$

$$\mathbf{L}_q = \begin{bmatrix} c_{n21}s_{n21} - c_{n13}s_{n13} & -c_{n21}s_{n21} + c_{n13}s_{n13} & -(c_{n21}^2 - s_{n21}^2) + (c_{n13}^2 - s_{n13}^2) \\ \frac{1}{2}(c_{n21}^2 x_{12} + c_{n13}^2 x_{31}) & \frac{1}{2}(s_{n21}^2 x_{12} + s_{n13}^2 x_{31}) & s_{n21}^2 y_{21} + s_{n13}^2 y_{13} \\ -\frac{1}{2}(c_{n21}^2 y_{21} + c_{n13}^2 y_{13}) & -\frac{1}{2}(s_{n21}^2 y_{21} + s_{n13}^2 y_{13}) & -c_{n21}^2 x_{12} - s_{n13}^2 x_{31} \\ c_{n32}s_{n32} - c_{n21}s_{n21} & -c_{n32}s_{n32} + c_{n21}s_{n21} & -(c_{n32}^2 - s_{n32}^2) + (c_{n21}^2 - s_{n21}^2) \\ \frac{1}{2}(c_{n32}^2 x_{23} + c_{n21}^2 x_{12}) & \frac{1}{2}(s_{n32}^2 x_{23} + s_{n21}^2 x_{12}) & s_{n32}^2 y_{32} + s_{n21}^2 y_{21} \\ -\frac{1}{2}(c_{n32}^2 y_{32} + c_{n21}^2 y_{21}) & -\frac{1}{2}(s_{n32}^2 y_{32} + s_{n21}^2 y_{21}) & -c_{n32}^2 x_{23} - s_{n21}^2 x_{12} \\ c_{n13}s_{n13} + c_{n32}s_{n32} & -c_{n13}s_{n13} - c_{n32}s_{n32} & -(c_{n13}^2 - s_{n13}^2) + (c_{n32}^2 - s_{n32}^2) \\ \frac{1}{2}(c_{n13}^2 x_{31} + c_{n32}^2 x_{23}) & \frac{1}{2}(s_{n13}^2 x_{31} + s_{n32}^2 x_{23}) & s_{n13}^2 y_{13} + s_{n32}^2 y_{32} \\ -\frac{1}{2}(c_{n13}^2 y_{13} + c_{n32}^2 y_{32}) & -\frac{1}{2}(s_{n13}^2 y_{13} + s_{n32}^2 y_{32}) & -c_{n13}^2 x_{31} - s_{n32}^2 x_{23} \end{bmatrix} \quad (28.18)$$

Here c_{nji} and s_{nji} denote the cosine and sine, respectively, of the angle formed by x and the exterior normal to side $i \rightarrow j$.

Matrix $\mathbf{L}_l$ was introduced by Bergan and Nygård [13] and $\mathbf{L}_q$ by Militello and Felippa [20].

A.3 The Higher Order Stiffness Template

For an element of constant $\mathbf{D}$, the higher order stiffness template is defined by

$$\mathbf{K}_h = \frac{A}{3} (\mathbf{B}_4^T \mathbf{D}_\chi \mathbf{B}_4 + \mathbf{B}_5^T \mathbf{D}_\chi \mathbf{B}_5 + \mathbf{B}_6^T \mathbf{D}_\chi \mathbf{B}_6) \quad (28.19)$$

where $\mathbf{D}_\chi = \mathbf{T}^T \mathbf{D} \mathbf{T}$ is the plate constitutive relation expressed in terms of natural curvatures and moments, and $\mathbf{B}_{\chi m}$ are the natural curvature-displacement matrices evaluated at the midpoints $m = 4, 5, 6$ opposing corners 3, 1, 2, respectively.

These matrices are parametrized as follows. Define the geometric invariants

$$\lambda_1 = \frac{x_{12}x_{13} + y_{12}y_{13}}{x_{21}^2 + y_{21}^2} - \frac{1}{2}, \quad \lambda_2 = \frac{x_{23}x_{21} + y_{23}y_{21}}{x_{32}^2 + y_{32}^2} - \frac{1}{2}, \quad \lambda_3 = \frac{x_{31}x_{32} + y_{31}y_{32}}{x_{13}^2 + y_{13}^2} - \frac{1}{2}. \quad (28.20)$$

These have a simple physical meaning as measures of triangle distortion (for an equilateral triangle, $\lambda_1 = \lambda_2 = \lambda_3 = 0$). In the following expressions, the β -derived coefficients γ_i and σ_i are selected so that the $\mathbf{B}_{\chi m}$ matrices are

exactly orthogonal to all rigid body modes and constant curvature states. This is a requirement of the fundamental stiffness decomposition.

$$\begin{aligned}
\beta_1 &= \beta_{10} + \beta_{11}\lambda_3 + \beta_{12}\lambda_1 + \beta_{13}\lambda_2, & \beta_2 &= \beta_{20} + \beta_{21}\lambda_3 + \beta_{22}\lambda_1 + \beta_{23}\lambda_2, & \beta_3 &= \beta_{30} + \beta_{31}\lambda_3 + \beta_{32}\lambda_1 + \beta_{33}\lambda_2, \\
\beta_4 &= \beta_{40} + \beta_{41}\lambda_3 + \beta_{42}\lambda_1 + \beta_{43}\lambda_2, & \beta_5 &= \beta_{50} + \beta_{51}\lambda_3 + \beta_{52}\lambda_1 + \beta_{53}\lambda_2, & \beta_6 &= \beta_{60} + \beta_{61}\lambda_3 + \beta_{62}\lambda_1 + \beta_{63}\lambda_2, \\
\beta_7 &= \beta_{70} + \beta_{71}\lambda_3 + \beta_{72}\lambda_1 + \beta_{73}\lambda_2, & \beta_8 &= \beta_{80} + \beta_{81}\lambda_3 + \beta_{82}\lambda_1 + \beta_{83}\lambda_2, & \beta_9 &= \beta_{90} + \beta_{91}\lambda_3 + \beta_{92}\lambda_1 + \beta_{93}\lambda_2, \\
\gamma_1 &= \beta_1 + \beta_3, & \gamma_2 &= \beta_3, & \gamma_3 &= \beta_2 + \beta_3, & \gamma_4 &= \beta_4 + \beta_9, & \gamma_5 &= \beta_9, & \gamma_6 &= \beta_6 + \beta_8, \\
\gamma_7 &= \beta_6 + \beta_7, & \gamma_8 &= \beta_6, & \gamma_9 &= \beta_5 + \beta_9, & \sigma_1 &= 2\gamma_3 - 2\gamma_1, & \sigma_2 &= 2\gamma_1, & \sigma_3 &= 2\gamma_3, \\
\sigma_4 &= 2\gamma_9 - 2\gamma_4, & \sigma_5 &= 2\gamma_4, & \sigma_6 &= -2\gamma_6, & \sigma_7 &= 2\gamma_6 - 2\gamma_7, & \sigma_8 &= 2\gamma_7, & \sigma_9 &= -2\gamma_9
\end{aligned}$$

$$\mathbf{B}_{\chi 4} = \begin{bmatrix} \sigma_5 & \gamma_4 y_{31} + \gamma_5 y_{23} & \gamma_4 x_{13} + \gamma_5 x_{32} & \sigma_9 & \beta_9 y_{31} + \gamma_9 y_{23} & \beta_9 x_{13} + \gamma_9 x_{32} \\ \sigma_8 & \gamma_7 y_{31} + \gamma_8 y_{23} & \gamma_7 x_{13} + \gamma_8 x_{32} & \sigma_6 & \beta_6 y_{31} + \gamma_6 y_{23} & \beta_6 x_{13} + \gamma_6 x_{32} \\ \sigma_2 & \gamma_1 y_{31} + \gamma_2 y_{23} & \gamma_1 x_{13} + \gamma_2 x_{32} & \sigma_3 & \beta_3 y_{31} + \gamma_3 y_{23} & \beta_3 x_{13} + \gamma_3 x_{32} \\ \sigma_4 & \beta_4 y_{31} + \beta_5 y_{23} & \beta_4 x_{13} + \beta_5 x_{32} \\ \sigma_7 & \beta_7 y_{31} + \beta_8 y_{23} & \beta_7 x_{13} + \beta_8 x_{32} \\ \sigma_1 & \beta_1 y_{31} + \beta_2 y_{23} & \beta_1 x_{13} + \beta_2 x_{32} \end{bmatrix} \quad (28.21)$$

$$\begin{aligned}
\beta_1 &= \beta_{10} + \beta_{11}\lambda_1 + \beta_{12}\lambda_2 + \beta_{13}\lambda_3, & \beta_2 &= \beta_{20} + \beta_{21}\lambda_1 + \beta_{22}\lambda_2 + \beta_{23}\lambda_3, & \beta_3 &= \beta_{30} + \beta_{31}\lambda_1 + \beta_{32}\lambda_2 + \beta_{33}\lambda_3, \\
\beta_4 &= \beta_{40} + \beta_{41}\lambda_1 + \beta_{42}\lambda_2 + \beta_{43}\lambda_3, & \beta_5 &= \beta_{50} + \beta_{51}\lambda_1 + \beta_{52}\lambda_2 + \beta_{53}\lambda_3, & \beta_6 &= \beta_{60} + \beta_{61}\lambda_1 + \beta_{62}\lambda_2 + \beta_{63}\lambda_3, \\
\beta_7 &= \beta_{70} + \beta_{71}\lambda_1 + \beta_{72}\lambda_2 + \beta_{73}\lambda_3, & \beta_8 &= \beta_{80} + \beta_{81}\lambda_1 + \beta_{82}\lambda_2 + \beta_{83}\lambda_3, & \beta_9 &= \beta_{90} + \beta_{91}\lambda_1 + \beta_{92}\lambda_2 + \beta_{93}\lambda_3, \\
\gamma_1 &= \beta_1 + \beta_3, & \gamma_2 &= \beta_3, & \gamma_3 &= \beta_2 + \beta_3, & \gamma_4 &= \beta_4 + \beta_9, & \gamma_5 &= \beta_9, & \gamma_6 &= \beta_6 + \beta_8, \\
\gamma_7 &= \beta_6 + \beta_7, & \gamma_8 &= \beta_6, & \gamma_9 &= \beta_5 + \beta_9, & \sigma_1 &= 2\gamma_3 - 2\gamma_1, & \sigma_2 &= 2\gamma_1, & \sigma_3 &= -2\gamma_3, \\
\sigma_4 &= 2\gamma_9 - 2\gamma_4, & \sigma_5 &= 2\gamma_4, & \sigma_6 &= -2\gamma_6, & \sigma_7 &= 2\gamma_6 - 2\gamma_7, & \sigma_8 &= 2\gamma_7, & \sigma_9 &= -2\gamma_9
\end{aligned}$$

$$\mathbf{B}_{\chi 5} = \begin{bmatrix} \sigma_1 & \beta_1 y_{12} + \beta_2 y_{31} & \beta_1 x_{21} + \beta_2 x_{13} & \sigma_2 & \gamma_1 y_{12} + \gamma_2 y_{31} & \gamma_1 x_{21} + \gamma_2 x_{13} \\ \sigma_4 & \beta_4 y_{12} + \beta_5 y_{31} & \beta_4 x_{21} + \beta_5 x_{13} & \sigma_5 & \gamma_4 y_{12} + \gamma_5 y_{31} & \gamma_4 x_{21} + \gamma_5 x_{13} \\ \sigma_7 & \beta_7 y_{12} + \beta_8 y_{31} & \beta_7 x_{21} + \beta_8 x_{13} & \sigma_8 & \gamma_7 y_{12} + \gamma_8 y_{31} & \gamma_7 x_{21} + \gamma_8 x_{13} \\ \sigma_3 & \beta_3 y_{12} + \gamma_3 y_{31} & \beta_3 x_{21} + \gamma_3 x_{13} \\ \sigma_9 & \beta_9 y_{12} + \gamma_9 y_{31} & \beta_9 x_{21} + \gamma_9 x_{13} \\ \sigma_6 & \beta_6 y_{12} + \gamma_6 y_{31} & \beta_6 x_{21} + \gamma_6 x_{13} \end{bmatrix} \quad (28.22)$$

$$\begin{aligned}
\beta_1 &= \beta_{10} + \beta_{11}\lambda_2 + \beta_{12}\lambda_3 + \beta_{13}\lambda_1, & \beta_2 &= \beta_{20} + \beta_{21}\lambda_2 + \beta_{22}\lambda_3 + \beta_{23}\lambda_1, & \beta_3 &= \beta_{30} + \beta_{31}\lambda_2 + \beta_{32}\lambda_3 + \beta_{33}\lambda_1, \\
\beta_4 &= \beta_{40} + \beta_{41}\lambda_2 + \beta_{42}\lambda_3 + \beta_{43}\lambda_1, & \beta_5 &= \beta_{50} + \beta_{51}\lambda_2 + \beta_{52}\lambda_3 + \beta_{53}\lambda_1, & \beta_6 &= \beta_{60} + \beta_{61}\lambda_2 + \beta_{62}\lambda_3 + \beta_{63}\lambda_1, \\
\beta_7 &= \beta_{70} + \beta_{71}\lambda_2 + \beta_{72}\lambda_3 + \beta_{73}\lambda_1, & \beta_8 &= \beta_{80} + \beta_{81}\lambda_2 + \beta_{82}\lambda_3 + \beta_{83}\lambda_1, & \beta_9 &= \beta_{90} + \beta_{91}\lambda_2 + \beta_{92}\lambda_3 + \beta_{93}\lambda_1, \\
\gamma_1 &= \beta_1 + \beta_3, & \gamma_2 &= \beta_3, & \gamma_3 &= \beta_2 + \beta_3, & \gamma_4 &= \beta_4 + \beta_9, & \gamma_5 &= \beta_9, & \gamma_6 &= \beta_6 + \beta_8, \\
\gamma_7 &= \beta_6 + \beta_7, & \gamma_8 &= \beta_6, & \gamma_9 &= \beta_5 + \beta_9, & \sigma_1 &= 2\gamma_3 - 2\gamma_1, & \sigma_2 &= 2\gamma_1, & \sigma_3 &= -2\gamma_3, \\
\sigma_4 &= 2\gamma_9 - 2\gamma_4, & \sigma_5 &= 2\gamma_4, & \sigma_6 &= -2\gamma_6, & \sigma_7 &= 2\gamma_6 - 2\gamma_7, & \sigma_8 &= 2\gamma_7, & \sigma_9 &= -2\gamma_9
\end{aligned}$$

$$\mathbf{B}_{\chi 6} = \begin{bmatrix} \sigma_6 & \beta_6 y_{23} + \gamma_6 y_{12} & \beta_6 x_{32} + \gamma_6 x_{21} & \sigma_7 & \beta_7 y_{23} + \beta_8 y_{12} & \beta_7 x_{32} + \beta_8 x_{21} \\ \sigma_3 & \beta_3 y_{23} + \gamma_3 y_{12} & \beta_3 x_{32} + \gamma_3 x_{21} & \sigma_1 & \beta_1 y_{23} + \beta_2 y_{12} & \beta_1 x_{32} + \beta_2 x_{21} \\ \sigma_9 & \beta_9 y_{23} + \gamma_9 y_{12} & \beta_9 x_{32} + \gamma_9 x_{21} & \sigma_4 & \beta_4 y_{23} + \beta_5 y_{12} & \beta_4 x_{32} + \beta_5 x_{21} \\ \sigma_8 & \gamma_7 y_{23} + \gamma_8 y_{12} & \gamma_7 x_{32} + \gamma_8 x_{21} \\ \sigma_2 & \gamma_1 y_{23} + \gamma_2 y_{12} & \gamma_1 x_{32} + \gamma_2 x_{21} \\ \sigma_5 & \gamma_4 y_{23} + \gamma_5 y_{12} & \gamma_4 x_{32} + \gamma_5 x_{21} \end{bmatrix} \quad (28.23)$$

Equations (28.19) through (28.23) complete the definition of the KPT-1-36 template.

29

Optimal Membrane Triangles with Drilling Freedoms

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A Study of Optimal Membrane Triangles with Drilling Freedoms

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SUMMARY

This article compares derivation methods for constructing optimal membrane triangles with corner drilling freedoms. The term “optimal” is used in the sense of exact inplane pure-bending response of rectangular mesh units of arbitrary aspect ratio. Following a comparative summary of element formulation approaches, the construction of an optimal 3-node triangle using the ANDES formulation is presented. The construction is based upon techniques developed by 1991 in student term projects, but taking advantage of the more general framework of templates developed since. The optimal element that fits the ANDES template is shown to be unique if energy orthogonality constraints are enforced *a priori*. Two other formulations are examined and compared with the optimal model. Retrofitting the conventional LST (Linear Strain Triangle) element by midpoint-migrating by congruential transformations is shown to be unable to produce an optimal element while rank deficiency is inevitable. Use of the quadratic strain field of the 1988 Allman triangle, or linear filtered versions thereof, is also unable to reproduce the optimal element. Moreover these elements exhibit serious aspect ratio lock. These predictions are verified on benchmark examples.

Keywords: finite elements, templates, high performance, drilling freedoms, triangles, membrane, plane stress, shell, assumed natural deviatoric strains, hierarchical models, signatures, clones.

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§29.1. Introduction

One active area of “finitelementology” is the development of high-performance (HP) elements. The definition of such creatures is subjective. The writer likes to use a result-oriented definition, as stated in [1]: “simple elements that provide results of engineering accuracy with coarse meshes.”

But what are “simple” elements? Again that term is subjective. The writer’s definition is: *elements with only corner nodes and physical degrees of freedom*. Following the high-order element frenzy of the late 1960s and 1970s, the back trend towards simplicity was noted as early as 1986 by the father of NASTRAN: “The limitations of higher order elements set out by Zienkiewicz have proved themselves in application. As a practical matter, the real choice is between lowest order elements (constant strain, probably with some linear strain terms) and next-lowest-order elements (linear strain, possibly with some quadratic strain terms), because these are the ones that developers of finite element programs have found to be commercially viable” [2, p. 89].

The trend has strenghtened since that statement because commercial FEM codes are now used by comparatively more novices, often as backend of CAD studies. These users have at best only a foggy notion of what goes on inside the black boxes. Hence the writer’s admonition in an introductory FEM course: “never, never, never use a higher order or special element unless you are absolutely sure of what you are doing.” The attraction of HP elements in the real world is understandable: to get reasonable answers with models that cannot stray too far from physics.

An *optimal element* is one whose performance cannot be improved for a given node-freedom configuration. The concept is fuzzy, however, unless one specifies precisely what is the optimality measure. There are often tradeoffs. For example, passing patch tests on any mesh may conflict with insensitivity to mesh distortion [2, p. 115].

One of the side effects of interest in high performance is the proliferation of elements with drilling degrees of freedom (DOFs). These are nodal rotations that are not taken as independent DOFs in conventional elements. Two well known examples are: (i) corner rotations normal to the plane of a membrane element (or to the membrane component of a shell element); (ii) three corner rotations added to solid elements. This paper considers only (i).

Why membrane drilling freedoms? Three reasons are given in the Introduction to [3]:

1. The element performance may be improved without adding midside nodes, keeping model preparation and mesh generation simple.
2. The extra degree of freedom is “free of charge” in programs that carry six DOFs per node, as is the case in most commercial codes.
3. It simplifies the treatment of shell intersections as well as connection of shells to beam elements.

The purpose of this paper is to review critically several approaches for the construction of these elements. To keep the exposition to a reasonable length, only triangular membrane elements with 3 corner nodes are studied.

§29.2. Element Derivation Approaches

The term *approach* is taken here to mean a combination of methods and empirical tools to achieve a given objective. In FEM work, isoparametric, stress-assumed-hybrid and ANS (Assumed Natural Strain) formulations are methods and not approaches. An approach may zig-zag through several methods.

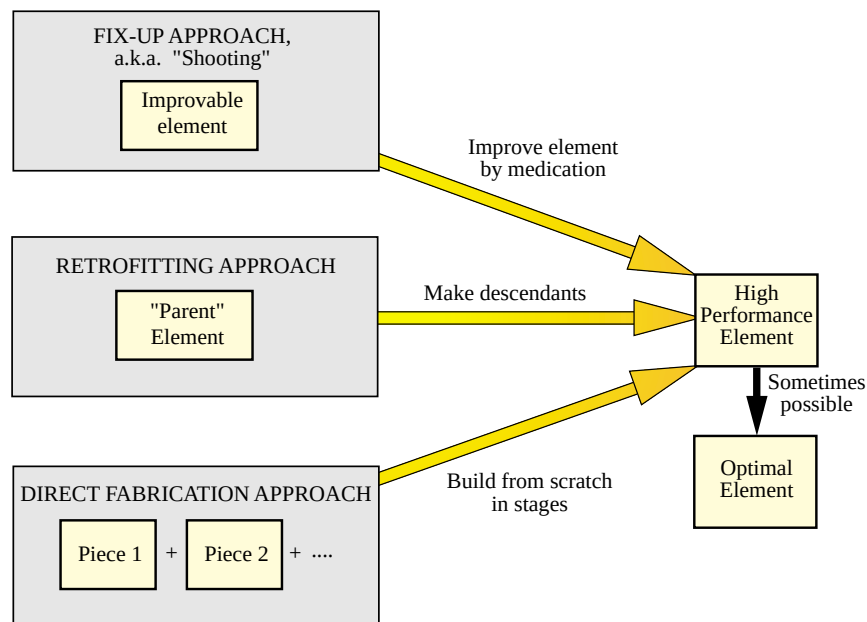


Figure 29.1. Element derivation approaches, not to be confused with methods.

FEM approaches range from heuristic to highly analytical. The experience of the writer in teaching advanced FEM courses is that even bright graduate students have trouble connecting different construction methods, much as undergraduates struggle to connect mathematics and laws of nature. To help students the writer has grouped element derivation approaches into those pictured in Figure 1.

Figure 1 makes an implicit assumption: the performance of an element of given geometry, node and freedom configuration can be improved. There are obvious examples where this is not possible. For example, constant-strain elements with translational freedoms only: 2-node bar, 3-node membrane triangle and 4-node elasticity tetrahedron. Those cases are excluded because it makes no sense to talk about high performance or optimality under those conditions.

§29.2.1. Fixing Up

Conventional element derivation methods, such as the isoparametric formulation, may produce bad or mediocre low-order elements. If that is the case two questions may be raised:

- (i) Can the element be improved?
- (ii) Is the improvement worth the trouble?

If the answer to both is yes, the fix-up approach tries to improve the performance by an array of remedies that may be collectively called the FEM pharmacy. Cures range from heuristic tricks such as reduced and selective integration to more scientifically based concoctions.

This approach accounts for most of the current publications in finiteelementology. Playing doctor can be fun. But also frustrating, as trying to find a black cat in a dark cellar at midnight. Inject these incompatible modes: oops! the patch test is violated. Make the Jacobian constant: oops! it locks in distortion. Reduce the integration order: oops! it lost rank sufficiency. Split the stress-strain equations and integrate selectively: oops! it is not observer invariant. And so on.

§29.2.2. Retrofitting

Retrofitting is a more sedated activity. One begins with a irreproachable parent element, free of obvious defects. Typically this is a higher order iso-P element constructed with a complete or bicomplete polynomial; for example the 6-node quadratic triangle or the 9-node Lagrange quadrilateral. The parent is fine but too complicated to be an HP element. Complexity is reduced by master-slave constraint techniques so as to fit the desired node-freedom configuration pattern.

This approach commonly makes use of node and freedom migration techniques. For example, drilling freedoms may be defined by moving translational midpoint or thirdpoint freedoms to corner rotations by kinematic constraints. The development of “descendants” of the LST element discussed in Section 7 fits this approach. Discrete Kirchhoff constraints and degeneration (3D→2D) for plate and shell elements provide another example. Retrofitting has the advantage of being easy to understand and teach. It occasionally produces useful elements but rarely high performance ones.

§29.2.3. Direct Fabrication

This approach relies on divide and conquer. To give an analogy: upon short training a FEM novice knows that a discrete system is decomposed into elements, which interact only through common freedoms. Going deeper, an element can be constructed as the superposition of components or pieces, with interactions limited through appropriate orthogonality conditions. (Mathematically, components are multifield subspaces [4].) Components are invisible to the user once the element is implemented.

Fabrication is done in stages. At the start there is nothing: the element is without form, and void. At each stage the developer injects another component (= subspace). Components may be done through different methods. The overarching principle is correct performance after each stage. If at any stage the element has problems (for example: it locks) no retroactive cure is attempted as in the fix-up approach. Instead the component is trashed and another one picked. One never uses more components than strictly needed: condensation is forbidden. Components may contain free parameters, which may be used to improve performance and eventually to try for optimality. One general scheme for direct fabrication is the template approach [5].

All applications of the direct fabrication method to date have been done in two stages, separating the element response into basic and higher order. This process is further elaborated in Section 4.3.

§29.2.4. A Warning

The classification of Figure 1 is based on approaches and not methods. A method may appear in more than one approach. For example, methods based on hybrid functionals may be used to retrofit or to fabricate, and even (more rarely) to fix up. Methods based on assumed strain or incompatible displacement fields may be used to do all three. This interweaving of methods and approaches is what makes so difficult to teach advanced FEM. While it is relatively easy to teach methods, choosing and pursuing an approach is a synthesis activity that relies on judgement, experience and luck.

§29.3. A Gallery of Triangles

This article looks at triangular membrane elements in several flavors organized along family lines. To keep track of parents and siblings it is convenient to introduce the following notational scheme for the configuration of an element:

$$\text{xST-}n/m[\text{variants}][\text{-application}] \quad (29.1)$$

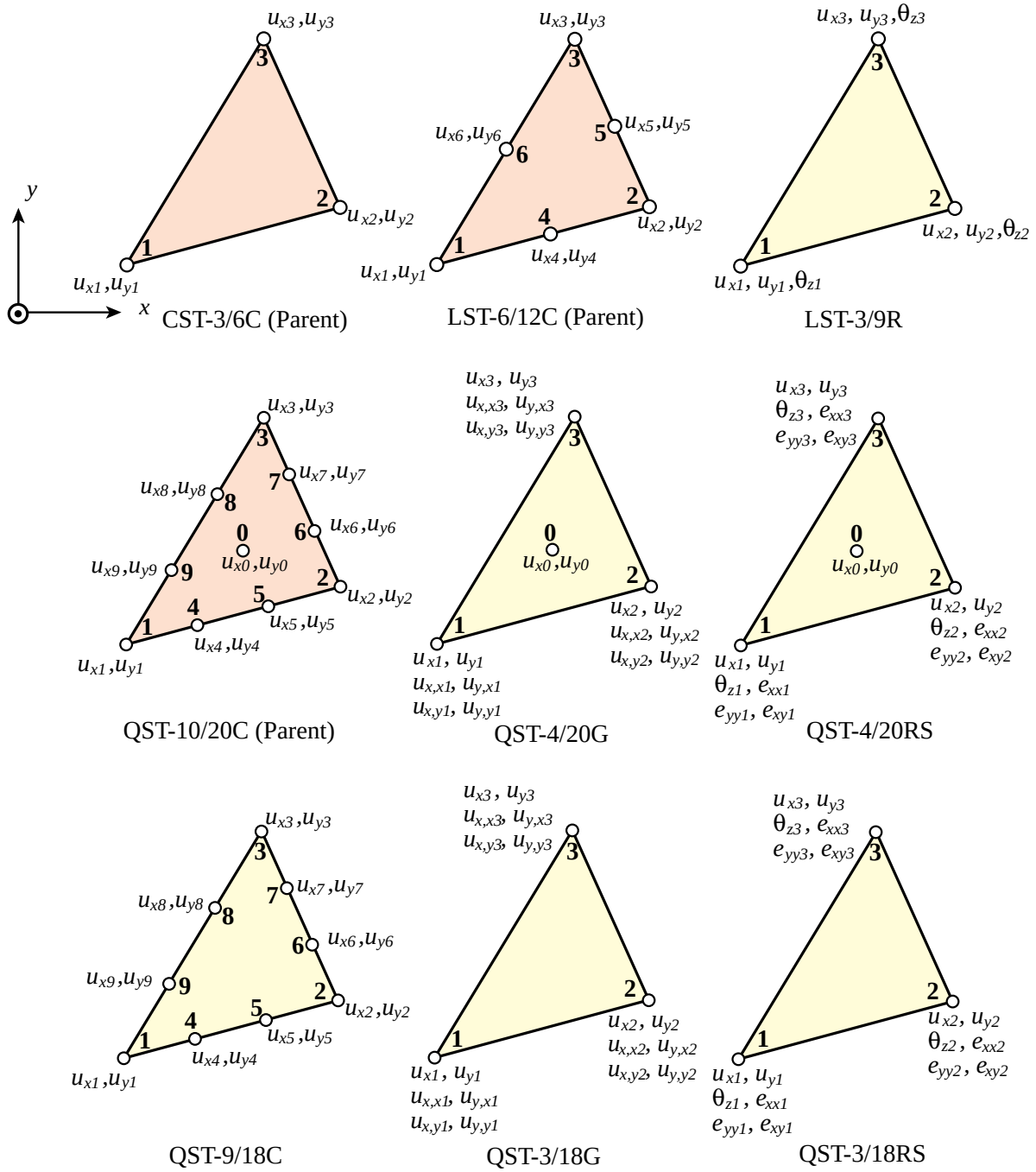


Figure 29.2. Node and freedom configuration of triangular membrane element families.
Only non-hierarchical models with Cartesian node displacements are shown.

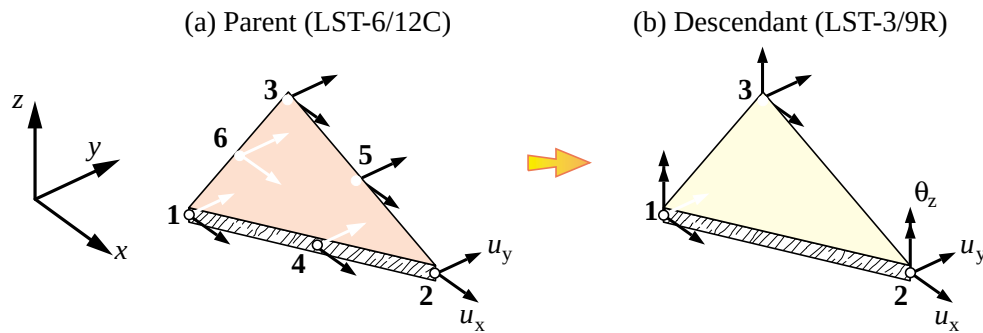


Figure 29.3. Node and freedom configuration of the membrane triangle LST-3/9R and its parent element LST-6/12C.

Lead letter x is C, L or Q, which fingers the parent element as indicated below. Integers n and m give the total number of nodes and freedoms, respectively. Further distinction is made by appending letters to identify variants. For example QST-10/20C, QST-3/20G and QST-3/20RS identify the QST parent and two descendants. Here C, G and RS stand for “conventional freedoms”, “gradient freedoms” and “rotational-plus-strain freedoms,” respectively. The reader may see examples of this identification scheme arranged in Figure 2.

The three parent elements shown there are generated by complete polynomials. They are:

1. *Constant Strain Triangle* or CST. Also called linear triangle and Turner triangle. Developed as plane stress element by Jon Turner, Ray Clough and Harold Martin in 1952–53 [6]; published 1956 [7].
2. *Linear Strain Triangle* or LST. Also called quadratic triangle and Veubeke triangle. Developed by B. Fraeijns de Veubeke in 1962–63 [8]; published 1965 [9].
3. *Quadratic Strain Triangle* or QST. Also called cubic triangle. Developed by the writer in 1965; published 1966 [10]. Shape functions for QST-10/20RS to QST-3/18G were presented there but used for plate bending instead of plane stress; e.g., QST-3/18G clones the BCIZ element [11].

Drilling freedoms in triangles were used in static and dynamic shell analysis in Carr’s thesis under Ray Clough [12,13], using QST-3/20RS as membrane component. The same idea was independently exploited for rectangular and quadrilateral elements, respectively, in the theses of Abu-Ghazaleh [14] and Willam [15], both under Alex Scordelis. A variant of the Willam quadrilateral, developed by Bo Almroth at Lockheed, has survived in the nonlinear shell analysis code STAGS as element 410 [16]. (For access to pertinent old-thesis material through the writer, see References section.)

The focus of this article is on LST-3/9R, shown in the upper right corner of Figure 2 and, in 3D view, in Figure 3(b). The whole development pertains to the membrane (plane stress) problem. Thus no additional identifiers are used. Should the model be applied to a different problem, for example plane strain or axisymmetric analysis, an application identifier would be necessary under scheme (29.1).

§29.4. The ANDES Triangle with Drilling Freedoms

As pictured in in Figure 3(b), the LST-3/9R membrane triangle has 3 corner nodes and 3 DOFs per node: two inplane translations and a drilling rotation. In the retrofitting approach studied in Section 7 the parent element is the conventional Linear Strain Triangle, which is technically identified as LST-6/12C. The direct fabrication approach was used in a three-part 1992 paper [3,17,18] to construct an optimal version of LST-3/9R. (This work grew out of student term projects in an advanced finite element course.) Two different techniques were used in that development:

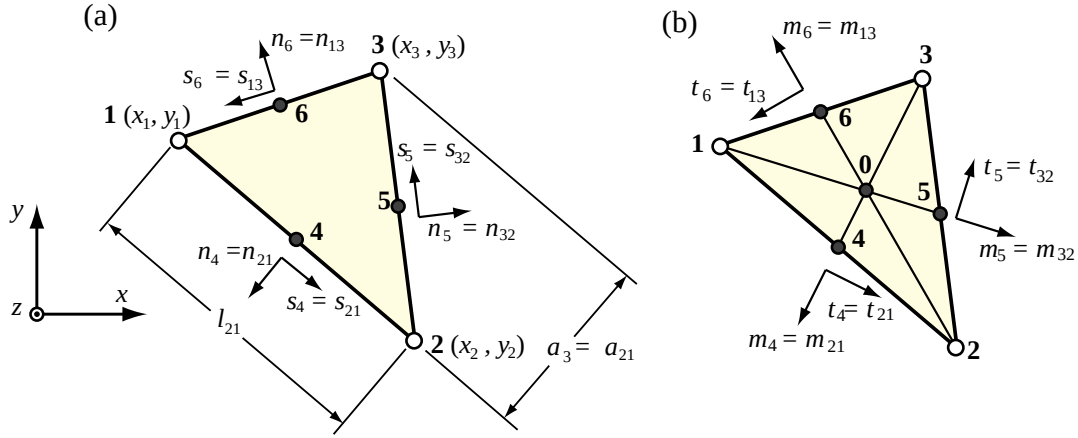


Figure 29.4. Triangle geometry.

EFF. The Extended Free Formulation, which is a variant of the Free Formulation (FF) of Bergan [19–27].

ANDES. The Assumed Natural DEviatoric Strain formulation, which combines the FF of Bergan and a variant of the Assumed Natural Strain (ANS) method due to Park and Stanley [28,29]. ANDES has also been used to develop plate bending and shell elements [30,31].

For the LST-3/9R, these techniques led to stiffness matrices with free parameters: 3 and 7 in the case of EFF and ANDES, respectively. Free parameters were optimized so that rectangular mesh units are exact in pure bending for arbitrary aspect ratios, a technique further discussed in Section 5. Surprisingly the same optimal element was found. In the nomenclature of templates summarized in Section 4.7 the two elements are said to be *clones*. This coalescence nurtured the feeling that the optimal form is unique. More recent studies reported in Section 5.5 verify uniqueness if certain orthogonality constraints are placed on the higher order response.

§29.4.1. Element Description

The membrane (plane stress) triangle shown in Figure 4 has straight sides joining the corners defined by the coordinates $\{x_i, y_i\}$, $i = 1, 2, 3$. Coordinate differences are abbreviated $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$. The signed area A is given by

$$2A = (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1) = y_{21}x_{13} - x_{21}y_{13} \quad (\text{and 2 others}). \quad (29.2)$$

In addition to the corner nodes 1, 2 and 3 we shall also use midpoints 4, 5 and 6 for derivations although these nodes do not appear in the final equations of the LST-3/9R. Midpoints 4, 5, 6 are located opposite corners 3, 1 and 2, respectively. The centroid is denoted by 0. As shown in Figure 4, two intrinsic coordinate systems are used over each side:

$$n_{21}, s_{21}, \quad n_{32}, s_{32}, \quad n_{13}, s_{13}, \quad (29.3)$$

$$m_{21}, t_{21}, \quad m_{32}, t_{32}, \quad m_{13}, t_{13}. \quad (29.4)$$

Here n and s are oriented along the external normal-to-side and side directions, respectively, whereas m and t are oriented along the triangle median and normal-to-median directions, respectively. The coordinate sets (29.3)–(29.4) align only for equilateral triangles. The origin of these systems is left

“floating” and may be adjusted as appropriate. If the origin is placed at the midpoints, subscripts 4, 5 and 6 may be used instead of 21, 32 and 13, respectively, as illustrated in Figure 4.

Other intrinsic dimensions of use in element derivations are

$$\ell_{ij} = \ell_{ji} = \sqrt{x_{ij}^2 + y_{ij}^2}, \quad a_{ij} = a_k = 2A/\ell_{ij}, \quad m_k = \frac{3}{2}\sqrt{x_{k0}^2 + y_{k0}^2}, \quad b_k = 2A/m_k, \quad (29.5)$$

Here j and k denote the positive cyclic permutations of i ; for example $i = 2, j = 3, k = 1$. The ℓ_{ij} 's are the lengths of the sides, $a_k = a_{ij}$ are triangle heights, m_k are the lengths of the medians, and b_k are side lengths projected on normal-to-median directions.

The well known triangle coordinates are denoted by ζ_1, ζ_2 and ζ_3 , which satisfy $\zeta_1 + \zeta_2 + \zeta_3 = 1$.

The degrees of freedom of LST-3/9R are collected in the node displacement vector

$$\mathbf{u}_R = [u_{x1} \ u_{y1} \ \theta_1 \ u_{x2} \ u_{y2} \ \theta_2 \ u_{x3} \ u_{y3} \ \theta_3]^T. \quad (29.6)$$

Here u_{xi} and u_{yi} denote the nodal values of the translational displacements u_x and u_y along x and y , respectively, and $\theta \equiv \theta_z$ are the “drilling rotations” about z (positive counterclockwise when looking down on the element midplane along $-z$). In continuum mechanics these rotations are defined by

$$\theta = \theta_z = \frac{1}{2} \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right). \quad (29.7)$$

The triangle will be assumed to have constant thickness h and uniform plane stress constitutive properties. These are defined by the 3×3 elasticity and compliance matrices arranged in the usual manner:

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}, \quad \mathbf{C} = \mathbf{E}^{-1} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}. \quad (29.8)$$

For later use six invariants of the elasticity tensor are listed here:

$$\begin{aligned} J_{E1} &= E_{11} + 2E_{12} + E_{22}, & J_{E2} &= -E_{12} + E_{33}, & J_{E3} &= (E_{11} - E_{22})^2 + 4(E_{13} + E_{23})^2, \\ J_{E4} &= (E_{11} - 2E_{12} + E_{22} - 4E_{33})^2 + 16(E_{13} - E_{23})^2, \\ J_{E5} &= \det(\mathbf{E}) = E_{11}E_{22}E_{33} + 2E_{12}E_{13}E_{23} - E_{11}E_{23}^2 - E_{22}E_{13}^2 - E_{33}E_{12}^2, \\ J_{E6} &= 2E_{13}^3 + E_{12}E_{13}E_{22} - E_{13}E_{22}^2 + E_{11}^2E_{23} + 2E_{13}^2E_{23} + E_{12}E_{22}E_{23} - 2E_{13}E_{23}^2 - 2E_{23}^3 + \\ &\quad 2E_{22}(E_{13} + E_{23})E_{33} - E_{11}(E_{12}E_{13} - E_{13}E_{22} + E_{12}E_{23} + E_{22}E_{23} + 2(E_{13} + E_{23})E_{33}). \end{aligned} \quad (29.9)$$

Of these J_{E1} , J_{E2} and J_{E5} are well known, while the others were found by *Mathematica*.

§29.4.2. Natural Strains

In the derivation of the higher order stiffness by ANDES [17] *natural strains* play a key role. These are extensional (direct) strains along three directions intrinsically related to the triangle geometry. Four possible choices are depicted in Figure 5. Choice (s): strains along the 3 side directions, was the one used in [17] because it matches the direction of neutral axes of the assumed inplane bending modes as discussed in Section 4.6.

The (s) natural strains are collected in the 3-vector

$$\boldsymbol{\epsilon} = [\epsilon_{21} \ \epsilon_{32} \ \epsilon_{13}]^T. \quad (29.10)$$

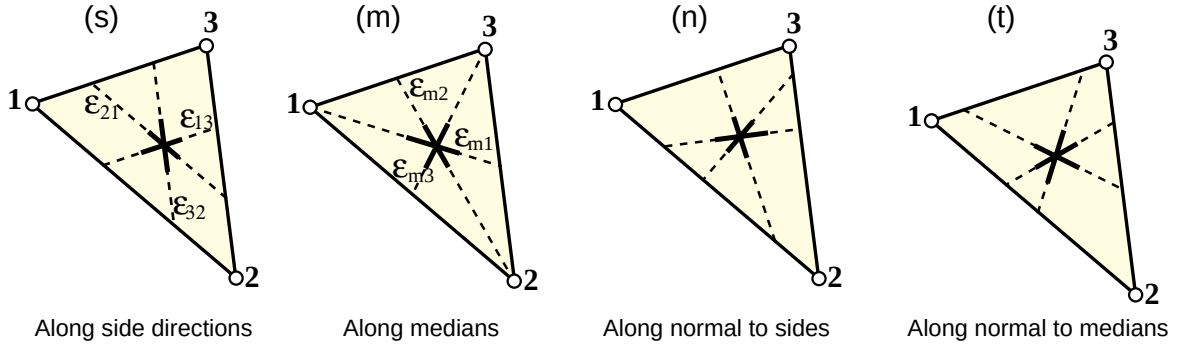


Figure 29.5. Four choices for natural strains. Labels (s) through (t) correlate with the notation (29.3)-(29.4). Although the “natural straining rosettes” are pictured at the centroid for viewing convenience, they may be placed at any point on the triangle.

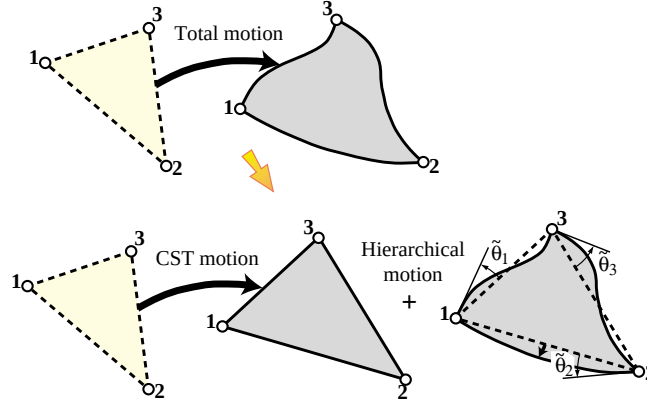


Figure 29.6. Decomposition of inplane motion into CST (linear displacement) + hierarchical. The same idea (in 2D or 3D) is also important in corotational formulations.

Vector ϵ at point i is denoted by ϵ_i . The natural strain ϵ_{jk} at point i will be written $\epsilon_{jk|i}$, the bar being used for reading convenience. The natural strains are related to Cartesian strains $\{e_{xx}, e_{yy}, 2e_{xy}\}$ by the “straingage rosette” transformation

$$\epsilon = \begin{bmatrix} \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{bmatrix} = \begin{bmatrix} x_{21}^2/\ell_{21}^2 & y_{21}^2/\ell_{21}^2 & x_{21}y_{21}/\ell_{21}^2 \\ x_{32}^2/\ell_{32}^2 & y_{32}^2/\ell_{32}^2 & x_{32}y_{32}/\ell_{32}^2 \\ x_{13}^2/\ell_{13}^2 & y_{13}^2/\ell_{13}^2 & x_{13}y_{13}/\ell_{13}^2 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{T}_e^{-1} \mathbf{e}, \quad (29.11)$$

in which $\ell_{ji}^2 = x_{ji}^2 + y_{ji}^2$. The inverse relation is

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{23}y_{13}\ell_{21}^2 & y_{31}y_{21}\ell_{32}^2 & y_{12}y_{32}\ell_{13}^2 \\ x_{23}x_{13}\ell_{21}^2 & x_{31}x_{21}\ell_{32}^2 & x_{12}x_{32}\ell_{13}^2 \\ (y_{23}x_{31} + x_{32}y_{13})\ell_{21}^2 & (y_{31}x_{12} + x_{13}y_{21})\ell_{32}^2 & (y_{12}x_{23} + x_{21}y_{32})\ell_{13}^2 \end{bmatrix} \begin{bmatrix} \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{bmatrix}, \quad (29.12)$$

or, in compact matrix notation, $\mathbf{e} = \mathbf{T}_e \epsilon$. Note that $\mathbf{T}_e$ is constant over the triangle. The natural stress-strain matrix $\mathbf{E}_{nat}$ is defined by

$$\mathbf{E}_{nat} = \mathbf{T}_e^T \mathbf{E} \mathbf{T}_e, \quad (29.13)$$

which is also constant over the triangle.

§29.4.3. Hierarchical Rotations

Hierarchical drilling freedoms are useful for compactly expressing the higher order behavior of the element. Their geometric interpretation is shown in Figure 6. To extract the hierarchical corner rotations $\tilde{\theta}_i$ from the total corner rotations θ_i , subtract the mean or CST rotation θ_0 :

$$\tilde{\theta}_i = \theta_i - \theta_0, \quad (29.14)$$

where $i = 1, 2, 3$ is the corner index and

$$\theta_0 = \frac{1}{4A} (x_{23}u_{x1} + x_{31}u_{x2} + x_{12}u_{x3} + y_{23}u_{y1} + y_{31}u_{y2} + y_{12}u_{y3}). \quad (29.15)$$

Applying (29.14)-(29.15) to the three corners we assemble the transformation

$$\tilde{\boldsymbol{\theta}} = \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} = \frac{1}{4A} \begin{bmatrix} x_{32} & y_{32} & 4A & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 4A & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 4A \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \\ u_{x3} \\ u_{y3} \\ \theta_3 \end{bmatrix} = \tilde{\mathbf{T}}_{\theta u} \mathbf{u}_R. \quad (29.16)$$

For some developments it is useful to complete this transformation with the identity matrix for the translational freedoms:

$$\tilde{\mathbf{u}}_R = \begin{bmatrix} u_{x1} \\ u_{y1} \\ \tilde{\theta}_1 \\ u_{x2} \\ u_{y2} \\ \tilde{\theta}_2 \\ u_{x3} \\ u_{y3} \\ \tilde{\theta}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{x_{32}}{4A} & \frac{y_{32}}{4A} & 1 & \frac{x_{13}}{4A} & \frac{y_{13}}{4A} & 0 & \frac{x_{21}}{4A} & \frac{y_{21}}{4A} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{x_{32}}{4A} & \frac{y_{32}}{4A} & 0 & \frac{x_{13}}{4A} & \frac{y_{13}}{4A} & 1 & \frac{x_{21}}{4A} & \frac{y_{21}}{4A} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{x_{32}}{4A} & \frac{y_{32}}{4A} & 0 & \frac{x_{13}}{4A} & \frac{y_{13}}{4A} & 0 & \frac{x_{21}}{4A} & \frac{y_{21}}{4A} & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \\ u_{x3} \\ u_{y3} \\ \theta_3 \end{bmatrix} = \tilde{\mathbf{T}}_R \mathbf{u}_R. \quad (29.17)$$

The inverse transformation $\tilde{\mathbf{T}}_R^{-1}$ that connects $\mathbf{u}_R = \tilde{\mathbf{T}}_R^{-1} \tilde{\mathbf{u}}_R$ is obtained by simply transposing the subscripts in the coordinate differences; $x_{32} \rightarrow -x_{32} = x_{23}$, etc. The foregoing transformation matrices are constant over the element.

§29.4.4. The Stiffness Template

The fundamental element stiffness decomposition of the two-stage direct fabrication method is

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_h \quad (29.18)$$

Here $\mathbf{K}_b$ is the basic stiffness, which takes care of consistency, and $\mathbf{K}_h$ is the higher order stiffness, which takes care of stability (rank sufficiency) and accuracy. This decomposition was found by Bergan

and Nygård [20] as part of the Free Formulation (FF), but actually holds for any element that passes the Individual Element Test (IET) of Bergan and Hanssen [32]. (The IET is a strong form of the patch test that demands pairwise cancellation of tractions between adjacent elements in constant stress states.) Similar elements were constructed by Belytschko and coworkers [33] using an hourglass stabilization approach. See also Hughes [34, §4.8].

Orthogonality conditions satisfied by $\mathbf{K}_h$ are discussed in [4,19,21–27,35–38].

The EFF and ANDES triangles derived in [3] and [17] initially carry along a set of free numerical parameters, most of which affect the higher order stiffness:

$$\mathbf{K}^{\text{EFF}'91}(\alpha_b, \alpha_h, \gamma) = \mathbf{K}_b(\alpha_b) + (1 - \gamma)\mathbf{K}_h^u(\alpha_h), \quad (29.19)$$

$$\mathbf{K}^{\text{ANDES}'91}(\alpha_b, \beta, \rho_1, \dots, \rho_5) = \mathbf{K}_b(\alpha_b) + \beta\mathbf{K}_h^u(\rho_1, \dots, \rho_5), \quad (29.20)$$

where $\mathbf{K}_h^u$ is the unscaled higher order stiffness. Both $\mathbf{K}_b$ and $\mathbf{K}_h$ must have rank 3. Algebraic forms such as (29.19)–(29.20) possessing free parameters are called *element stiffness templates* or simply *templates*.

The basic stiffness $\mathbf{K}_b(\alpha_b)$ is identical for both (29.19) and (29.20). In fact, patch test and template theory [5,35–38] says that $\mathbf{K}_b(\alpha_b)$ must be shared by all elements that pass the IET although α_b may vary for different models. However α_b must be the same for all LST-3/9R elements connected in an assembly, for otherwise the patch test would be violated. This is called a *mixability condition*. Parameters other than α_b may, in principle, vary from element to element without affecting convergence.

§29.4.5. The Basic Stiffness

An explicit form of the basic stiffness for the LST-3/9R configuration was obtained in 1984 and published the following year [21]. It can be expressed as

$$\mathbf{K}_b = V^{-1}\mathbf{L}\mathbf{E}\mathbf{L}^T. \quad (29.21)$$

where $V = Ah$ is the element volume, and $\mathbf{L}$ is a 3×9 matrix that contains a free parameter α_b :

$$\mathbf{L} = \frac{1}{2}h \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ \frac{1}{6}\alpha_b y_{23}(y_{13} - y_{21}) & \frac{1}{6}\alpha_b x_{32}(x_{31} - x_{12}) & \frac{1}{3}\alpha_b(x_{31}y_{13} - x_{12}y_{21}) \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ \frac{1}{6}\alpha_b y_{31}(y_{21} - y_{32}) & \frac{1}{6}\alpha_b x_{13}(x_{12} - x_{23}) & \frac{1}{3}\alpha_b(x_{12}y_{21} - x_{23}y_{32}) \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \\ \frac{1}{6}\alpha_b y_{12}(y_{32} - y_{13}) & \frac{1}{6}\alpha_b x_{21}(x_{23} - x_{31}) & \frac{1}{3}\alpha_b(x_{23}y_{32} - x_{31}y_{13}) \end{bmatrix}. \quad (29.22)$$

In the FF this is called a *force-lumping matrix*, hence the symbol $\mathbf{L}$. Under certain conditions $\mathbf{L}$ can be related to the mean strain-displacement matrix $\mathbf{B}_0$ or $\tilde{\mathbf{B}}$ used in one-point reduced integration schemes: $\mathbf{B}_0 = \mathbf{L}^T/V$, for specific choices of α_b . This matrix also appears in the so-called “B-bar” formulation [34]. If $\alpha_b = 0$ the basic stiffness reduces to the total stiffness matrix of the CST-3/6C, in which case the rows and columns associated with the drilling rotations vanish.

One interesting result is that

$$\mathbf{L}^T = \mathbf{L}^T \tilde{\mathbf{T}}_R, \quad (\text{also } \mathbf{B}_0 = \mathbf{B}_0 \tilde{\mathbf{T}}_R), \quad (29.23)$$

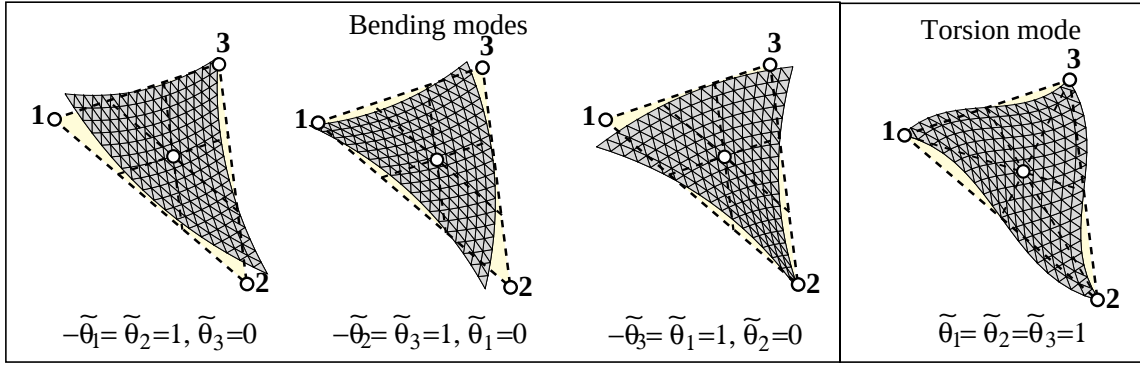


Figure 7. Patterns chosen to build the higher order stiffness of the ANDES template (31): three bending-along-sides modes plus a torsion mode. Although pictured as displacement motions for visualization convenience, the bending modes were initially assumed in natural strains as described in Appendix A. The “neutral axes” of the bending modes are parallel to the sides and pass through the centroid.

for any α_b , which shows that the transformation (29.17) projects out the higher order behavior.

The deep significance of this development is: *the basic stiffness of any element with this node-freedom configuration that passes the IET must have the form (29.21)–(29.22)*. Most derivation methods produce the total stiffness $\mathbf{K}$ directly, with $\mathbf{K}_b$ concealed behind the scenes. This is one of the reasons accounting for the capricious nature of the fix-up approach. In the direct fabrication approach the decomposition (29.18) is explicitly used in the two-stage construction of the element: first $\mathbf{K}_b$ and then $\mathbf{K}_h$.

§29.4.6. The Higher Order Stiffness

We describe here essentially the ANDES form of $\mathbf{K}_h$ developed in [17], with some generalizations in the set of free parameters discussed at the end of this subsection. The higher order stiffness matrix is

$$\mathbf{K}_h = c_{fac} \tilde{\mathbf{T}}_{\theta u}^T \mathbf{K}_\theta \tilde{\mathbf{T}}_{\theta u}. \quad (29.24)$$

where $\mathbf{K}_\theta$ is the 3×3 higher order stiffness in terms of the hierarchical rotations $\tilde{\theta}$ of (29.14), $\tilde{\mathbf{T}}_{\theta u}$ is the matrix (29.16), and c_{fac} is a scaling factor to be determined later. To construct $\mathbf{K}_\theta$ by ANDES one picks deviatoric natural strain patterns, in which “deviatoric” means change from the constant strain states.

Since the main objective is to have good inplane bending behavior, it is logical to begin by assuming patterns associated with three bending-like modes. A key question is, along which directions? For a triangle, four choices — already depicted in Figure 5 as regards the definition of natural strains — satisfy observer invariance:

$$\text{Along the side directions } s_4, s_5, s_6 \quad (29.24)$$

$$\text{Along the normal directions } n_4, n_5, n_6 \quad (29.25)$$

$$\text{Along the median directions } m_4, m_5, m_6 \quad (29.26)$$

$$\text{Along the normal-to-the-median directions } t_4, t_5, t_6 \quad (29.27)$$

Choice (29.24) was adopted in [17]. The three bending strain patterns are sketched on the left of Figure 7 as displacement modes for visualization convenience. (The bending shapes pictured there were obtained

by integrating the assumed strain fields and adjusting for rigid body motions.) It turns out that the three patterns are not linearly independent: their sum vanishes for any triangle geometry. Thus use of only those modes would produce a rank deficient $\mathbf{K}_h$.

To attain the correct rank of 3 the “torsion” pattern depicted on the right of Figure 7 is adjoined. This can be visualized as produced by applying identical hierarchical rotations $\tilde{\theta}_1 = \tilde{\theta}_2 = \tilde{\theta}_3$. A cubic displacement pattern was constructed from the QST-4/20G interpolation. The associated quadratic strain pattern was transformed to natural strains and filtered to a linear variation by midpoint collocation. Those derivations are presented in Appendix A for readers interested in the technical details.

To express $\mathbf{K}_\theta$ compactly, introduce the following matrices, which depend on nine free dimensionless parameters, β_1 through β_9 :

$$\mathbf{Q}_1 = \frac{2A}{3} \begin{bmatrix} \frac{\beta_1}{\ell_{21}^2} & \frac{\beta_2}{\ell_{21}^2} & \frac{\beta_3}{\ell_{21}^2} \\ \frac{\beta_4}{\ell_{32}^2} & \frac{\beta_5}{\ell_{32}^2} & \frac{\beta_6}{\ell_{32}^2} \\ \frac{\beta_7}{\ell_{13}^2} & \frac{\beta_8}{\ell_{13}^2} & \frac{\beta_9}{\ell_{13}^2} \end{bmatrix}, \quad \mathbf{Q}_2 = \frac{2A}{3} \begin{bmatrix} \frac{\beta_9}{\ell_{21}^2} & \frac{\beta_7}{\ell_{21}^2} & \frac{\beta_8}{\ell_{21}^2} \\ \frac{\beta_3}{\ell_{32}^2} & \frac{\beta_1}{\ell_{32}^2} & \frac{\beta_2}{\ell_{32}^2} \\ \frac{\beta_6}{\ell_{13}^2} & \frac{\beta_4}{\ell_{13}^2} & \frac{\beta_5}{\ell_{13}^2} \end{bmatrix}, \quad \mathbf{Q}_3 = \frac{2A}{3} \begin{bmatrix} \frac{\beta_5}{\ell_{21}^2} & \frac{\beta_6}{\ell_{21}^2} & \frac{\beta_4}{\ell_{21}^2} \\ \frac{\beta_8}{\ell_{32}^2} & \frac{\beta_9}{\ell_{32}^2} & \frac{\beta_7}{\ell_{32}^2} \\ \frac{\beta_3}{\ell_{13}^2} & \frac{\beta_1}{\ell_{13}^2} & \frac{\beta_2}{\ell_{13}^2} \end{bmatrix}. \quad (29.28)$$

The scaling by $2A/3$ is for convenience in correlating with prior developments. Matrix $\mathbf{Q}_i$ relates the natural strains ϵ_i at corner i to the deviatoric corner curvatures $\tilde{\theta}$. At a point of triangular coordinates $\{\zeta_1, \zeta_2, \zeta_3\}$, $\epsilon = \mathbf{Q} \tilde{\theta}$, where $\mathbf{Q} = \mathbf{Q}_1 \zeta_1 + \mathbf{Q}_2 \zeta_2 + \mathbf{Q}_3 \zeta_3$. Evaluate this at the midpoints:

$$\mathbf{Q}_4 = \frac{1}{2}(\mathbf{Q}_1 + \mathbf{Q}_2), \quad \mathbf{Q}_5 = \frac{1}{2}(\mathbf{Q}_2 + \mathbf{Q}_3), \quad \mathbf{Q}_6 = \frac{1}{2}(\mathbf{Q}_3 + \mathbf{Q}_1) \quad (29.29)$$

Then

$$\mathbf{K}_\theta = h (\mathbf{Q}_4^T \mathbf{E}_{nat} \mathbf{Q}_4 + \mathbf{Q}_5^T \mathbf{E}_{nat} \mathbf{Q}_5 + \mathbf{Q}_6^T \mathbf{E}_{nat} \mathbf{Q}_6), \quad (29.30)$$

and $\mathbf{K}_h = \frac{3}{4} \beta_0 \mathbf{T}_{\theta u}^T \mathbf{K}_\theta \mathbf{T}_{\theta u}$, where β_0 is an overall scaling coefficient. (This coefficient could be absorbed into the β_i but it is left separate to simplify the incorporation of material behavior into the optimal element.) So finally $\mathbf{K}_R$ assumes a template form with 11 free parameters: $\alpha_b, \beta_0, \beta_1, \dots, \beta_9$:

$$\mathbf{K}_R(\alpha_b, \beta_0, \beta_1, \dots, \beta_9) = V^{-1} \mathbf{L} \mathbf{E} \mathbf{L}^T + \frac{3}{4} \beta \tilde{\mathbf{T}}_{\theta u}^T \mathbf{K}_\theta \tilde{\mathbf{T}}_{\theta u}. \quad (29.31)$$

The factor $\frac{3}{4}$ in $\mathbf{K}_h$ comes from “historical grandfathering”: as shown in Section 5 the optimal β_0 for isotropic material with $\nu = 0$ becomes $\frac{1}{2}$, same as in the 1984 FF element [21]. The template (29.31) will be called the “LST-3/9R ANDES Template” to distinguish it from others alluded to in Section 4.9.

It is easily checked that if the 3×3 matrix with $\{\beta_1, \beta_2, \beta_3\}$, $\{\beta_4, \beta_5, \beta_6\}$ and $\{\beta_7, \beta_8, \beta_9\}$ as rows is nonsingular, then $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ have full rank for $A \neq 0$ and nonzero side lengths. With the usual restrictions on the elasticity matrix $\mathbf{E}$, $\mathbf{K}_\theta$ has full rank of 3 and $\mathbf{K}_R$ is rank sufficient.

As remarked previously, the parameter set in (29.28) is more general than that used in [17]. That development carried only five free parameters in the $\mathbf{Q}_i$ matrices: ρ_1 through ρ_5 , cf. (29.20), which enforced *a priori* the triangular symmetry conditions

$$\beta_7 = -\beta_1, \quad \beta_8 = -\beta_3, \quad \beta_9 = -\beta_2. \quad (29.32)$$

These constraints may be derived, for example, by taking an equilateral triangle in which $\ell_{21} = \ell_{32} = \ell_{13}$ and looking at symmetries about the medians as the $\tilde{\theta}_i$ are applied to each corner in turn. Furthermore the torsional mode was not separately parametrized. The present parameter set is able to encompass elements, such as the retrofitted LST, where that mode is missing.

Table 1. Identifier of Triangle Element Instances

Name	Description	See
ALL-3I	Allman 88 element integrated by 3-point interior rule.	Section 8
ALL-3M	Allman 88 element integrated by 3-midpoint rule.	Section 8
ALL-EX	Allman 88 element, exactly integrated	Section 8
ALL-LS	Allman 88 element, least-square strain fit.	Section 8
CST	Constant strain triangle CST-3/6C.	Ref. [7]
FF84	1984 Free Formulation element of Bergan and Felippa.	Ref. [21]
LST-Ret	Retrofitted LST with $\alpha_b = 4/3$.	Section 7
OPT	Optimal ANDES Template.	Sections 5.2 and 5.3

Table 2. Signatures of Some LST-3/9R Instances Befitting the ANDES Template (31)

Name	α_b	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
ALL-3I	1	4/9	1/12	5/12	1/2	0	1/3	-1/3	-1/12	-1/2	-5/12
ALL-3M	1	4/9	1/4	5/4	3/2	0	1	-1	-1/4	-3/2	-5/4
ALL-EX	not an instance of ANDES template										
ALL-LS	1	4/9	3/20	3/4	9/10	0	3/5	-3/5	-3/20	-9/10	-3/4
CST	0	any	0	0	0	0	0	0	0	0	0
FF84	not an instance of ANDES template										
LST-Ret	4/3	1/2	2/3	-2/3	0	0	-4/3	4/3	-2/3	0	2/3
OPT	3/2	see §5.2	1	2	1	0	1	-1	-1	-1	-2

§29.4.7. Instances, Signatures, Clones

An element generated by specifying numerical values to the parameters $\{\alpha_b, \beta_0, \beta_1, \dots, \beta_9\}$ is a *template instance*. The set of parameter values is the *template signature*. Two elements with the same signature, possibly derived through different methods, are called *clones*.

Table 1 lists triangular elements compared later in this paper. Table 2 defines their signatures if they happen to be instances of the ANDES template (29.31).

By construction all template instances verify exactly the IET for rigid body modes and uniform strain/stress states. Here we see the key advantage of the direct fabrication approach: any template instance that keeps the correct rank is guaranteed to be consistent and stable. Since surprises are mitigated the task of optimizing the element, covered in Section 5, is straightforward.

§29.4.8. Energy Orthogonality

For future use the following definition is noted. An element with linearly varying higher order strains is called *energy orthogonal* in the sense of Bergan [19] if $\mathbf{Q} = \mathbf{Q}_1\zeta_1 + \mathbf{Q}_2\zeta_2 + \mathbf{Q}_3\zeta_3$ vanishes at the

centroid $\zeta_1 = \zeta_2 = \zeta_3 = 1/3$. This gives the algebraic condition

$$\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 = \mathbf{0}. \quad (29.33)$$

For the matrices (29.28), condition (29.33) translates to $\beta_1 + \beta_5 + \beta_9 = \beta_2 + \beta_6 + \beta_7 = \beta_3 + \beta_4 + \beta_8 = 0$. These conditions are not enforced *a priori*. The optimal element derived in Section 5, however, is found to satisfy energy orthogonality.

For more general strain variations energy orthogonality conditions are discussed in [19,21–27,35–38].

§29.4.9. Other Templates

Three more templates for $\mathbf{K}_h$ may be generated by choosing bending patterns according to the prescriptions (29.24), (29.25) and (29.26) for the bending modes. This has not been done to date and remains an open research problem. The closest attempt in this direction was the development of the 1984 FF element described in [21,22] by assuming bending modes along the 3 median directions. Because the FF was used, the modes were initially constructed in displacement form. An advantage of this choice is that the three modes are linearly independent and there is no need to adjoin the torsional mode. But perfect optimality in the sense discussed below was not attainable. As there are indications that the optimal element derivable from (29.31) is unique, as discussed in Section 5.5, there seems to be no compelling incentive for exploring other templates.

§29.5. Finding the Best

A template such as (29.31) generates an infinity of element instances by assigning numeric values to the free parameters. The obvious question is: among all those instances, is there a best one? Because all template instances pass the IET for basic modes (rigid body motions and constant strain states) any optimality criterion must necessarily rely on higher order patch tests. The obvious tests involve the response of regular mesh units to inplane bending along the side directions. This leads to element bending tests expressed as energy ratios. These have been used since 1984 to tune up the higher order stiffness of triangular elements [21,24].

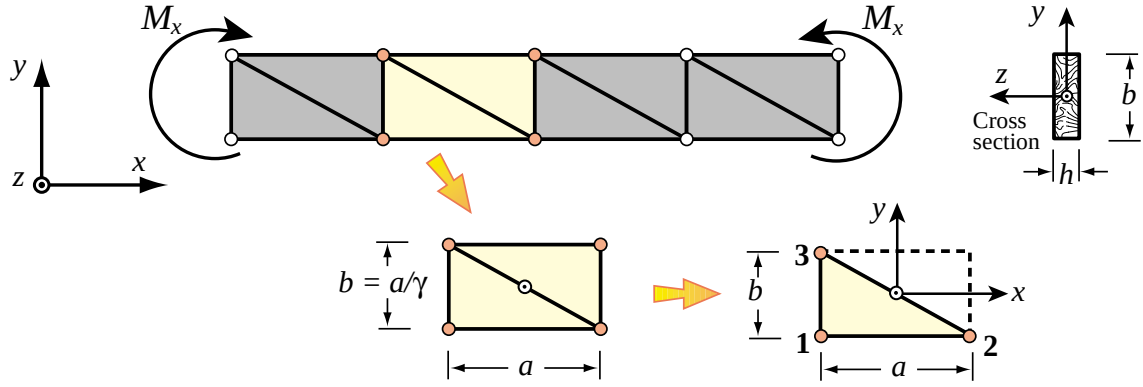
§29.5.1. The Bending Test

The x -bending test is defined in Figure 8. A Bernoulli-Euler plane beam of thin rectangular cross-section with span L , height b and thickness h (normal to the plane of the figure) is bent under applied end moments M_x . The beam is fabricated of isotropic material with elastic modulus E and Poisson's ratio ν . Except for possible end effects the exact solution of the beam problem (from both the theory-of-elasticity and beam-theory standpoints) is a constant bending moment $M(x) = M_x$ along the span. The stress field is $\sigma_{xx} = M_x y / I_{zz}$, $\sigma_{yy} = \sigma_{xy} = 0$, where $I_{zz} = \frac{1}{12}hb^3$. Computing the strain field $\mathbf{e} = \mathbf{E}^{-1}\boldsymbol{\sigma}$ and integrating it one finds the associated displacement field

$$u_x = -\kappa xy, \quad u_y = \frac{1}{2}\kappa(x^2 + \nu y^2), \quad (29.34)$$

where κ is the deformed beam curvature M_x / EI_{zz} . The internal energy taken up by a Bernoulli-Euler beam segment of length a is $U_x^{\text{beam}} = M_x \kappa a = 6aEM_x^2 / (b^3h)$.

To test the ANDES template, the beam is modeled with one layer of identical rectangular mesh units dimensioned $a \times b$ and made up of two LST-3/9R triangles, as illustrated in Figure 9. The aspect ratio a/b is called γ . All rectangles will undergo the same deformations and stresses. We can therefore consider a typical mesh unit. Both triangles will absorb the same energy so it is sufficient to take one

Figure 8. Constant-moment inplane bending test along the x direction.

triangle and multiply by two. For simplicity begin by taking $\nu = 0$. Evaluating (29.34) at nodes 1-2-3 of the triangle shown at the bottom right of Figure 8 we get the node displacement vector

$$\mathbf{u}_x^{\text{trig}} = \frac{3M_x E \gamma^2}{a^2 h} [-a \quad \frac{1}{2}\gamma a \quad -2\gamma \quad a \quad \frac{1}{2}\gamma a \quad 2\gamma \quad a \quad \frac{1}{2}\gamma a \quad -2\gamma]^T \quad (29.35)$$

The strain energy absorbed by the triangle under these applied node displacements is $U_x^{\text{trig}} = \frac{1}{2}(\mathbf{u}_x^{\text{trig}})^T \mathbf{K} \mathbf{u}_x^{\text{trig}}$. That absorbed by the two-triangle mesh unit is $U_x^{\text{quad}} = 2U_x^{\text{trig}}$. The bending energy ratio computed by *Mathematica* can be expressed as

$$r = \frac{U_x^{\text{quad}}}{U_x^{\text{beam}}} = c_0 + c_2 \gamma^2 + c_4 \gamma^4, \quad (29.36)$$

where c_0 , c_2 and c_4 are only functions of the free parameters. For the ensuing derivation we use the parameters of (29.31), but under the symmetry constraints (29.32) that effectively reduce the 11 parameters to 8: $\alpha_b, \beta_0, \beta_1, \dots, \beta_6$. Introduce the 6-vector $\beta = [\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 \beta_6]$. Then a compact form of the coefficients is

$$c_0 = \frac{1}{3}(\alpha_b - 6)\alpha_b + \frac{\beta_0}{64}(\beta^T \mathbf{C}_0 \beta), \quad c_2 = \frac{2}{3}(\alpha_b - 3)\alpha_b + \frac{\beta_0}{64}(\beta^T \mathbf{C}_2 \beta), \quad c_4 = \frac{\beta_0}{64}(\beta^T \mathbf{C}_4 \beta), \quad (29.37)$$

in which $\mathbf{C}$ are the symmetric matrices

$$\mathbf{C}_0 = \begin{bmatrix} 13 & -11 & -1 & 2 & 2 & -6 \\ -11 & 13 & -1 & -2 & -2 & 6 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & 1 & -3 \\ 2 & -2 & 0 & 1 & 1 & -3 \\ -6 & 6 & 0 & -3 & -3 & 9 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 26 & -20 & -4 & -10 & 12 & -6 \\ -20 & 22 & -2 & 8 & -14 & 8 \\ -4 & -2 & 6 & 0 & 2 & 0 \\ -10 & 8 & 0 & 5 & -5 & 1 \\ 12 & -14 & 2 & -5 & 9 & -5 \\ -6 & 8 & 0 & 1 & -5 & 5 \end{bmatrix}, \quad \mathbf{C}_4 = \begin{bmatrix} 1 & 1 & -3 & 0 & 0 & 0 \\ 1 & 1 & -3 & 0 & 0 & 0 \\ -3 & -3 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & -3 & -3 \\ 0 & 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{bmatrix} \quad (29.38)$$

The energy ratio (29.36) happens to be the ratio of the exact (beam) displacement solution to that of the 2D solution. Hence $r = 1$ means that we get the exact answer, that is, the LST-3/9R element is x -bending exact. If $r > 1$ or $r < 1$ the triangle is overstiff or overflexible in x bending, respectively.

In particular, if $r \gg 1$ as $a/b = \gamma$ grows the element is said to experience *aspect ratio locking* along the x direction.

The treatment of energy balance in y bending for rectangular mesh units stacked in the y direction only entails replacing γ by $1/\gamma$. Therefore if the element is x -bending optimal in the sense discussed below it is also y -bending optimal and the analysis need not be repeated.

§29.5.2. Optimality for Isotropic Material

If $r = 1$ for any aspect ratio γ the element is called *bending optimal*. From (29.36) optimality requires

$$c_0 = 1, \quad c_2 = 0, \quad c_4 = 0, \quad \text{for all } \gamma = a/b \text{ and real parameter values.} \quad (29.39)$$

The last proviso means that complex solutions for template parameters are not admissible. The solution method is explained in Appendix B. It gives the optimal parameter set

$$\alpha_b = \frac{3}{2}, \quad \beta_0 = \frac{1}{2}, \quad \beta_1 = \beta_3 = \beta_5 = 1, \quad \beta_2 = 2, \quad \beta_4 = 0, \quad \beta_6 = \beta_7 = \beta_8 = -1, \quad \beta_9 = -2, \quad (29.40)$$

and the $\mathbf{Q}_i$ matrices found in [17] are recovered. It is easily verified that $r = r_b + r_h = 3/4 + 1/4$, where r_b and r_h come from energy taken by the basic and higher-order stiffnesses, respectively. That is, for the optimal element the basic energy accounts for 75% of the exact beam energy.

The symbolic analysis for an arbitrary ν is similar and shows that only β_0 needs to be changed:

$$\alpha_b = \frac{3}{2}, \quad \beta_0 = \frac{1}{2}(1 - 4\nu^2), \quad \beta_1 = \beta_3 = \beta_5 = 1, \quad \beta_2 = 2, \quad \beta_4 = 0, \quad \beta_6 = \beta_7 = \beta_8 = -1, \quad \beta_9 = -2. \quad (29.41)$$

In this case $r_h = \frac{1}{4}(1 - 4\nu^2)$ so for $\nu = \frac{1}{2}$ the basic stiffness takes up all the bending energy. Since for $\nu = \frac{1}{2}$ the optimal β_0 is 0, the higher order stiffness would vanish and the element is rank deficient. To maintain stability one sets a tiny minimum β_0 , for example $\beta_0 = \max(\frac{1}{2}(1 - 4\nu^2), 0.01)$ is used in our shell codes. The case of a non-isotropic material is treated in the next subsection.

Table 3 gives bending ratios for the elements listed in Table 1, along with numerical values for $\nu = 1/4$ and $\gamma = 1, 2, 4, 16$. Those quoted for elements other than OPT are derived in Sections 7 and 8.

§29.5.3. Optimality for Non-Isotropic Material

If the elasticity matrix takes up the general form (29.8) the analysis of bending optimality becomes more elaborate, but follows essentially the same steps. Only the final results are stated here. For optimal bending behavior along a direction x_φ that forms an angle φ (positive ccw) from x , the optimal parameter set is still given by (29.41) except for the overall scaling parameter β_0 :

$$\begin{aligned} \beta_{0\varphi} &= \frac{2}{\Upsilon} - \frac{3}{2}, \quad \Upsilon = E_{11\varphi} C_{11\varphi}, \\ E_{11\varphi} &= E_{11} c_\varphi^4 + 4E_{13} c_\varphi^3 s_\varphi + (2E_{12} + 4E_{33}) c_\varphi^2 s_\varphi^2 + 4E_{23} c_\varphi s_\varphi^3 + E_{22} s_\varphi^4, \\ C_{11\varphi} &= C_{11} c_\varphi^4 + 2C_{13} c_\varphi^3 s_\varphi + (2C_{12} + C_{33}) c_\varphi^2 s_\varphi^2 + 2C_{23} c_\varphi s_\varphi^3 + C_{22} s_\varphi^4, \end{aligned} \quad (29.42)$$

with $s_\varphi = \sin \varphi$ and $c_\varphi = \cos \varphi$. Here $E_{11\varphi}$ and $C_{11\varphi}$ are the elasticity and compliance along x_φ , respectively, when (29.8) are rotated by φ . For isotropic material of Poisson's ratio ν , $\Upsilon = 1/(1 - \nu^2)$ and the rule (29.41) is recovered for any φ . Two difficulties, however, arise in the general case:

1. The optimal β_0 depends on orientation of bending actions with respect the element axis. That information is not likely to be known *a priori*.

Table 3 Bending Energy Ratios r for Triangular Elements of Table 1

Triangle	Bending ratio r for isotropic material	$\gamma = 0$ $\nu = \frac{1}{4}$	$\gamma = 1$ $\nu = \frac{1}{4}$	$\gamma = 4$ $\nu = \frac{1}{4}$	$\gamma = 16$ $\nu = \frac{1}{4}$
ALL-3I	$\frac{584 + (79 - 91\nu)\gamma^2 + 6\gamma^4}{432(1 - \nu^2)}$	1.442	1.595	7.457	1007.901
ALL-3M	$\frac{24 + (5 - 9\nu)\gamma^2 + 2\gamma^4}{16(1 - \nu^2)}$	1.600	1.916	38.667	8786.667
ALL-EX	$\frac{84 + (15 - 19\nu)\gamma^2 + 2\gamma^4}{60(1 - \nu^2)}$	1.493	1.711	13.511	2378.311
ALL-LS	$\frac{1672 + (263 - 371\nu)\gamma^2 + 54\gamma^4}{1200(1 - \nu^2)}$	1.486	1.686	16.196	3185.956
CST	$\frac{6 + 3(1 - \nu)\gamma^2}{2(1 - \nu^2)}$	3.200	4.400	22.400	310.400
FF84	$\frac{3}{4} + \frac{13}{96(1 - \nu)} + \frac{13 + 54\gamma^2 + 119\gamma^4 + 70\gamma^6 + 13\gamma^8}{96(1 + 3\gamma^2 + \gamma^4)^2(1 + \nu)}$	1.039	1.020	1.035	1.039
LST-Ret	$\frac{34 + 5(1 - \nu)\gamma^2}{27(1 - \nu^2)}$	1.343	1.491	3.714	39.269
OPT	1	1.000	1.000	1.000	1.000

2. There is no guarantee that $\Upsilon < 4/3$, so $\beta_{0\varphi}$ may turn out to be negative.

The first obstacle is overcome by adopting an invariant measure that involves the average of $E_{11\varphi}C_{11\varphi}$ over a 2π sweep:

$$\bar{\Upsilon} = \frac{1}{2\pi} \int_0^{2\pi} E_{11\varphi} C_{11\varphi} d\varphi = \frac{W}{128 \det(\mathbf{E})}, \quad \bar{\beta}_0 = \frac{2}{\bar{\Upsilon}} - \frac{3}{2}. \quad (29.43)$$

Mathematica gives W as the complicated expression listed in Figure 12. In terms of the elasticity tensor invariants listed in (29.9), $8W = 9J_{E1}^3 + 48J_{E1}^2J_{E2} + J_{E1}(80J_{E2}^2 - 10J_{E3} + J_{E4}) + 8(16J_{E2}^3 - J_{E2}(J_{E3} + J_{E4}) + 72J_{E5})$ so the invariance of $\bar{\beta}_0$ is confirmed. The second difficulty is handled by checking whether $\bar{\beta}_0$ is less than a positive threshold, say, 0.01 and if so setting $\bar{\beta}_0 = 0.01$.

What is the effect of setting a β_0 that is not exactly optimal? Choosing $\alpha_b, \beta_1, \dots, \beta_9$ as listed in (29.40) or (29.41) guarantees that $c_2 = c_4 = 0$ for any $\mathbf{E}$. Consequently the element cannot lock as the aspect ratio γ increases. On the other hand c_0 will generally differ from one so suboptimal performance can be expected if the material is not isotropic.

§29.5.4. Multiple Element Layers

Results of the energy bending test can be readily extended to predict the behavior of $2n$ ($n = 1, 2, \dots$) identical layers of elements symmetrically placed through the beam height. If γ stays constant, the energy ratio becomes

$$r^{(2n)} = \frac{2^{2n} - 1 + r}{2^{2n}}, \quad (29.44)$$

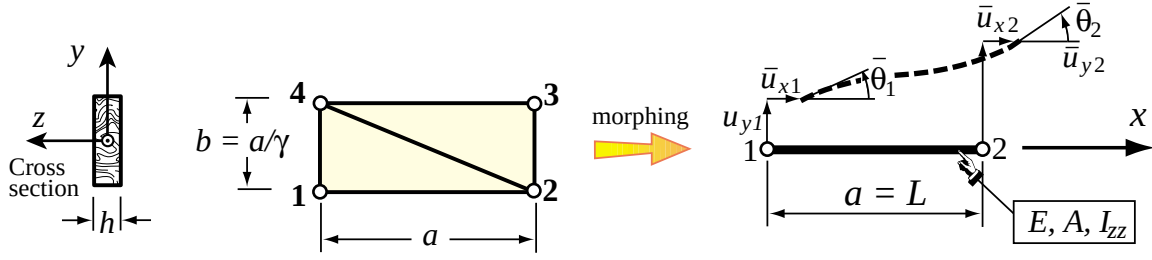


Figure 9. Morphing a 12-DOF two-triangle mesh unit to a 6-DOF beam-column element.

where r is the ratio (29.38) for one layer, as in the configuration of Figure 8. If $r \equiv 1$, $r^{2n} \equiv 1$ so bending exactness is maintained, as can be expected. For example, if $n = 1$ (two element layers), $r^{(2)} = (3 + r)/4$. This is actually the ratio reported in Table 2 of [18].

§29.5.5. Is the Optimal Element Unique?

To investigate whether the optimal element is unique, the common factor A in the matrices (29.30) was replaced by nine values A_{jk} , $j = 1, 2, 3$, $k = 1, 2, 3$. These have dimensions of area but are otherwise arbitrary. A_{jk} is assigned to the $\{j, k\}$ entry of $\mathbf{Q}_1$ and then cyclically carried through $\mathbf{Q}_2$ and $\mathbf{Q}_3$. If the energy orthogonality condition (29.36) is enforced *a priori*, a complete symbolic analysis of the bending ratio was possible with *Mathematica*. Except for β_0 the previous solution (29.43) re-emerges for $r \equiv 1$ in the sense that the $A_{jk} = A$ and the β_i are recovered except for a scaling factor. Absorbing this factor into β_0 the same element is obtained.

If the orthogonality constraint (29.36) is not enforced *a priori*, the energy balance conditions become highly complicated (a system of three quartic polynomial equations emerges with unknown terms $\beta_i \beta_j A_{k\ell} A_{mn}$) and no simple solution was found. Thus the question of whether non-energy-orthogonal optimal elements of this configuration exist remains open.

§29.5.6. Morphing

Morphing means transforming an individual element or macroelement into a simpler model using kinematic constraints. Often the simpler element has lower dimensionality. For example a plate bending macroelement may be morphed to a beam or a torqued shaft [5]. To illustrate the idea consider transforming the rectangular panel of Figure 9 into the two-node Bernoulli-Euler beam-column element shown on the right of that Figure. The length, cross sectional area and moment of inertia of the beam-column element, respectively, are denoted by $L = a$, $A = bh$ and $I_{zz} = b^3h/12 = a^3h/(12\gamma^3)$, respectively.

The transformation between the freedoms of the mesh unit and those of the beam-column is

$$\mathbf{u}_R = \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \\ u_{x3} \\ u_{y3} \\ \theta_3 \\ u_{x4} \\ u_{y4} \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2}b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2}b \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2}b \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -\frac{1}{2}b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{\theta}_1 \\ \bar{u}_{x2} \\ \bar{u}_{y2} \\ \bar{\theta}_2 \end{bmatrix} = \mathbf{T}_m \bar{\mathbf{u}}_m. \quad (29.45)$$

where a superposed bar distinguishes the beam-column freedoms grouped in array $\bar{\mathbf{u}}_m$. Let $\mathbf{K}_R^{unit}$ denote the 12×12 stiffness of the mesh unit of Figure 9 assembled with two optimal LST-3/9R triangles. For isotropic material with $\nu = 0$ a symbolic calculation gives the morphed stiffness

$$\mathbf{K}_m = \mathbf{T}_m^T \mathbf{K}_R^{unit} \mathbf{T}_m = \frac{E}{L} \begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & 12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & -12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L \\ 0 & 6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & -6c_{23}I_{zz}/L & 4c_{33}I_{zz} \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & 12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & -12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L \\ 0 & 6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & -6c_{23}I_{zz}/L & 4c_{33}I_{zz} \end{bmatrix}, \quad (29.46)$$

in which $c_{22} = c_{23} = (1 + 8\gamma^2 + \gamma^4)/12$ and $c_{33} = (5 + 8\gamma^2 + \gamma^4)/16$. The entries in rows/columns 1 and 4 form the well known two-node bar stiffness. Those in rows and columns 2, 3, 5 and 6 are dimensionally homogeneous to those of a C^1 beam, and may be grouped into the following matrix configuration:

$$\mathbf{K}_m^{beam} = \frac{EI_{zz}}{L} \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \bar{\beta} \begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 3 & -6/L & 3 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 3 & -6/L & 3 \end{bmatrix} \right) \quad (29.47)$$

with $\bar{\beta} = c_{22} = c_{23} = (4c_{33} - 1)/3 = (1 + 8\gamma^2 + \gamma^4)/12$. For arbitrary Poisson's ratio, $\bar{\beta} = ((1 - 4\nu^2)(1 + \gamma^4) + \gamma^2(8 - 9\nu + 8\nu^2 - 12\nu^3))/(12(1 - \nu^2))$.

Now (29.47) happens to be the universal template of a prismatic beam, first presented in [4] and further studied, for the C^1 case, in [39,40] using Fourier methods. The basic stiffness on the left characterizes the pure-bending symmetric response to a uniform bending moment, whereas the higher-order stiffness on the right characterizes the antisymmetric response to a linearly-varying, bending moment of zero mean. For the C^1 Bernoulli-Euler beam constructed with cubic shape functions, $\bar{\beta} = 1$. For the C^0 Timoshenko beam, the exact equilibrium model [41, p. 80] is matched by $\bar{\beta} = 1/(1 + \phi)$, $\phi = 12EI_z/(GA_sL^2)$, in which $A_s = 5bh/6$ is the shear area and $G = \frac{1}{2}E/(1 + \nu)$ the shear modulus.

As γ grows the morphed beam template shows that the antisymmetric response, as scaled by $\bar{\beta}$, stiffens rapidly. However, the symmetric response is exact for any γ , which confirms the optimality of the triangular macroelement. Observe also that what was a higher order patch test on the two-triangle mesh

unit becomes a basic (constant-moment) patch test on the morphed element. This is typical of morphing transformations that reduce spatial dimensionality.

What is the difference between morphing and retrofitting? They share reduction techniques but have different goals: the morphed element is not used as a product but as a way to learn about the source element.

§29.5.7. Strain and Stress Recovery

Once node displacements are computed, element strains and stresses can be recovered using the following scheme. Let $\mathbf{u}^e$ denote the compute node displacements. The Cartesian stresses at a point $\{\zeta_1, \zeta_2, \zeta_3\}$ are $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$, in which the Cartesian strains are computed from

$$\mathbf{e} = (\mathbf{L}^T / (Ah)) \mathbf{u}^e + \mathbf{T}_e \beta_0^e (\mathbf{Q}_1 \zeta_1 + \mathbf{Q}_2 \zeta_2 + \mathbf{Q}_3 \zeta_3) \mathbf{T}_{\theta_u} \mathbf{u}^e, \quad (29.48)$$

Here $\mathbf{L}$ is the lumping matrix (29.22) with $\alpha_b = 3/2$, and $\mathbf{Q}_i$ are the matrices (29.28) computed with the optimal parameters (29.41). However β_0^e is not that recommended for $\mathbf{K}^{(e)}$. Least-square fit studies using the method outlined in [42] suggest using $\beta_0^e = 3/2$ for isotropic material. (For the non-isotropic case the best $\beta_0^{(e)}$ is still unknown.) This value is used in the stress results reported in Sections 9ff.

§29.6. A Mathematica Implementation

Figure 10 lists a *Mathematica* implementation of (29.31) as Module LST39RMemTemplateStiffness. The four module arguments are the node coordinates `xycoord` ordered $\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}\}$, the 3×3 stress-strain matrix `Emat`, the thickness `h` and the set of free parameters ordered $\{\alpha_b, \beta_0, \beta_1, \beta_2, \dots, \beta_9\}$. The module returns matrices $\mathbf{K}_b$ and $\mathbf{K}_h$ as list $\{\mathbf{K}_b, \mathbf{K}_h\}$, a separate return of the two matrices being useful for research work.

This implementation emphasizes readability and may be further streamlined. A carefully coded Fortran or C implementation can form $\mathbf{K}_b + \mathbf{K}_h$ in about 500 floating point operations. A f77 version is available from the writer by e-mail. On a 3GHz P4 processor under SUSE Linux, that version was clocked at over 380000 triangles per second. A 18-DOF shell element using this triangle as membrane component can be formed in roughly 2400 floating-point operations.

To expedite “cloning tests” discussed in the Conclusions, the optimal stiffness of a triangle with $x_1 = y_1 = 0$, $x_2 = 4.08$, $y_2 = -3.44$, $x_3 = 3.4$, $y_3 = 1.14$, $E = 120$, $\nu = 1/4$ and $h = 1/8$ is formed and displayed by the statements in Figure 11. (Module LST39ANDESTemplateSignature referenced therein is listed in Figure 12.) `SetPrecision` keeps output entries down to 5 significant places so matrices fit across the page. The results are

$$\mathbf{K}_b = \begin{bmatrix} 10.350 & 0.95258 & 7.7327 & -2.1309 & 1.7629 & 2.1745 & -8.2194 & -2.7155 & -9.9073 \\ 0.95258 & 4.0758 & 8.1695 & 2.7629 & 0.17327 & -5.7458 & -3.7155 & -4.2491 & -2.4237 \\ 7.7327 & 8.1695 & 19.723 & 3.9414 & 1.3377 & -9.4943 & -11.674 & -9.5072 & -10.229 \\ -2.1309 & 2.7629 & 3.9414 & 2.7575 & -1.1855 & -4.1179 & -0.62660 & -1.5774 & 0.17657 \\ 1.7629 & 0.17327 & 1.3377 & -1.1855 & 5.8957 & -4.6390 & -0.57737 & -6.0690 & 3.3013 \\ 2.1745 & -5.7458 & -9.4943 & -4.1179 & -4.6390 & 13.777 & 1.9434 & 10.385 & -4.2825 \\ -8.2194 & -3.7155 & -11.674 & -0.62660 & -0.57737 & 1.9434 & 8.8460 & 4.2928 & 9.7307 \\ -2.7155 & -4.2491 & -9.5072 & -1.5774 & -6.0690 & 10.385 & 4.2928 & 10.318 & -0.87754 \\ -9.9073 & -2.4237 & -10.229 & 0.17657 & 3.3013 & -4.2825 & 9.7307 & -0.87754 & 14.512 \end{bmatrix} \quad (29.49)$$


```

LST39RMembTemplateStiffness [xycoor_, Emat_, h_, fpars_] := Module[
  {x1, y1, x2, y2, x3, y3, x12, y12, x21, y21, x23, y23, x32, y32, x31, y31,
   x13, y13, A, A4, V, LL21, LL32, LL13, αb, β0, β1, β2, β3, β4, β5, β6, β7, β8, β9,
   Te, Tθu, Q1, Q2, Q3, Q4, Q5, Q6, c, L, Enat, Kθ, Kh, Kb, Ke},
  {{x1, y1}, {x2, y2}, {x3, y3}} = xycoor;
  {αb, β0, β1, β2, β3, β4, β5, β6, β7, β8, β9} = fpars;
  x12 = x1 - x2; x23 = x2 - x3; x31 = x3 - x1; x21 = -x12; x32 = -x23; x13 = -x31;
  y12 = y1 - y2; y23 = y2 - y3; y31 = y3 - y1; y21 = -y12; y32 = -y23; y13 = -y31;
  A = (y21*x13 - x21*y13)/2; A2 = 2*A; A4 = 4*A;
  L = {{y23, 0, x32}, {0, x32, y23},
        {y23*(y13 - y21), x32*(x31 - x12), (x31*y13 - x12*y21)*2}*αb/6,
        {y31, 0, x13}, {0, x13, y31},
        {y31*(y21 - y32), x13*(x12 - x23), (x12*y21 - x23*y32)*2}*αb/6,
        {y12, 0, x21}, {0, x21, y12},
        {y12*(y32 - y13), x21*(x23 - x31), (x23*y32 - x31*y13)*2}*αb/6}*h/2;
  Kb = (L . Emat . Transpose[L])/(h*A);
  Tθu = {{x32, y32, A4, x13, y13, 0, x21, y21, 0},
          {x32, y32, 0, x13, y13, A4, x21, y21, 0},
          {x32, y32, 0, x13, y13, 0, x21, y21, A4}}/A4;
  LL21 = x21^2 + y21^2; LL32 = x32^2 + y32^2; LL13 = x13^2 + y13^2;
  Te = {{y23*y13*LL21, y31*y21*LL32, y12*y32*LL13},
        {x23*x13*LL21, x31*x21*LL32, x12*x32*LL13},
        {(y23*x31 + x32*y13)*LL21, (y31*x12 + x13*y21)*LL32,
         (y12*x23 + x21*y32)*LL13}}/(A*A4);
  Q1 = {{β1, β2, β3}/LL21, {β4, β5, β6}/LL32, {β7, β8, β9}/LL13}*A2/3;
  Q2 = {{β9, β7, β8}/LL21, {β3, β1, β2}/LL32, {β6, β4, β5}/LL13}*A2/3;
  Q3 = {{β5, β6, β4}/LL21, {β8, β9, β7}/LL32, {β2, β3, β1}/LL13}*A2/3;
  Q4 = (Q1 + Q2)/2; Q5 = (Q2 + Q3)/2; Q6 = (Q3 + Q1)/2;
  Enat = Transpose[Te] . Emat . Te;
  Kθ = (3/4)*β0*h*A*(Transpose[Q4] . Enat . Q4 + Transpose[Q5] . Enat . Q5 +
    Transpose[Q6] . Enat . Q6);
  Kh = Transpose[Tθu] . Kθ . Tθu;
  Return[{Kb, Kh}];

```

Figure 10. A Mathematica implementation of the LST-3/9R ANDES template (31). A f77 version clocked at over 380000 trigs/sec on a 3GHz P4 is available from the writer by e-mail.

$$\mathbf{K}_h = \begin{bmatrix}
0.041538 & -0.27977 & -0.63728 & 0.20769 & 0.069638 & -0.52781 & -0.24923 & 0.21014 & -0.83208 \\
-0.27977 & 1.8844 & 4.2923 & -1.3989 & -0.46903 & 3.5549 & 1.6786 & -1.4153 & 5.6043 \\
-0.63728 & 4.2923 & 12.833 & -3.1864 & -1.0684 & 7.7151 & 3.8237 & -3.2239 & 10.092 \\
0.20769 & -1.3989 & -3.1864 & 1.0385 & 0.34819 & -2.6390 & -1.2462 & 1.0507 & -4.1604 \\
0.069638 & -0.46903 & -1.0684 & 0.34819 & 0.11675 & -0.88485 & -0.41783 & 0.35229 & -1.3950 \\
-0.52781 & 3.5549 & 7.7151 & -2.6390 & -0.88485 & 7.9515 & 3.1668 & -2.6701 & 9.7104 \\
-0.24923 & 1.6786 & 3.8237 & -1.2462 & -0.41783 & 3.1668 & 1.4954 & -1.2608 & 4.9925 \\
0.21014 & -1.4153 & -3.2239 & 1.0507 & 0.35229 & -2.6701 & -1.2608 & 1.0630 & -4.2093 \\
-0.83208 & 5.6043 & 10.092 & -4.1604 & -1.3950 & 9.7104 & 4.9925 & -4.2093 & 20.204
\end{bmatrix} \quad (29.50)$$

$$\mathbf{K}_R = \mathbf{K}_b + \mathbf{K}_h =$$

$$\begin{bmatrix} 10.392 & 0.67281 & 7.0955 & -1.9232 & 1.8325 & 1.6467 & -8.4687 & -2.5053 & -10.739 \\ 0.67281 & 5.9602 & 12.462 & 1.3640 & -0.29577 & -2.1909 & -2.0368 & -5.6644 & 3.1805 \\ 7.0955 & 12.462 & 32.557 & 0.75498 & 0.26929 & -1.7792 & -7.8504 & -12.731 & -0.13702 \\ -1.9232 & 1.3640 & 0.75498 & 3.7959 & -0.83734 & -6.7570 & -1.8728 & -0.52670 & -3.9838 \\ 1.8325 & -0.29577 & 0.26929 & -0.83734 & 6.0125 & -5.5238 & -0.99520 & -5.7167 & 1.9063 \\ 1.6467 & -2.1909 & -1.7792 & -6.7570 & -5.5238 & 21.728 & 5.1102 & 7.7147 & 5.4278 \\ -8.4687 & -2.0368 & -7.8504 & -1.8728 & -0.99520 & 5.1102 & 10.341 & 3.0320 & 14.723 \\ -2.5053 & -5.6644 & -12.731 & -0.52670 & -5.7167 & 7.7147 & 3.0320 & 11.381 & -5.0869 \\ -10.739 & 3.1805 & -0.13702 & -3.9838 & 1.9063 & 5.4278 & 14.723 & -5.0869 & 34.716 \end{bmatrix} \quad (29.51)$$

```
xycoor=N[{0,0},{102/25,-86/25},{85/25,285/250}];
Print["xycoor=",xycoor];
Em=120; v=1/4; h=1/8;
Emat=Em/(1-v^2)*{{1,v,0},{v,1,0},{0,0,(1-v)/2}};
fpars= LST39RANDESTemplateSignature["OPT", Emat];
{Kb,Kh}=LST39RMembTemplateStiffness [xycoor,Emat,h,fpars];
Print["Kb=",SetPrecision[N[Kb],5]//MatrixForm];
Print["Kh=",SetPrecision[N[Kh],5]//MatrixForm];
Print["Ke=",SetPrecision[N[Kb+Kh],5]//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Kb+Kh]]]]];
```

Figure 11. Test statements that produce the matrices (29.49)–(29.51).

```

LST39RANDETemplateSignature[name_, Emat_] := Module[{fpars,
  E11, E22, E33, E12, E13, E23, Edet, W, E11C11avg,  $\beta$ 0},
  fpars={0,0,0,0,0,0,0,0,0,0,0,0,0}; (* CST *)
  If [name=="OPT",
    {{E11, E12, E13}, {E12, E22, E23}, {E13, E23, E33}}=Emat;
    Edet=E11*E22*E33+2*E12*E13*E23-E11*E23^2-E22*E13^2-E33*E12^2;
    W=-6*E12^3+5*E11^2*E22-5*E12^2*E22-E22*(75*E13^2+
    14*E13*E23+3*E23^2)+2*E12*(7*E13^2+46*E13*E23+7*E23^2)-
    E11*(5*E12^2+3*E13^2-6*E12*E22-5*E22^2+14*E13*E23+75*E23^2)+
    (3*E11^2+82*E11*E22+3*E22^2-4*(6*E12^2+5*E13^2-6*E13*E23+
    5*E23^2))*E33+4*(5*E11-6*E12+5*E22)*E33^2;
    E11C11avg=W/(128*Edet);  $\beta$ 0=Max[2/E11C11avg-3/2, 1/100];
    fpars={3/2,  $\beta$ 0, 1, 2, 1, 0, 1, -1, -1, -1, -2}}];
  If [name=="ALL3I",
    fpars={ 1, 4/9, 1/12, 5/12, 1/2, 0, 1/3, -1/3, -1/12, -1/2, -5/12}];
  If [name=="ALL3M",
    fpars={ 1, 4/9, 1/4, 5/4, 3/2, 0, 1, -1, -1/4, -3/2, -5/4}];
  If [name=="ALLLS",
    fpars={ 1, 4/9, 3/20, 3/4, 9/10, 0, 3/5, -3/5, -3/20, -9/10, -3/4}];
  If [name=="LSTRet",
    fpars={4/3, 1/2, 2/3, -2/3, 0, 0, -4/3, 4/3, -2/3, 0, 2/3}];
  Return[fpars]];

```

Figure 12. Module to return signature given a template instance name.

The eigenvalues of $\mathbf{K}_R$ to 5 places are [52.913 43.834 26.434 11.181 1.8722 0.64900 0 0 0].

When doing element comparison studies as in Section 9 it is convenient to pass from a user supplied mnemonic name to the set of free parameters (template signature). Module `LST39ANDESTemplateSignature`, listed in Figure 12, returns the template signature given an

mnemonic type name. The second argument, which is the stress-strain matrix $\mathbf{E}$, is only used for type "OPT". For example `LST39ANDESTemplateSignature["LSTRet",0]` returns `{4/3, 1/2, 2/3, -2/3, 0, 0, -4/3, 4/3, -2/3, 0, 2/3}` as the set of free parameters for the retrofitted LST derived in the next section.

§29.7. Retrofitting LST

The process of building templates by direct fabrication may look like black magic to the uninitiated. By comparison, retrofitting a well known “parent” element appears straightforward because kinematic constraints and congruential transformations are taught in introductory FEM courses.

§29.7.1. Midpoint Migration Migraines

The idea explored in this section is to start with the conventional Linear Strain Triangle depicted in Figure 3(a). The LST-3/12C has 12 DOFs arranged as

$$\mathbf{u}_C = [u_{x1} \ u_{y1} \ u_{x2} \ u_{y2} \ u_{x3} \ u_{y3} \ u_{x4} \ u_{y4} \ u_{x5} \ u_{y5} \ u_{x6} \ u_{y6}]^T. \quad (29.52)$$

The triangle has straight sides, with nodes 4, 5 and 6 at the midpoints. (It makes no sense to start with the more general curved-side iso-P element since these nodes are eventually eliminated.) The stiffness of this superparametric triangle is readily computed in closed form since the Jacobian is constant. For constant $\mathbf{E}$ and h , a very simple expression, discovered in 1966 [10], is

$$\mathbf{K}_C = \frac{1}{3}Ah (\mathbf{B}_1^T \mathbf{E} \mathbf{B}_1 + \mathbf{B}_2^T \mathbf{E} \mathbf{B}_2 + \mathbf{B}_3^T \mathbf{E} \mathbf{B}_3) \quad (29.53)$$

in which A is the triangle area and

$$\begin{aligned} \mathbf{B}_1 &= \frac{1}{2A} \begin{bmatrix} y_{32} & 0 & y_{31} & 0 & y_{12} & 0 & 2y_{23} & 0 & 2y_{32} & 0 & 2y_{23} & 0 \\ 0 & x_{23} & 0 & x_{13} & 0 & x_{21} & 0 & 2x_{32} & 0 & 2x_{23} & 0 & 2x_{32} \\ x_{23} & y_{32} & x_{13} & y_{31} & x_{21} & y_{12} & 2x_{32} & 2y_{23} & 2x_{23} & 2y_{32} & 2x_{32} & 2y_{23} \end{bmatrix} \\ \mathbf{B}_2 &= \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{13} & 0 & y_{12} & 0 & 2y_{31} & 0 & 2y_{31} & 0 & 2y_{13} & 0 \\ 0 & x_{32} & 0 & x_{31} & 0 & x_{21} & 0 & 2x_{13} & 0 & 2x_{13} & 0 & 2x_{31} \\ x_{32} & y_{23} & x_{31} & y_{13} & x_{21} & y_{12} & 2x_{13} & 2y_{31} & 2x_{13} & 2y_{31} & 2x_{31} & 2y_{13} \end{bmatrix} \\ \mathbf{B}_3 &= \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{21} & 0 & 2y_{21} & 0 & 2y_{12} & 0 & 2y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{12} & 0 & 2x_{12} & 0 & 2x_{21} & 0 & 2x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{12} & y_{21} & 2x_{12} & 2y_{21} & 2x_{21} & 2y_{12} & 2x_{21} & 2y_{12} \end{bmatrix} \end{aligned} \quad (29.54)$$

Next, establish by some method a “midpoint migration” 12×9 transformation matrix $\mathbf{T}_{CR}$ that links $\mathbf{u}_C = \mathbf{T}_{CR} \mathbf{u}_R$, where the latter is configured as per (29.6). Then the LST-3/9R stiffness is

$$\mathbf{K}_L = \mathbf{T}_{CR}^T \mathbf{K}_C \mathbf{T}_{CR} \quad (29.55)$$

where the subscript distinguishes this stiffness from that derived in the previous section. One advantage of this technique is that $\mathbf{T}_{CR}$ can be reused to produce consistent mass, geometric stiffness matrices and consistent forces from those of the LST-6/12C, which are well known and available in many codes.

If this sounds too good to be true, it is. Three things may go wrong. First, there is no guarantee that (29.55) will pass the patch test. Generally it will not: as shown below, the $\mathbf{T}_{CR}$ form that does it while avoiding the second problem is very special. Second, the transformation may “blow up” for some triangle geometries. Third, the element may be rank deficient.

There are no easy cures for any of these mishaps. Repeated failures during the 1970s prompted Irons and Ahmad in 1980 to proclaim, a bit prematurely, that trying to construct membrane elements with drilling freedoms was a waste of time [43, p. 289]. To be fair, some of the tools used here were not known at the time. In particular the fundamental decomposition (29.18), which facilitates passing the patch test *a priori*, came after their book.

§29.7.2. Divide and Conquer

Begin by decomposing the stiffness $\mathbf{K}_C$ of (29.53) into basic and higher order: $\mathbf{K}_C = \mathbf{K}_{Cb} + \mathbf{K}_{Ch}$. It is easily checked that for constant $\mathbf{E}$ and h

$$\mathbf{K}_{Cb} = A h \mathbf{B}_0^T \mathbf{E} \mathbf{B}_0, \quad \mathbf{K}_{Ch} = \frac{1}{3} A h (\mathbf{B}_{h1}^T \mathbf{E} \mathbf{B}_{h1} + \mathbf{B}_{h2}^T \mathbf{E} \mathbf{B}_{h2} + \mathbf{B}_{h3}^T \mathbf{E} \mathbf{B}_{h3}). \quad (29.56)$$

in which $\mathbf{B}_0 = \frac{1}{3}(\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3)$ and $\mathbf{B}_{hi} = \mathbf{B}_i - \mathbf{B}_0$, $i = 1, 2, 3$. Explicit forms of the strain-displacement matrices are

$$\begin{aligned} \mathbf{B}_0 &= \frac{1}{6A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 & 4y_{21} & 0 & 4y_{32} & 0 & 4y_{13} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} & 0 & 4x_{12} & 0 & 4x_{23} & 0 & 4x_{31} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} & 4x_{12} & 4y_{21} & 4x_{23} & 4y_{32} & 4x_{31} & 4y_{13} \end{bmatrix} \\ \mathbf{B}_{h1} &= \frac{1}{3A} \begin{bmatrix} 2y_{32} & 0 & y_{31} & 0 & y_{12} & 0 & 2y_{13}+y_{23} & 0 & y_{32} & 0 & 2y_{21}+y_{23} & 0 \\ 0 & 2x_{23} & 0 & x_{13} & 0 & x_{21} & 0 & x_{13}+2x_{23} & 0 & x_{23} & 0 & 2x_{12}+x_{32} \\ 2x_{23} & 2y_{32} & x_{13} & y_{31} & x_{21} & y_{12} & x_{13}+2x_{23} & 2y_{13}+y_{23} & x_{23} & y_{32} & 2x_{12}+x_{32} & 2y_{21}+y_{23} \end{bmatrix} \\ \mathbf{B}_{h2} &= \frac{1}{3A} \begin{bmatrix} y_{23} & 0 & 2y_{13} & 0 & y_{12} & 0 & y_{31}+2y_{32} & 0 & 2y_{21}+y_{31} & 0 & y_{13} & 0 \\ 0 & x_{32} & 0 & 2x_{31} & 0 & x_{21} & 0 & x_{13}+2x_{23} & 0 & 2x_{12}+x_{13} & 0 & x_{31} \\ x_{32} & y_{23} & 2x_{31} & 2y_{13} & x_{21} & y_{12} & x_{13}+2x_{23} & y_{31}+2y_{32} & 2x_{12}+x_{13} & 2y_{21}+y_{31} & x_{31} & y_{13} \end{bmatrix} \\ \mathbf{B}_{h3} &= \frac{1}{3A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & 2y_{21} & 0 & y_{21} & 0 & 2y_3+y_{12} & 0 & y_{12}+2y_{32} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & 2x_{12} & 0 & x_{12} & 0 & x_{21}+2x_{31} & 0 & x_{21}+2x_{23} \\ x_{32} & y_{23} & x_{13} & y_{31} & 2x_{12} & 2y_{21} & x_{12} & y_{21} & x_{21}+2x_{31} & 2y_3+y_{12} & x_{21}+2x_{23} & y_{12}+2y_{32} \end{bmatrix} \end{aligned} \quad (29.57)$$

While $\mathbf{B}_0$ is unique, there are several $\mathbf{B}_{hi}$ matrices that produce the same $\mathbf{K}_{Ch}$ through the formula in (29.56). This indeterminacy causes no problems, however, since $\mathbf{K}_{Ch}$ is unique because $\mathbf{K}_C$ and $\mathbf{K}_{Cb}$ are. Insertion of $\mathbf{K}_C = \mathbf{K}_{Cb} + \mathbf{K}_{Ch}$ into (29.55) shows that

$$\mathbf{K}_L = \mathbf{K}_{Lb} + \mathbf{K}_{Lh}, \quad \mathbf{K}_{Lb} = \mathbf{T}_{CR}^T \mathbf{K}_{Cb} \mathbf{T}_{CR}, \quad \mathbf{K}_{Lh} = \mathbf{T}_{CR}^T \mathbf{K}_{Ch} \mathbf{T}_{CR}. \quad (29.58)$$

Next, we make use of the hierarchical form of the LST introduced in [44, p. 222]. This variant, labelled LST-6/20CH, has the same nodes as the LST-6/12C but the midpoint freedoms are deviations from linearity:

$$\tilde{\mathbf{u}}_C = [u_{x1} \ u_{y1} \ u_{x2} \ u_{y2} \ u_{x3} \ u_{y3} \ \tilde{u}_{x4} \ \tilde{u}_{y4} \ \tilde{u}_{x5} \ \tilde{u}_{y5} \ \tilde{u}_{x6} \ \tilde{u}_{y6}]^T. \quad (29.59)$$

where $\tilde{u}_{x4} = u_{x4} - \frac{1}{2}(u_{x1} + u_{x2})$, $\tilde{u}_{y4} = u_{y4} - \frac{1}{2}(u_{y1} + u_{y2})$, etc. The freedoms of the conventional and hierarchical LST are related by the transformation

$$\mathbf{u}_C = \mathbf{T}_{CH} \tilde{\mathbf{u}}_C, \quad \tilde{\mathbf{u}}_C = \mathbf{T}_{HC} \mathbf{u}_C, \quad (29.60)$$

in which

$$\mathbf{T}_{CH} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (29.61)$$

The inverse $\mathbf{T}_{HC} = \mathbf{T}_{CH}^{-1}$ is obtained by changing all $\frac{1}{2}$ to $-\frac{1}{2}$. It follows that

$$\tilde{\mathbf{K}}_C = \mathbf{T}_{HC}^T \mathbf{K}_C \mathbf{T}_{HC} = \tilde{\mathbf{K}}_{Cb} + \tilde{\mathbf{K}}_{Ch}, \quad \tilde{\mathbf{K}}_{Cb} = \mathbf{T}_{HC}^T \mathbf{K}_{Cb} \mathbf{T}_{HC}, \quad \tilde{\mathbf{K}}_{Ch} = \mathbf{T}_{HC}^T \mathbf{K}_{Ch} \mathbf{T}_{HC}. \quad (29.62)$$

From the Free Formulation let us borrow a couple of modal-basis matrices:

$$\mathbf{G}_{Cb} = \begin{bmatrix} 1 & 0 & y_1 & x_1 & 0 & y_1 \\ 0 & 1 & -x_1 & 0 & y_1 & x_1 \\ 1 & 0 & y_2 & x_2 & 0 & y_2 \\ 0 & 1 & -x_2 & 0 & y_2 & x_2 \\ 1 & 0 & y_3 & x_3 & 0 & y_3 \\ 0 & 1 & -x_3 & 0 & y_3 & x_3 \\ 1 & 0 & y_4 & x_4 & 0 & y_4 \\ 0 & 1 & -x_4 & 0 & y_4 & x_4 \\ 1 & 0 & y_5 & x_5 & 0 & y_5 \\ 0 & 1 & -x_5 & 0 & y_5 & x_5 \\ 1 & 0 & y_6 & x_6 & 0 & y_6 \\ 0 & 1 & -x_6 & 0 & y_6 & x_6 \end{bmatrix}, \quad \mathbf{G}_{Rb} = \begin{bmatrix} 1 & 0 & y_1 & x_1 & 0 & y_1 \\ 0 & 1 & -x_1 & 0 & y_1 & x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & y_2 & x_2 & 0 & y_2 \\ 0 & 1 & -x_2 & 0 & y_2 & x_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & y_3 & x_3 & 0 & y_3 \\ 0 & 1 & -x_3 & 0 & y_3 & x_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (29.63)$$

and their hierarchical counterparts:

$$\tilde{\mathbf{G}}_{Cb} = \mathbf{G}_{Cb} \mathbf{T}_{CR} = \begin{bmatrix} 1 & 0 & y_1 & x_1 & 0 & y_1 \\ 0 & 1 & -x_1 & 0 & y_1 & x_1 \\ 1 & 0 & y_2 & x_2 & 0 & y_2 \\ 0 & 1 & -x_2 & 0 & y_2 & x_2 \\ 1 & 0 & y_3 & x_3 & 0 & y_3 \\ 0 & 1 & -x_3 & 0 & y_3 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{G}}_{Rb} = \mathbf{G}_{Rb} \mathbf{T}_{u\bar{u}} = \begin{bmatrix} 1 & 0 & y_1 & x_1 & 0 & y_1 \\ 0 & 1 & -x_1 & 0 & y_1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & y_2 & x_2 & 0 & y_2 \\ 0 & 1 & -x_2 & 0 & y_2 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & y_3 & x_3 & 0 & y_3 \\ 0 & 1 & -x_3 & 0 & y_3 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (29.64)$$

The first three columns of each matrix span the rigid body modes and the last three the constant strain modes evaluated at the nodes (these bases are not orthonormalized since that property is not required here.) Collectively these six columns are called the basic kinematic modes, hence the subscript.

The hierarchical form of $\mathbf{T}_{CR}$ is $\tilde{\mathbf{T}}_{CR}$, connecting $\tilde{\mathbf{K}}_L = \tilde{\mathbf{T}}_{CR}^T \mathbf{K}_R \tilde{\mathbf{T}}_{CR}$. This 12×9 matrix has 108 entries. But 90 are immediately set to 1 or 0 by the obvious assumption: the hierarchical rotations $\tilde{\theta}_i$, $i = 1, 2, 3$ are defined only by the hierarchical midpoint displacements $\tilde{u}_{xm}$, $\tilde{u}_{ym}$, $m = 4, 5, 6$. Consequently $\tilde{\mathbf{T}}_{CR}$ must have the form

$$\tilde{\mathbf{T}}_{CR} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & q_{12} & 0 & 0 & q_{21} & 0 & 0 & q_{33} \\ 0 & 0 & p_{21} & 0 & 0 & p_{12} & 0 & 0 & p_{33} \\ 0 & 0 & q_{11} & 0 & 0 & q_{23} & 0 & 0 & q_{32} \\ 0 & 0 & q_{11} & 0 & 0 & p_{32} & 0 & 0 & p_{23} \\ 0 & 0 & q_{13} & 0 & 0 & q_{22} & 0 & 0 & q_{31} \\ 0 & 0 & p_{31} & 0 & 0 & p_{22} & 0 & 0 & p_{13} \end{bmatrix} \quad (29.65)$$

Here p_{ij} and q_{ij} are 18 undetermined coefficients with dimension of length. Now $\tilde{\mathbf{T}}_{CR}$ must preserve rigid body modes and constant strain states in the hierarchical elements. This requirement is expressed as

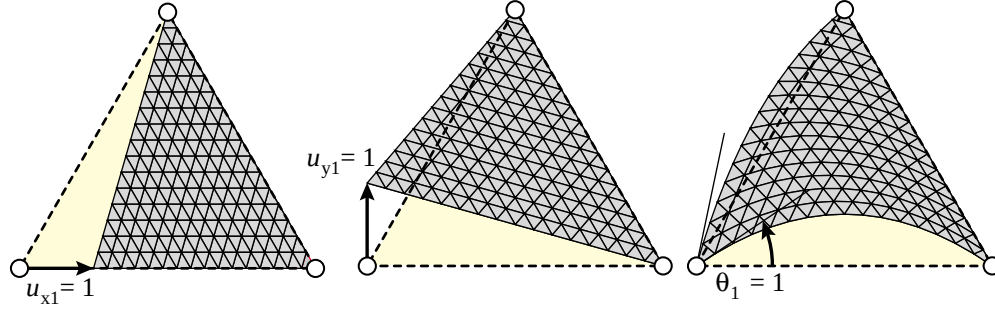
$$\tilde{\mathbf{G}}_{Cb} = \tilde{\mathbf{T}}_{CR} \tilde{\mathbf{G}}_{Rb} \quad (29.66)$$

which gives six nontrivial conditions. These are satisfied by taking $q_{11} = -(q_{23} + q_{32})$, $p_{11} = -(p_{23} + p_{32})$, $q_{22} = -(q_{13} + q_{31})$, $p_{22} = -(p_{13} + p_{31})$, $q_{33} = -(q_{12} + q_{21})$ and $p_{33} = -(p_{12} + p_{21})$.

Transforming to non-hierarchical freedoms gives $\mathbf{T}_{CR} = \mathbf{T}_{HC} \tilde{\mathbf{T}}_{CR}$. Matrix $\mathbf{T}_{CR}$ gains twelve $\frac{1}{2}$ entries at midpoint locations but still has 12 undetermined coefficients.

Finally, require that the Individual Element Test (IET) be passed by forcing the basic stiffness matrices to coalesce: $\mathbf{T}_{RC}^T \mathbf{K}_{Cb} \mathbf{T}_{RC} = V \mathbf{T}_{RC}^T \mathbf{B}_0^T \mathbf{E} \mathbf{B}_0 \mathbf{T}_{RC} = V^{-1} \mathbf{L} \mathbf{E} \mathbf{L}^T$. Hence $V \mathbf{B}_0 \mathbf{T}_{RC} = \mathbf{L}^T$, where $\mathbf{L}$ is given in equation (29.24). The solution for p_{ij} and q_{ij} contain divisors that can vanish for certain triangular geometries unless one sets $p_{21} = -p_{12}$, $q_{21} = -q_{12}$, $p_{32} = -p_{23}$, $q_{32} = -q_{23}$, $p_{31} = -p_{13}$ and $q_{31} = -q_{13}$, whence $p_{11} = q_{11} = p_{22} = q_{22} = p_{33} = q_{33} = 0$. On doing that the unique solution is found to be $p_{12} = \frac{1}{8} \alpha_b x_{12}$, $p_{23} = \frac{1}{8} \alpha_b x_{23}$, $p_{13} = \frac{1}{8} \alpha_b x_{13}$, $q_{12} = \frac{1}{8} \alpha_b y_{12}$, $q_{13} = \frac{1}{8} \alpha_b y_{13}$, $q_{23} = \frac{1}{8} \alpha_b y_{23}$. Hence the only transformation that avoids singularities while satisfying the IET is

$$\mathbf{T}_{CR} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{8} \alpha_b y_{12} & 0 & 0 & \frac{1}{8} \alpha_b y_{21} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{8} \alpha_b x_{21} & 0 & 0 & \frac{1}{8} \alpha_b x_{12} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{8} \alpha_b y_{23} & \frac{1}{2} & 0 & \frac{1}{8} \alpha_b y_{32} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} \alpha_b x_{32} & 0 & \frac{1}{2} & \frac{1}{8} \alpha_b x_{23} \\ \frac{1}{2} & 0 & \frac{1}{8} \alpha_b y_{13} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{8} \alpha_b y_{31} \\ 0 & \frac{1}{2} & \frac{1}{8} \alpha_b x_{31} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} \alpha_b x_{13} \end{bmatrix} \quad (29.67)$$

Figure 13. Shape functions for corner 1 of retrofitted LST, $\alpha_b = 1$.

By inspection, if $\alpha_b \neq 0$ this matrix has the nontrivial null vector:

$$\mathbf{T}\mathbf{u}_R^{null} = \mathbf{0}, \quad \mathbf{u}_R^{null} = [0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1]^T. \quad (29.68)$$

If $\alpha_b = 0$ the dimension of the null space grows to three.

The shape functions of this element are obtained on postmultiplying the shape functions of the LST-6/12 by $\mathbf{T}_{CR}$. The interpolation is

$$\begin{aligned} u_x &= N_{xx1}u_{x1} + N_{xy2}u_{y1} + N_{x\theta1}\theta_1 + \dots + N_{x\theta3}\theta_3, \\ u_y &= N_{yx1}u_{x1} + N_{yy2}u_{y1} + N_{y\theta1}\theta_1 + \dots + N_{y\theta3}\theta_3. \end{aligned} \quad (29.69)$$

where $N_{xxi} = N_{yyi} = \zeta_i$ ($i = 1, 2, 3$), $N_{xyi} = N_{yxi} = 0$ ($i = 1, 2, 3$), $N_{x\theta1} = \alpha\zeta_1(y_{12}\zeta_2 - y_{31}\zeta_3)/2$, $N_{x\theta2} = \alpha\zeta_2(y_{23}\zeta_3 - y_{12}\zeta_1)/2$, $N_{x\theta3} = \alpha\zeta_3(y_{31}\zeta_1 - y_{23}\zeta_2)/2$, $N_{y\theta1} = \alpha\zeta_1(x_{21}\zeta_2 - x_{13}\zeta_3)/2$, $N_{y\theta2} = \alpha\zeta_2(x_{32}\zeta_3 - x_{21}\zeta_1)/2$, $N_{y\theta3} = \alpha\zeta_3(x_{13}\zeta_1 - x_{32}\zeta_2)/2$. The shape functions for the three freedoms of corner node 1 are plotted in Figure 13. Note that the translational node displacement shape functions are the same as those of the CST-3/6C, and thus conforming. The shape functions for the rotational freedoms are quadratic and nonconforming.

§29.7.3. Stiffness Matrix Assessment

It is easily verified that $\mathbf{K}_L$ is an instance of the ANDES template (29.31), with

$$\alpha_b = \alpha, \quad \beta_0 = \frac{1}{2}, \quad \beta_1 = \beta_9 = \frac{1}{2}\alpha, \quad \beta_2 = \beta_7 = -\beta_1, \quad \beta_3 = \beta_4 = \beta_8 = 0, \quad \beta_5 = -\alpha, \quad \beta_6 = \alpha. \quad (29.70)$$

Computing the bending energy ratio gives for an isotropic material

$$r = c_0 + c_2\gamma^2, \quad \text{with} \quad c_0 = \frac{144 - 96\alpha_b + 25\alpha_b^2}{48(1 - \nu^2)}, \quad c_2 = \frac{144 - 192\alpha_b + 73\alpha_b^2}{96(1 + \nu)}. \quad (29.71)$$

The α_b roots of $c_2 = 0$ are imaginary, so the element cannot be bending optimal. However, the dependence of r on γ is fairly mild compared to that of the elements with $c_4 \neq 0$. Coefficient c_2 is minimized by taking $\alpha_b = 96/73 = 1.31507 \dots \approx 4/3$.

Because of (29.68) the element is rank deficient by one for all $\alpha_b > 0$ with $\mathbf{u}_R^{null}$ as spurious mode. This could be cured by adding the torsional mode of Figure 7 as stabilization mode. This stabilization leads to the cubic elements studied in the next Section. The bad news is that stabilization worsens significantly the aspect ratio locking.

§29.7.4. Deriving a Mass Matrix

An advantage previously noted for the retrofitting approach is that it readily produces mass matrices, geometric stiffness matrices and consistent load vectors from those of the parent element. We give here an application of this approach for deriving a mass matrix. The consistent mass matrix of a LST-6/12C triangle of uniform thickness and mass density ρ is

$$\mathbf{M}_C = \frac{hA\rho}{180} \begin{bmatrix} 6 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 6 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -4 & 0 & 0 \\ -1 & 0 & 6 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & -1 & 0 & 6 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -4 \\ -1 & 0 & -1 & 0 & 6 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 6 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 32 & 0 & 16 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 32 & 0 & 16 & 0 & 16 \\ -4 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 32 & 0 & 16 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 32 & 0 & 16 \\ 0 & 0 & -4 & 0 & 0 & 0 & 16 & 0 & 16 & 0 & 32 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 16 & 0 & 16 & 0 & 32 \end{bmatrix} \quad (29.72)$$

Using the transformation (29.67), a possible mass matrix for the LST-6/12R configuration is

$$\mathbf{M}_L = \mathbf{T}_{CR}^T \mathbf{M}_C \mathbf{T}_{CR}, \quad (29.73)$$

which is easily obtained in closed form. This mass matrix has a rank deficiency of one if $\alpha \neq 0$, with the null eigenvector (29.68). However rank deficiency of mass matrices is not that important as in stiffness matrices, unless they are used in explicit transient analysis.

The mass matrix $\mathbf{M}_L$ depends on the parameter α present in $\mathbf{T}_{RC}$. To find the best value α_M when this mass is paired to the OPT stiffness it is convenient to morph the mass into that of a Bernoulli-Euler beam following the technique outlined in Section 5.6: $\mathbf{M}_{beam} = \mathbf{T}_m^T \mathbf{M}_L \mathbf{T}_m$. Then the plane-wave Fourier analysis described in [39,40] on a repeating beam lattice is carried out. The end result for the computed nondimensional acoustic frequency is the dispersion relation

$$\omega_a^2 = k^4 + \frac{1 + (17 + 2\alpha)\gamma^2 + (16\alpha - 23)\gamma^4 + (2\alpha - 3)\gamma^6}{12\gamma^2(1 + 8\gamma^2 + \gamma^4)} k^6 + O(k^8) \quad (29.74)$$

where k is the wavenumber. The exact nondimensional acoustic frequency is $\omega_a^2 = k^4$. Because of the dependence on γ , the term in k^6 cannot be made to vanish for a specific α . But it rapidly tends to zero for $\gamma \gg 1$ if $\alpha_M = \alpha = 3/2$, which is the recommended value. (Note that this differs from the recommended value for the retrofitted stiffness, which is $\alpha_b = 4/3$. But the mass-stiffness pairing of $\mathbf{M}_L$ is not with $\mathbf{K}_L$, but with that of the optimal element.)

§29.8. The Allman 1988 Triangle

In 1984 Allman published [45] a drilling freedom triangle based on quadratic shape functions. This element is a clone of that presented in Section 7 if one takes $\alpha_b = 1$. It is therefore rank deficient. This was corrected in a subsequent publication [46].

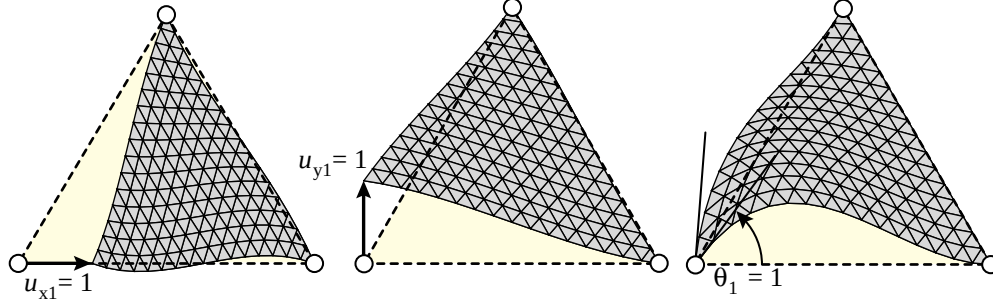


Figure 14. Shape functions for corner 1 of Allman 1988 triangle.

§29.8.1. Shape Functions

The displacement interpolation used by Allman in the 1988 element is

$$\begin{aligned} u_x &= N_{xx1}u_{x1} + N_{xy1}u_{y1} + N_{x\theta1}\theta_1 + \dots + N_{x\theta3}\theta_3, \\ u_y &= N_{yx1}u_{x1} + N_{yy1}u_{y1} + N_{y\theta1}\theta_1 + \dots + N_{y\theta3}\theta_3. \end{aligned} \quad (29.75)$$

where the shape functions are

$$\begin{aligned} N_{xx1} &= \zeta_1 - x_{23}x_{r0}, \quad N_{xy1} = -y_{23}x_{r0}, \quad N_{xx2} = \zeta_2 - x_{31}x_{r0}, \quad N_{xy2} = -y_{31}x_{r0}, \\ N_{xx3} &= \zeta_3 - x_{12}x_{r0}, \quad N_{xy3} = -y_{12}x_{r0}, \quad N_{x\theta1} = \frac{1}{2}(-g_1\zeta_{12} + g_5\zeta_{31} + g_1\zeta_{1221} + g_5\zeta_{3113}), \\ N_{x\theta2} &= \frac{1}{2}(-g_3\zeta_{23} + g_1\zeta_{12} + g_3\zeta_{2332} + g_1\zeta_{1221}), \quad N_{x\theta3} = \frac{1}{2}(-g_5\zeta_{31} + g_3\zeta_{23} + g_5\zeta_{3113} + g_3\zeta_{2332}), \\ N_{yx1} &= -x_{23}y_{r0}, \quad N_{yy1} = \zeta_1 - y_{23}y_{r0}, \quad N_{yx2} = -x_{31}y_{r0}, \quad N_{yy2} = \zeta_2 - y_{31}y_{r0}, \\ N_{yx3} &= -x_{12}y_{r0}, \quad N_{yy3} = \zeta_3 - y_{12}y_{r0}, \quad N_{y\theta1} = \frac{1}{2}(-g_2\zeta_{12} + g_6\zeta_{31} + g_2\zeta_{1221} + g_6\zeta_{3113}), \\ N_{y\theta2} &= \frac{1}{2}(-g_4\zeta_{23} + g_2\zeta_{12} + g_4\zeta_{2332} + g_2\zeta_{1221}), \quad N_{y\theta3} = \frac{1}{2}(-g_6\zeta_{31} + g_4\zeta_{23} + g_6\zeta_{3113} + g_4\zeta_{2332}). \end{aligned} \quad (29.76)$$

in which

$$\begin{aligned} \zeta_{12} &= \zeta_1\zeta_2, \quad \zeta_{23} = \zeta_2\zeta_3, \quad \zeta_{31} = \zeta_3\zeta_1, \quad \zeta_{1221} = \zeta_{12}(\zeta_2 - \zeta_1), \quad \zeta_{2332} = \zeta_{23}(\zeta_3 - \zeta_2), \quad \zeta_{3113} = \zeta_{31}(\zeta_1 - \zeta_3), \\ a_{12} &= 2A/L_{12}, \quad b_{12} = (x_{12}x_{13} + y_{12}y_{13})/L_{12}, \quad a_{23} = 2A/L_{23}, \\ b_{23} &= (x_{23}x_{21} + y_{23}y_{21})/L_{23}, \quad a_{31} = 2A/L_{31}, \quad b_{31} = (x_{31}x_{32} + y_{31}y_{32})/L_{31}, \\ x_{p12} &= x_{21}b_{12}/L_{12} + x_1, \quad y_{p12} = y_{21}b_{12}/L_{12} + y_1, \quad x_{p23} = x_{32}b_{23}/L_{23} + x_2, \\ y_{p23} &= y_{32}b_{23}/L_{23} + y_2, \quad x_{p31} = x_{13}b_{31}/L_{31} + x_3, \quad y_{p31} = y_{13}b_{31}/L_{31} + y_3, \\ g_1 &= L_{12}(x_{p12} - x_3)/a_{12}, \quad g_2 = L_{12}(y_{p12} - y_3)/a_{12}, \quad g_3 = L_{23}(x_{p23} - x_1)/a_{23}, \\ g_4 &= L_{23}(y_{p23} - y_1)/a_{23}, \quad g_5 = L_{31}(x_{p31} - x_2)/a_{31}, \quad g_6 = L_{31}(y_{p31} - y_2)/a_{31}, \\ x_{r0} &= (g_1\zeta_{1221} + g_3\zeta_{2332} + g_5\zeta_{3113})/(4A), \quad y_{r0} = (g_2\zeta_{1221} + g_4\zeta_{2332} + g_6\zeta_{3113})/(4A), \end{aligned} \quad (29.77)$$

The shape functions for the three freedoms of corner node 1 are plotted in Figure 14. In contrast to those of Figure 13, all shape functions are nonconforming. The strain-displacement matrix $\mathbf{e} = \mathbf{B}\mathbf{u}$ is readily computed by differentiation of the shape functions. The stiffness matrix is then obtained as

$$\mathbf{K}_A = \int_{\Omega^e} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega, \quad (29.78)$$

where $d\Omega$ is the element of area. Variants of this element result according to the integration rule adopted in (29.78). These are studied next.

§29.8.2. Variants

Since the shape functions (29.75) are cubic, the derived strains vary quadratically. For uniform $\mathbf{E}$ and h the integrand of (29.78) varies quartically. Exact integration may be achieved with the 7-point Gauss rule for triangles. For an isotropic material the exactly integrated element, has the bending energy ratio listed in Table 3 under label ALL-EX. This has γ^4 growth and hence rapidly locks for high aspect ratios. Since the optimal element of Section 5 has linearly varying strains, it is of interest to see whether filtering the strains to a linear variation improves the bending behavior. One way of linearizing strains is by reducing the integration rule. Consider the 3-point triangle quadrature rule defined parametrically by

$$\int_A F(\zeta_1, \zeta_2, \zeta_3) dA \approx \frac{A}{3} \left[F(\xi, \xi, 1-2\xi) + F(\xi, 1-2\xi, \xi) + F(1-2\xi, \xi, \xi) \right] \quad (29.79)$$

where $0 \leq \xi \leq \frac{1}{2}$. The two useful rules of this type are $\xi = 1/6$ (the interior-three-point rule) and $\xi = 1/2$ (the midpoint rule), both of which exhibit quadratic accuracy. But in the present context it is instructive to leave initially ξ free. A symbolic analysis with *Mathematica* shows that if the 1988 Allman element is numerically integrated by (29.79), the stiffness matrix is an instance of the ANDES template (29.31) with

$$\begin{aligned} \alpha_b &= 4\xi(2-3\xi), & \beta_0 &= \frac{4}{9}, & \beta_1 &= \frac{1}{2}\xi, & \beta_2 &= \frac{5}{2}\xi, & \beta_3 &= 3\xi, \\ \beta_4 &= 0, & \beta_5 &= 2\xi, & \beta_6 &= -2\xi, & \beta_7 &= -\frac{1}{2}\xi, & \beta_8 &= -3\xi, & \beta_9 &= -\frac{5}{2}\xi. \end{aligned} \quad (29.80)$$

Inserting these into (29.36)–(29.38) the bending energy ratio r is readily obtained. Results for the particular cases $\xi = 1/2$ and $\xi = 1/6$ are listed in Table 3 under labels ALL-3M and ALL-3I, respectively.

Yet another way to fit a linear strain pattern to Allman's quadratic strain fit is by minimizing a dislocation potential, which results in a least-square-like fit [42]. The resulting element is called ALL-LS. One finds that this element is an instance of the template (31) with the parameters shown in Table 2. The energy ratio is given in Table 3.

It can be seen that all variants of the Allman element exhibit catastrophic aspect ratio locking with r growing as γ^4 for $\gamma \gg 1$. Of the four variants shown in Tables 1–3, ALL-3I is preferable since it has the lowest r as γ grows. But all of them are worse than the humble CST when $\gamma > 9$.

§29.9. Numerical Examples

References [17,21] contain extensive comparisons of triangular elements with drilling freedoms, as does the thesis of Nygård [23] for both triangles and quadrilaterals. Only three numerical examples are presented here. The first two are standard benchmarks that focus on the effect of aspect ratio on the bending response. The third (Cook's problem) is included since there are published results for many element types.

§29.9.1. Example 1: Cantilever under End Moment

The slender cantilever beam of Figure 15 is subjected to an end moment $M = 100$. The modulus of elasticity is set to $E = 768$ so that the exact tip deflection $\delta_{tip} = ML/(2EI)$ is 100. Regular meshes ranging from 32×2 to 2×2 are used, each rectangle mesh unit being composed of four half-thickness overlaid triangles. The element aspect ratios vary from 1:1 through 16:1. The root clamping condition is imposed by setting

$$u_{x1} = u_{x2} = u_{x3} = 0, \quad u_{y2} = 0, \quad \theta_{x1} = \theta_{x2} = \theta_{x3} = 0, \quad (29.81)$$

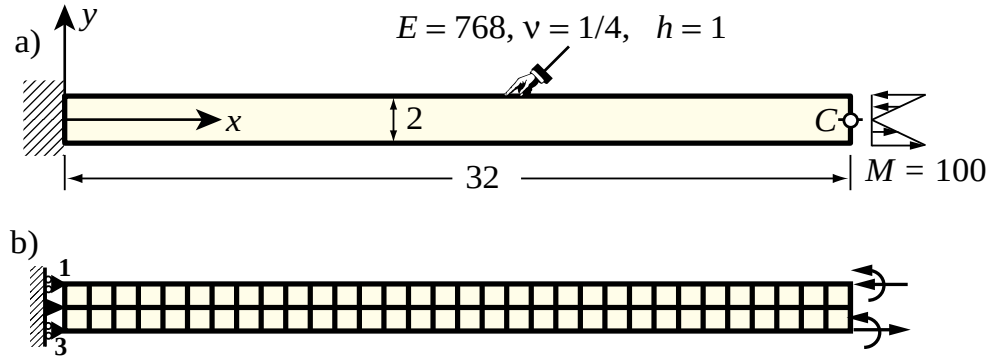


Figure 15. Slender cantilever beam under end moment: root contraction allowed; four-overlaid-triangle mesh units; a 32×2 mesh is shown in (b).

Table 4 Tip Deflections (exact=100) for Cantilever under End Moment

Element	Load Lumping	Mesh: x -subdivisions $\times$ y -subdivisions				
		32×2 ($\gamma = 1$)	16×2 ($\gamma = 2$)	8×2 ($\gamma = 4$)	4×2 ($\gamma = 8$)	2×2 ($\gamma = 16$)
ALL-3I	EBQ	87.08	76.48	38.32	5.42	0.39
ALL-3M	EBQ	81.36	53.57	9.59	0.71	0.04
ALL-EX	EBQ	84.90	69.09	24.23	2.47	0.16
ALL-LS	EBQ	85.36	68.25	20.83	1.89	0.12
CST	LI	54.05	36.36	15.75	4.82	1.28
FF84	EBQ	98.36	97.17	96.58	96.34	96.27
LST-Ret	EBQ	89.05	81.04	59.58	28.93	9.46
OPT	EBQ	99.99	99.99	99.99	99.96	100.07

where 1, 2, 3 are the root nodes, numbered from the top. It is important to leave u_{y1} and u_{y3} unrestrained for $\nu \neq 0$. This allows for the Poisson's contraction at the root and makes the exact solution merge with the displacement solution given in Section 5.1 over the entire beam.

Table 4 reports computed tip deflections (y displacement at C) for several element types and five aspect ratios. The identifiers in the “load lumping” column define ways in which the applied tractions at the free end are transformed to node forces; this topic is elaborated upon in [18].

Because two elements through the height are used, the computed deflections should be $100/r^{(2)}$, where $r^{(2)} = (3+r)/4$ as per equation (29.44). This provides a valuable numerical confirmation of the energy ratios listed in Table 3. Discrepancies from that prediction are due to load lumping schemes. For example, the results for OPT should be exactly 100.00 for any γ . And in fact they are if another load lumping scheme labeled EBZ in [18] is used. But the effect of the load lumping is slight, affecting only the fourth place of the computed deflections. The tiny deviations from 100.00 are due to scheme EBQ not being in exact energy balance, as explained in that reference.

The FF84 element maintains good but not perfect accuracy. The Allman 88 triangles perform well for unit aspect ratios but rapidly become over stiff for $\gamma > 2$; all variants are inferior to the CST for $\gamma > 9$. Of the four variants listed in Table 4 ALL-3I is consistently superior.

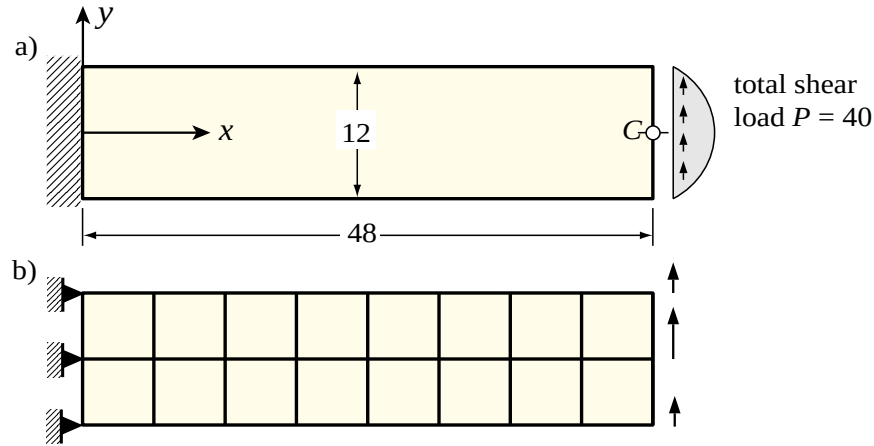


Figure 16. Cantilever under end shear: $E = 30000$, $\nu = 1/4$, $h = 1$; root contraction not allowed; four-overlaid-triangle mesh units; a 8×2 mesh is shown in (b).



Figure 17. Intensity contour plot of σ_{xy} given by the 64×16 OPT mesh. Stress node values averaged between adjacent elements. The root singularity pattern is visible.

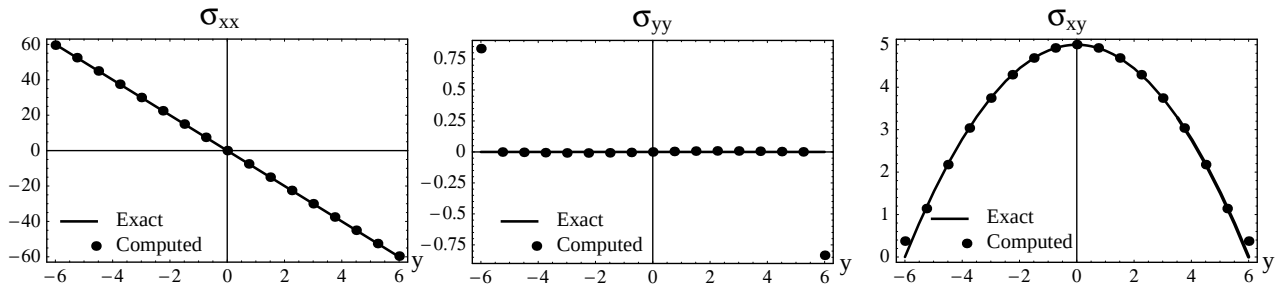


Figure 18. Distributions of σ_{xx} , σ_{yy} and σ_{xy} at $x = 12$ given by the 16×64 OPT mesh. Stress node values averaged between adjacent elements. Note different stress scales. Deviations at $y = \pm 6$ (free edges) due to “upwinded” y averaging.

§29.9.2. Example 2: The Shear-Loaded Short Cantilever

The shear-loaded cantilever beam defined in Figure 16 has been selected as a test problem for plane stress elements by many investigators since originally proposed in [10]. A full root-clamping condition is implemented by constraining both displacement components to zero at nodes located at the root section $x = 0$. Drilling rotations must not be constrained at the root because the term $\partial u_y / \partial x$ in the continuum-mechanics definition is nonzero there. The applied shear load varies parabolically over the end section and is consistently lumped at the nodes.

Table 5 Tip Deflections (exact = 100) for Short Cantilever under End Shear

Element	Mesh: x -subdivisions $\times$ y -subdivisions				
	8×2	16×4	32×8	64×16	128×32
ALL-3I	96.41	98.59	99.59	99.91	99.99
ALL-3M	82.70	94.78	98.57	99.62	99.91
ALL-EX	89.43	96.88	99.16	99.79	99.96
ALL-LS	89.72	96.94	99.17	99.79	99.96
CST	55.09	82.59	94.90	98.65	99.66
FF84	99.15	99.71	99.87	99.96	99.99
LST-Ret [†]	70.86	91.10	97.90	99.56	99.90
OPT	101.68	100.30	100.03	100.00	100.00
	4×2	8×4	16×8	32×16	64×32
ALL-3I	82.27	93.22	97.86	99.38	99.83
ALL-3M	54.23	81.84	94.52	98.50	99.61
ALL-EX	70.71	89.63	96.93	99.15	99.77
ALL-LS	69.97	89.30	96.94	99.17	99.79
CST	37.85	69.86	90.04	97.25	99.28
FF84	94.27	97.85	99.23	99.74	99.92
LST-Ret [†]	79.58	93.53	98.14	99.49	99.83
OPT	96.68	98.44	99.37	99.78	99.93
	2×2	4×4	8×8	16×16	32×32
ALL-3I	42.53	72.66	90.72	97.32	99.27
ALL-3M	12.39	31.81	63.68	87.24	96.41
ALL-EX	26.16	56.93	83.54	95.14	98.69
ALL-LS	23.02	52.37	80.84	94.22	98.45
CST	17.83	43.84	75.01	92.13	97.86
FF84	89.26	96.37	98.66	99.50	99.83
LST-Ret [†]	56.71	83.79	95.14	98.63	99.62
OPT	92.24	96.99	98.70	99.48	99.81

[†] Requires one drilling freedom to be fixed, else stiffness is singular.

The comparison value is the tip deflection $\delta_C = u_{yC}$ at the center of the end-loaded cross section. One perplexing question concerns the analytical value of δ_C . An approximate solution derived from 2-D elasticity, based on a polynomial Airy stress function, gives $\delta_{el} = 0.34133 + 0.01400 = 0.35533$, where the first term comes from the bending deflection $PL^3/3EI_{zz}$, $I_{zz} = hH^3/12$, and the second from a y -quadratic shear field. The shear term coefficient in the second term results from assuming a warping-allowed root-clamping condition that is more “relaxed” than the fully-clamped prescription for the FE model. Consequently in [10] it was argued that δ_{el} should be an upper bound, which was verified by the conforming FE models tested at that time. The finest grid results in [21] gave, however, $\delta_C \approx 0.35587$, which exceeds that “bound” in the fourth place. The finest OPT mesh ran here (128×32) gave a still larger value: 0.35601. The apparent explanation for this paradox is that if $\nu \neq 0$, a mild singularity in σ_{yy} and τ_{xy} , induced by the restraint $u_y|_{x=0} = 0$, develops at the corners of the root section, as depicted in Figure 17. This singularity “clouds” convergence of digits 4-5. (In retrospect it would have been better to allow for lateral contraction effects as in Example 1 to avoid this singularity.) The percentage results in Tables 3-5 of [21] therefore contain errors in the 4th place.

Table 5 gives computed deflections for rectangular mesh units with aspect ratios of 1, 2 and 4. Mesh units consist of four half-thickness overlaid triangles. For reporting purposes the load was scaled by $100/0.35601$ so that the “theoretical solution” becomes 100.00.

The data in Table 5 generally follows the patterns of the previous example; the main difference being the lack of drastically small deflections because element aspect ratios only go up to 4:1. Of the four Allman triangle versions again ALL-3I outperformed the others. The results for FF84 and OPT triangles are very similar, without the latter displaying the clear advantages of Example 1. The data for FF84 and CST changes slightly from that of Tables 3-5 of [21] on two accounts: four-triangle, rather than two-triangle, macroelements are used to eliminate y -directionality, and the normalizing “theoretical” solution changes by $+0.00014$ as explained above.

Figure 18 plots averaged node stress values at section $x = 12$ computed from the 64×16 OPT mesh. The recovery is based on (29.48) with $\beta_0^e = 3/2$. The agreement with the standard beam stress distribution (the section is sufficiently away from the root) is very good at interior points but less so at the free edges $y = \pm 6$ since the averaging becomes biased.

§29.9.3. Example 3: Cook’s Problem

Table 6 gives results computed for the plane stress problem defined in Figure 19. This problem was proposed by Cook [47] as a test case for nonrectangular quadrilateral elements. There is no known analytical solution but the OPT results for the 64×64 mesh may be used for comparison purposes. (Extrapolation of the OPT results using Wynn’s ϵ algorithm [48] yields $u_{yC} = 23.956$.) The last six lines in Table 6 pertain to quadrilateral elements. Results for HL, HG and Q4 are taken from [47] whereas those for Q6 and QM6 are taken from [49]. Results for the Free Formulation quadrilateral FFQ are taken from Nygård’s thesis [23]. Further data on other elements is provided in [50].

For triangle tests, quadrilaterals were assembled with two triangles in the shortest-diagonal-cut layout as illustrated in Figure 16 for a 2×2 mesh. Cutting the quadrilaterals the other way or using four-overlaid-triangle macroelements yields stiffer results.

Since element aspect ratios are reasonable near the root (where the action is), the performance of the seven LST-3/9R models tested can be expected to be similar, and indeed it was. Of the seven ALL-3I is best followed closely by LST-Ret, OPT and FF84. It should be noted that accuracy of the FF84 and OPT triangles for this problem is dominated by the basic stiffness response. Consequently the deflection values provided by the FF84 and OPT elements, which share the same basic stiffness, are very close.

The overall stress distribution for this problem is rarely reported. Figure 18 displays a *Mathematica*-generated intensity contour plot of the von Mises stress computed from the 32×32 OPT mesh, using the recovery formula (29.48) with $\beta_0^e = 3/2$. A singularity pattern at point $\{x = 0, y = 44\}$, which is located at a fixed entrant corner, is evident.

§29.10. Discussion and Conclusions

The conclusions are posted below in a Q&A format so that readers can scan subjects quickly.

Has the OPT triangle been used much?

The optimal LST-3/9R element has been used since 1991 as membrane component of a shell element with 6 DOF per corner. The presence of the drilling freedoms facilitates modeling of surface intersections and stiffeners. In fact the shell element is so fast and versatile that it has largely replaced beams in modeling complete aircraft structures. Figure 21 shows portion of the interior structure of an F-16 used

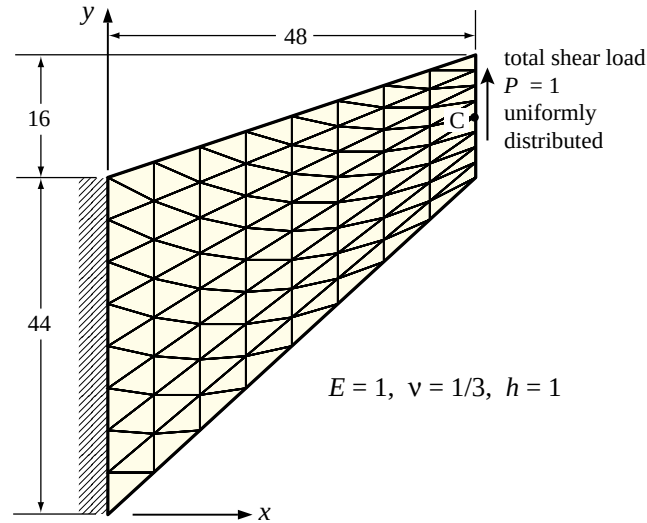


Figure 19. Cook's problem: clamped trapezoid under end shear. A 8×8 mesh is shown.

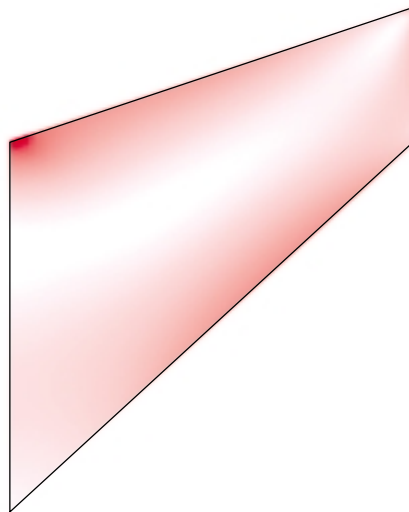


Figure 20. von Mises stress intensity distribution from 32×32 OPT element mesh. Peak occurs at upper left corner.

in aeroelastic studies by Farhat's team [51]: note that triangles are used for any thin wall component. A corotational version developed by Haugen [52] is used in the FEDEM multibody dynamics system. The FF84 ancestor is used in codes developed at Trondheim by Pål Bergan and his colleagues.

When is exact pure-bending response important?

Bending response exactness for any aspect ratio γ is important in modeling thin and composite aerospace structures, such as the stiffened panel depicted in Figure 22. If the longitudinal direction called x in the bending test is set along the panel, that Figure shows a very high γ in a flange and a small γ in a joint. Aspect ratio locking in such mesh units can adversely affect the response of the whole structure.

What are the main advantages of templates?

Table 6 Results for Cook's Problem

<i>Element</i>	<i>Vertical deflection at C for subdivision</i>					
	2×2	4×4	8×8	16×16	32×32	64×64
ALL-3I	21.61	23.00	23.66	23.88	23.94	
ALL-3M	16.61	21.05	23.02	23.69	23.87	
ALL-EX	19.01	21.83	23.43	23.81	23.91	
ALL-LS	19.43	22.32	23.44	23.82	23.92	
CST	11.99	18.28	22.02	23.41		
FF84	20.36	22.42	23.41	23.79	23.91	
LST-Ret [†]	19.82	22.62	23.58	23.86	23.94	
OPT	20.56	22.45	23.43	23.80	23.91	23.95
FFQ	21.66	23.11	23.79	23.88	23.94	
HL	18.17	22.03		23.81		
HG	22.32	23.23		23.91		
Q4	11.85	18.30		23.43		
Q6	22.94	23.48				
QM6	21.05	23.02				

[†] Requires one drilling DOF to be fixed, else stiffness is singular

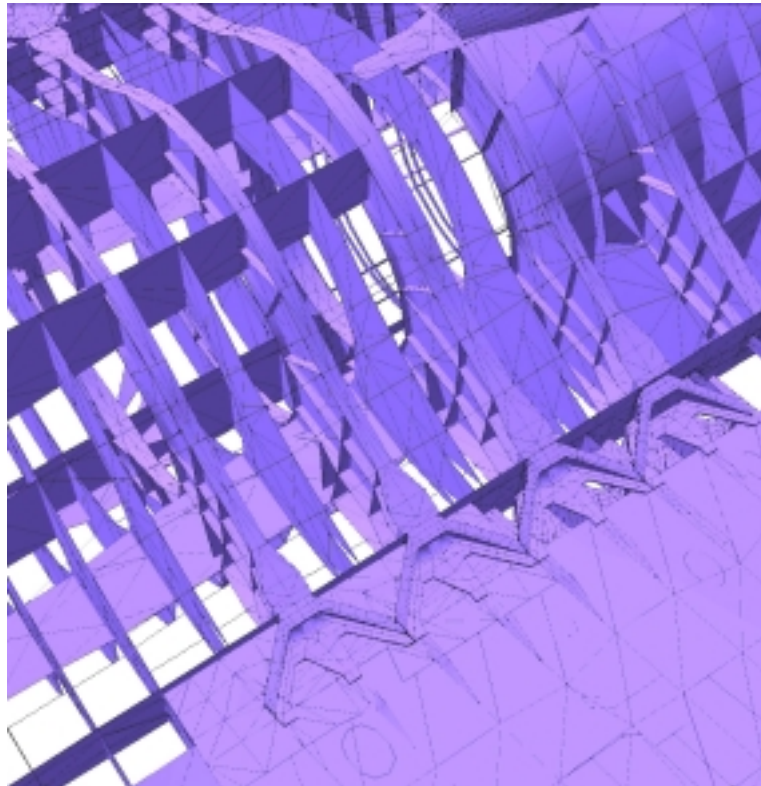


Figure 21. Internal structure of an F-16 modeled with triangle shell elements (courtesy Greg Brown).

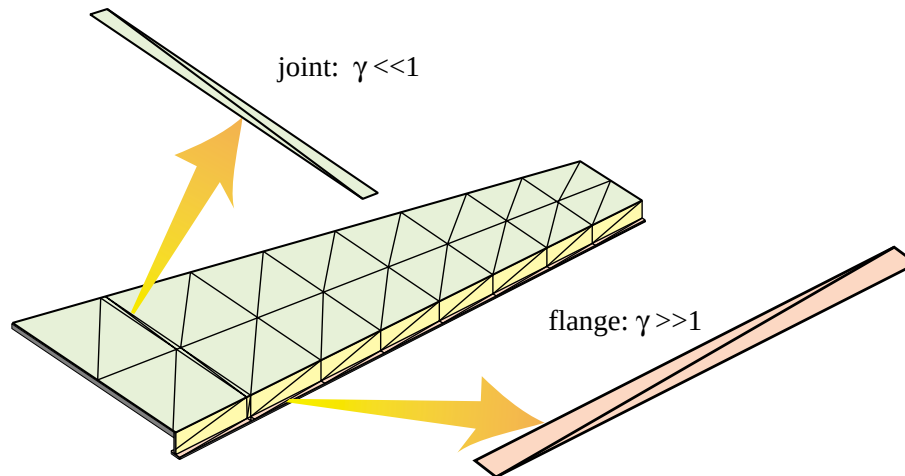


Figure 22. Stiffened panels modeled by facet shell elements are a common source of high aspect ratio elements.

The obvious one is the possibility of searching for optimal or custom element instances, without worry as to fixing up bad elements along the way. They are also useful in research studies because a template spans an infinity of possible elements including elements already published. This unification facilitates comparison of previous and new elements using a single implementation, as in Figure 10.

Why are strains good choice for higher order trial spaces?

Strains are intermediate variables between displacements and stresses. Unlike them, no boundary conditions can be applied on strains. This mediatory nature tends to produce elements of balanced behavior: neither too stiff nor too flexible. In non-mechanical problems, the same role can be assigned to the intermediate variable that appears in Tonti diagrams of the variational formulation [4].

Do templates supersede conventional element derivation methods?

No. The template configuration, by which it is meant the sequence of matrix products and the functional dependence of matrix entries on geometry and constitutive properties, has to be established by conventional methods. For example, the ANDES formulation leads to the forms (28)–(31), which could not be guessed *a priori*. Parameters are injected as weights of algebraic terms as appropriate.

I have derived an alledgedly new LST-3/9R triangle. How do I check if it is optimal?

The first step is to run the bending ratio test of Section 5.1, numerically or (better) symbolically. If: (i) the ratio $r = 1$ for all γ , (ii) the element passes the ordinary patch tests and (iii) is rank sufficient, it is indeed optimal. But is it new? The next step should be to try the test geometry of Section 6 and compare $\mathbf{K}$ to (29.52). If it matches, the “new triangle” is likely a clone of the OPT element. This can be rigorously proven by extracting its template signature, which is not a easy process if the element was fabricated as a whole. If it does not match you have a different optimal element. As discussed in Section 5.5, this cannot happen if the element is energy orthogonal.

Should parameters be left as arguments of element implementations?

Only for research studies, and to find clones (as in the scenario of the foregoing question). In production implementations parameters should be hardwired to a name, as illustrated by the module listed in Figure 12. Few users have the knowledge or interest to play around with parameter values.

Why is it worth republishing the optimal element?

Two reasons. First, the fact that a membrane element with this freedom configuration that is insensitive to aspect ratio can be constructed does not seem to be generally known. Second, the derivation can be now comfortably placed within the framework of finite element templates, which was an embryonic concept ten years ago. As a result, several elements in common use can be exhibited as instances of the ANDES template, facilitating unified implementation and testing.

The paper discusses three models, but more have been published. Why the omissions?

The three models are intended to illustrate the derivation approaches of Figure 1. The ANDES template exemplifies the direct fabrication approach. The retrofitted LST is a textbook example of, well, the retrofitting approach. The Allman 1988 triangle displays the typical tribulations of the fixed-up approach in that several variants are possible but none cures aspect ratio locking. Including more models would have lengthened the exposition without achieving appreciable benefits over the optimal one.

Appendix A. The Higher Order Strain Field

For completeness the construction of the higher order strain field carried out in Ref [17] is summarized here. Split the hierarchical rotations into mean and deviatoric: $\tilde{\theta}_1 = \bar{\theta} + \theta'_1$, $\theta_2 = \bar{\theta} + \theta'_2$, $\theta_3 = \bar{\theta} + \theta'_3$, where $\bar{\theta} = \frac{1}{3}(\tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3)$. In matrix form: $\tilde{\theta} = \bar{\theta} + \theta'$ where $\bar{\theta} = \bar{\theta} [1 \ 1 \ 1]^T$ and $\theta' = [\theta'_1 \ \theta'_2 \ \theta'_3]^T$. The deviatoric corner rotations define the linear deviatoric-rotation field:

$$\theta' = \theta'_1 \zeta_1 + \theta'_2 \zeta_2 + \theta'_3 \zeta_3, \quad (29.82)$$

which integrates to zero over the element. For subsequent use we note the matrix relation

$$\begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \\ \bar{\theta} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix}. \quad (29.83)$$

The splitting (29.83) translates to a similar decomposition of the higher order strains: $\mathbf{e}_d = \mathbf{e}_b + \mathbf{e}_t$, where subscripts b and t identify “pure bending” and “torsional” strain fields, respectively. These are generated by the deviatoric rotations θ' and the mean hierarchical rotation $\bar{\theta}$, respectively.

A.1 The Pure Bending Field

This field is produced by the deviatoric corner rotations θ'_i , $i = 1, 2, 3$, inducing the side-aligned natural strains

$$\epsilon_b = [\epsilon_{b21} \ \epsilon_{b32} \ \epsilon_{b13}]^T. \quad (29.84)$$

pertaining to choice (s) in Figure 5. The *straingage locations* are chosen at the triangle corners. The natural strain ϵ_{jk} at corner i is written $\epsilon_{jk|i}$, the bar being used for reading convenience. Vector ϵ_b at corner i is denoted by ϵ_{bi} . The goal is to construct the 3×3 matrices $\mathbf{Q}_{bi}$ that relate natural straingage readings to the deviatoric rotations:

$$\epsilon_{b1} = \mathbf{Q}_{b1} \theta', \quad \epsilon_{b2} = \mathbf{Q}_{b2} \theta', \quad \epsilon_{b3} = \mathbf{Q}_{b3} \theta', \quad (29.85)$$

Once these are known the natural bending strains can be obtained by linear interpolation over the triangle: $\epsilon_b = (\mathbf{Q}_{b1} \zeta_1 + \mathbf{Q}_{b2} \zeta_2 + \mathbf{Q}_{b3} \zeta_3) \theta' = \mathbf{Q}_b \theta'$. Consider $\epsilon_{b21}(P)$ at an arbitrary point P of the triangle. Denote by $d_{21|P}$ the signed distance from the centroid to P measured along the internal normal to side 21, and specialize P to corners:

$$d_{21|3} = \frac{4A}{3\ell_{21}}, \quad d_{21|1} = d_{21|2} = -\frac{1}{2}d_{21|3} = -\frac{2A}{3\ell_{12}}. \quad (29.86)$$

Assume that $\epsilon_{b21|P}$ depends only on $d_{21|P}$ divided by the side length ℓ_{21} , which introduces a distance scaling. These dimensionless ratios will be called $\chi_{21|P} = d_{21|P}/\ell_{21}$, which specialized to the corners become

$$\chi_{21|3} = \frac{4A}{3\ell_{21}^2}, \quad \chi_{21|1} = \chi_{21|2} = -\frac{2A}{3\ell_{21}^2}. \quad (29.87)$$

Formulas for corners 2 and 3 are obtained by cyclic permutation. According to the foregoing assumption, the natural straingage readings $\epsilon_{b21|i}$ at corner i depend only on $\chi_{21|i}$, multiplied by as yet unknown weighting factors. In matrix form:

$$\epsilon_{b1} = \begin{bmatrix} \epsilon_{b21|1} \\ \epsilon_{b32|1} \\ \epsilon_{b13|1} \end{bmatrix} = \frac{2A}{3} \begin{bmatrix} \frac{\psi_1}{\ell_{21}^2} & \frac{\psi_2}{\ell_{21}^2} & \frac{\psi_3}{\ell_{21}^2} \\ \frac{\psi_4}{\ell_{32}^2} & \frac{\psi_5}{\ell_{32}^2} & \frac{\psi_6}{\ell_{32}^2} \\ \frac{\psi_7}{\ell_{13}^2} & \frac{\psi_8}{\ell_{13}^2} & \frac{\psi_9}{\ell_{13}^2} \end{bmatrix} \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{bmatrix} = \mathbf{Q}_{b1} \theta'. \quad (29.88)$$

Here ψ_1 through ψ_9 are dimensionless weight parameters. [In reference [17] five parameters called ρ_1 through ρ_5 were used instead as shown in equation (29.20); these account *a priori* for the triangular symmetries (29.32).] Relations for corners 2 and 3 are constructed by cyclic permutation.

Pictures of the unweighted bending modes are shown in Figure 7. These were obtained by integrating the strain field into displacements, which is possible because both natural and Cartesian strains vary linearly.

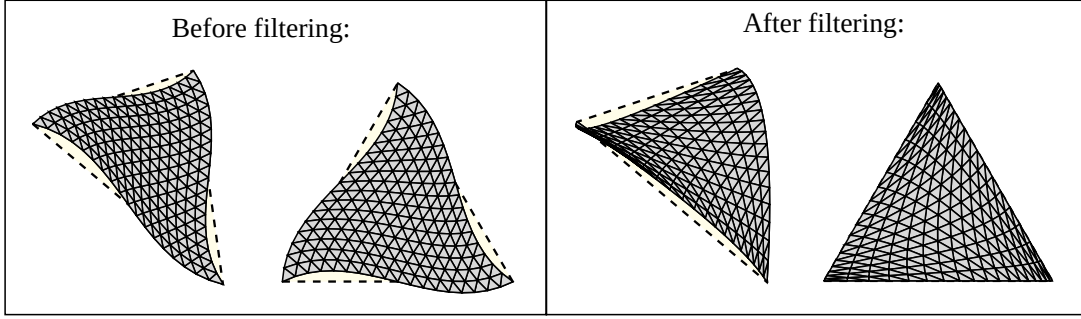


Figure 23. The torsional displacement mode before and after strain filtering. Filtered patterns were obtained by integrating the strain field (29.92). Note that for an equilateral triangle the filtered pattern becomes a bubble mode.

A.2 The Torsional Field

The higher order stiffness produced by the pure bending field alone is rank deficient by one because of the deviatoric constraint $\theta'_1 + \theta'_2 + \theta'_3 = 0$. To get rank sufficiency it is necessary to build a strain field associated with $\tilde{\theta}_i = \bar{\theta}$, others zero. This may be viewed as forcing each corner of the triangle to rotate by the same amount while the corner displacements are precluded. A displacement-based solution to this problem is provided by the cubic field of the QST4-20G shown in the center of Figure 2. From [10, p. 30] the shape function interpolation for u_x is

$$u_x(\zeta_1, \zeta_2, \zeta_3) = \begin{bmatrix} u_{x1} \\ u_{x,x|1} \\ u_{x,y|1} \\ u_{x2} \\ u_{x,x|2} \\ u_{x,y|2} \\ u_{x3} \\ u_{x,x|3} \\ u_{x,y|3} \\ u_{x0} \end{bmatrix}^T \begin{bmatrix} \zeta_1^2(3 - 2\zeta_1) - 7\zeta_1\zeta_2\zeta_3 \\ \zeta_1^2(x_{21}\zeta_2 - x_{13}\zeta_3) + (x_{13} - x_{21})\zeta_1\zeta_2\zeta_3 \\ \zeta_1^2(y_{21}\zeta_2 - y_{13}\zeta_3) + (y_{13} - y_{21})\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(3 - 2\zeta_2) - 7\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(x_{32}\zeta_3 - x_{21}\zeta_1) + (x_{21} - x_{32})\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(y_{32}\zeta_3 - y_{21}\zeta_1) + (y_{21} - y_{32})\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(3 - 2\zeta_3) - 7\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(x_{13}\zeta_1 - x_{32}\zeta_2) + (x_{32} - x_{13})\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(y_{13}\zeta_1 - y_{32}\zeta_2) + (y_{32} - y_{13})\zeta_1\zeta_2\zeta_3 \\ 27\zeta_1\zeta_2\zeta_3 \end{bmatrix} \quad (29.89)$$

where $u_{x,x|i}$ and $u_{x,y|i}$ denote $\partial u_x / \partial x$ and $\partial u_x / \partial y$, respectively, evaluated at node i , with $i = 0$ for the centroid. The same shape functions interpolate $u_y(\zeta_1, \zeta_2, \zeta_3)$. The torsional mode with unit rotations $\theta_i = \bar{\theta} = 1$ is imposed by setting the nodal displacements to

$$u_{xi} = u_{yi} = u_{x,x|i} = u_{y,y|i} = 0, \quad u_{x,y|i} = -\bar{\theta}, \quad u_{y,x|i} = \bar{\theta}, \quad i = 0, 1, 2, 3, \quad j = 1, 2, 3. \quad (29.90)$$

Differentiating the QST interpolation with respect to x and y , collapsing freedoms via (29.90) and transforming to natural strains via (11) yields

$$\epsilon_t = \begin{bmatrix} \epsilon_{t21} \\ \epsilon_{t32} \\ \epsilon_{t13} \end{bmatrix} = 3 \begin{bmatrix} \chi_{21|3} (\zeta_1 - \zeta_2) \zeta_3 \\ \chi_{32|1} (\zeta_2 - \zeta_3) \zeta_1 \\ \chi_{13|2} (\zeta_3 - \zeta_1) \zeta_2 \end{bmatrix} \bar{\theta}, \quad (29.91)$$

This quadratic field was found unable to produce optimal elements in conjunction with the foregoing bending field. A midpoint-filtered fit that permits optimality is obtained by collocating (29.91) at the midpoints and interpolating linearly over the triangle:

$$\epsilon_t^m = \begin{bmatrix} \epsilon_{t21}^m \\ \epsilon_{t32}^m \\ \epsilon_{t13}^m \end{bmatrix} = \frac{3}{2} \begin{bmatrix} \chi_{21|3} (\zeta_1 - \zeta_2) \\ \chi_{32|1} (\zeta_2 - \zeta_3) \\ \chi_{13|2} (\zeta_3 - \zeta_1) \end{bmatrix} \bar{\theta} \stackrel{\text{def}}{=} \frac{4A}{3} \psi_0 \begin{bmatrix} (\zeta_1 - \zeta_2)/\ell_{21}^2 \\ (\zeta_2 - \zeta_3)/\ell_{32}^2 \\ (\zeta_3 - \zeta_1)/\ell_{13}^2 \end{bmatrix} \bar{\theta}. \quad (29.92)$$

The effect of this strain filtering is pictured in Figure 23 using integrated displacement patterns.

As indicated on the right of (29.92) the field is scaled by a weight coefficient ψ_0 chosen so that $\psi_0 = 1$ for the parameter set that produces the optimal element. Evaluating (29.92) at corner 1, combining with (29.88) and using the rotational transformation (29.83) gives

$$\epsilon_1 = \frac{2A}{3} \begin{bmatrix} \frac{\psi_1}{\ell_{21}^2} & \frac{\psi_2}{\ell_{21}^2} & \frac{\psi_3}{\ell_{21}^2} & \frac{4\psi_0}{\ell_{21}^2} \\ \frac{\psi_4}{\ell_{32}^2} & \frac{\psi_5}{\ell_{32}^2} & \frac{\psi_6}{\ell_{32}^2} & 0 \\ \frac{\psi_7}{\ell_{13}^2} & \frac{\psi_8}{\ell_{13}^2} & \frac{\psi_9}{\ell_{13}^2} & \frac{-4\psi_0}{\ell_{13}^2} \end{bmatrix} \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \\ \tilde{\theta} \end{bmatrix} = \frac{2A}{3} \begin{bmatrix} \frac{\beta_1}{\ell_{21}^2} & \frac{\beta_2}{\ell_{21}^2} & \frac{\beta_3}{\ell_{21}^2} \\ \frac{\beta_4}{\ell_{32}^2} & \frac{\beta_5}{\ell_{32}^2} & \frac{\beta_6}{\ell_{32}^2} \\ \frac{\beta_7}{\ell_{13}^2} & \frac{\beta_8}{\ell_{13}^2} & \frac{\beta_9}{\ell_{13}^2} \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} = \mathbf{Q}_1 \tilde{\boldsymbol{\theta}}, \quad (29.93)$$

in which $\beta_1 = \frac{1}{3}(4\psi_0 + 2\psi_1 - \psi_2 - \psi_3)$, $\beta_2 = \frac{1}{3}(4\psi_0 - \psi_1 + 2\psi_2 - \psi_3)$, $\beta_3 = \frac{1}{3}(4\psi_0 - \psi_1 - \psi_2 + 2\psi_3)$, $\beta_4 = \frac{1}{3}(2\psi_4 - \psi_5 - \psi_6)$, $\beta_5 = \frac{1}{3}(-\psi_4 + 2\psi_5 - \psi_6)$, $\beta_6 = \frac{1}{3}(-\psi_4 - \psi_5 + 2\psi_6)$, $\beta_7 = \frac{1}{3}(-4\psi_0 + 2\psi_7 - \psi_8 - \psi_9)$, $\beta_8 = \frac{1}{3}(-4\psi_0 - \psi_7 + 2\psi_8 - \psi_9)$ and $\beta_9 = \frac{1}{3}(-4\psi_0 - \psi_7 - \psi_8 + 2\psi_9)$. Matrices $\mathbf{Q}_2$ and $\mathbf{Q}_3$ are obtained by cyclic permutation. These matrices are used in Section 4.6 to construct the ANDES template (29.31).

Appendix B. Solving Polynomial Equations for Template Optimality

In early work with HP elements (1984-1990) the writer searched for optimal free parameters using mathematical programming methods, by minimizing squared deviations of energy ratios from unity. This approach has a serious disadvantage: numerical studies require specific material and geometric data. The MP libraries gave answers but no solutions. The following approach has been found to be highly effective in symbolic work, which provides complete solutions.

Let $\mathbf{p} = [p_1 \ p_2 \ \dots \ p_n]^T$ be a n -vector of template parameters. While seeking template optimality under high order patch tests one must usually deal with a polynomial energy ratio of the form

$$r(\mathbf{p}) = c_0 + c_2\gamma^2 + \dots + c_k\gamma^k, \quad (29.94)$$

where k is even, γ is an element aspect ratio, and coefficients $c_j = c_j(\mathbf{p})$ for $j = 0, 2 \dots k$, take on one of the quadratic forms

$$c_j = \mathbf{p}^T \mathbf{A}_j \mathbf{p}, \quad c_j = \mathbf{p}^T \mathbf{A}_j \mathbf{p} + 2\mathbf{b}_j^T \mathbf{p}, \quad \text{or} \quad c_j = \mathbf{p}^T \mathbf{A}_j \mathbf{p} + 2\mathbf{b}_j^T \mathbf{p} + d_j. \quad (29.95)$$

The *kernel matrices* $\mathbf{A}_j$, are $n \times n$ symmetric matrices whereas $\mathbf{b}_j$ is an n -vector. The second and third forms of (29.95) may be reduced to the homogeneous form by the obvious augmentation

$$\mathbf{p} \rightarrow \hat{\mathbf{p}} = [1 \ p_1 \ p_2 \ \dots \ p_k], \quad \mathbf{A}_j \rightarrow \hat{\mathbf{A}}_j = \begin{bmatrix} d_j & \mathbf{b}_j \\ \mathbf{b}_j^T & \mathbf{A}_j \end{bmatrix}, \quad c_j = \hat{\mathbf{p}}^T \hat{\mathbf{A}}_j \hat{\mathbf{p}}. \quad (29.96)$$

The optimization conditions are $c_0 = 1$ and $c_j = 0$ for $j = 2, \dots, k$. One is interested only in solutions $\mathbf{p} = [p_1 \ \dots \ p_n]^T$ or $\hat{\mathbf{p}} = [1 \ \hat{p}_1 \ \dots \ \hat{p}_n]^T$ with *real* entries. Preferably the solutions should be rational if the entries of $\mathbf{A}_j$, $\mathbf{b}_j$ and d_j are, as is often the case. If $n(k+2) > 4$, a brute force solution as a system of polynomial equations in exact rational arithmetic may be hopeless. It is observed in practice, however, that matrices $\mathbf{A}_j$ or $\hat{\mathbf{A}}_j$ are highly singular and nonnegative. This allows an efficient staged reduction scheme in which most of the steps involve only the solution of linear equations. The method will be explained by example, using the optimization of the ANDES template undertaken in Section 5.2 as case study.

The coefficients of the energy ratio r of (29.36)–(29.38): $r = c_0 + c_2\gamma^2 + c_4\gamma^4$ can be expressed as

$$c_0 - 1 = \mathbf{p}^T \mathbf{A}_0 \mathbf{p}, \quad c_2 = \mathbf{p}^T \mathbf{A}_2 \mathbf{p}, \quad c_4 = \mathbf{p}^T \mathbf{A}_4 \mathbf{p}, \quad (29.97)$$

in which $\mathbf{p}$ is the 8-vector $[1 \ \alpha_b \ \beta_1 \ \beta_2 \ \beta_3 \ \beta_4 \ \beta_5 \ \beta_6]$. This is actually the augmented vector denoted by $\hat{\mathbf{p}}$ above, with the hat suppressed for brevity, and likewise over the $\mathbf{A}_j$ s. The kernel matrices are

$$\mathbf{A}_0 = \frac{1}{3} \begin{bmatrix} 9 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\beta_0}{32} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13 & -11 & -1 & 2 & 2 & -6 \\ 0 & 0 & -11 & 13 & -1 & -2 & -2 & 6 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 1 & 1 & -3 \\ 0 & 0 & 2 & -2 & 0 & 1 & 1 & -3 \\ 0 & 0 & -6 & 6 & 0 & -3 & -3 & 9 \end{bmatrix}, \quad (29.98)$$

$$\mathbf{A}_2 = \frac{1}{6} \begin{bmatrix} -9 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\beta_0}{32} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 26 & -20 & -4 & -10 & 12 & -6 \\ 0 & 0 & -20 & 22 & -2 & 8 & -14 & 8 \\ 0 & 0 & -4 & -2 & 6 & 0 & 2 & 0 \\ 0 & 0 & -10 & 8 & 0 & 5 & -5 & 1 \\ 0 & 0 & 12 & -14 & 2 & -5 & 9 & -5 \\ 0 & 0 & -6 & 8 & 0 & 1 & -5 & 5 \end{bmatrix}, \quad (29.99)$$

$$\mathbf{A}_4 = \frac{\beta_0}{64} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & -3 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 & 1 & 1 \end{bmatrix} \quad (29.100)$$

The optimality conditions are $c_0 - 1 = 0$, $c_2 = 0$, $c_4 = 0$. Taking the higher order scaling factor $\beta_0 = 1/2$ for convenience, matrices $\mathbf{A}_0$, $\mathbf{A}_2$ and $\mathbf{A}_4$ have the eigenvalues

$$\begin{aligned} \text{eigs of } \mathbf{A}_0: & [10/3 \ 3/64 \ (35 + \sqrt{521})/128 \ (35 - \sqrt{521})/128 \ 0 \ 0 \ 0 \ 0] \\ \text{eigs of } \mathbf{A}_2: & [13/6 \ 8053/8904 \ 1063/6333 \ 1774/25955 \ 0 \ 0 \ 0 \ 0] \\ \text{eigs of } \mathbf{A}_4: & [11/64 \ 11/64 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \end{aligned} \quad (29.101)$$

(For $\mathbf{A}_2$ the listed eigenvalues #2 through #4 are rational approximants within 10^{-8} .) Consequently the conditions stated previously are met. Begin with $\mathbf{A}_4$, which has rank 2, rank deficiency 6, and spectral decomposition

$$\mathbf{A}_4 = \mathbf{V}_4 \mathbf{\Lambda}_4 \mathbf{V}_4^T, \quad \mathbf{\Lambda}_4 = \begin{bmatrix} \frac{11}{64} & 0 \\ 0 & \frac{11}{64} \end{bmatrix}, \quad \mathbf{V}_4 = \frac{1}{\sqrt{11}} \begin{bmatrix} 0 & 0 & -1 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 1 & 1 \end{bmatrix}^T. \quad (29.102)$$

Since $\mathbf{\Lambda}_4$ is positive definite the only real solutions of $c_4 = \mathbf{p}^T \mathbf{A}_4 \mathbf{p} = \mathbf{p}^T \mathbf{V}_4 \mathbf{\Lambda}_4 \mathbf{V}_4^T \mathbf{p} = 0$ are those of $\mathbf{V}_4^T \mathbf{p} = \mathbf{0}$. This is an underdetermined *linear* system of 2 equations in 8 variables, from which 2 entries in $\mathbf{p}$ are designated as dependent and eliminated: $\beta_1 = 3\beta_3 - \beta_2$ and $3\beta_4 = \beta_5 + \beta_6$. Replacing these relations into $c_2 = \mathbf{p}^T \mathbf{A}_2 \mathbf{p} = 0$ reduces $\mathbf{p}$ to six entries and $\mathbf{A}_2$ to 6×6 . The reduced $\mathbf{A}_2$ is nonnegative definite and has rank 4. This allows 4 more variables to be eliminated. Repeating the spectral analysis yields $\alpha_b = 3/2$, $\beta_2 = -2\beta_6$, $\beta_3 = -\beta_6$ and $\beta_5 = -\beta_6$. Finally, replacing into $c_0 - 1 = 0$ gives $\beta_6^2 = 1$. Choosing $\beta_6 = -1$ the complete solution is

$$\alpha_b = \frac{3}{2}, \quad \beta_0 = \frac{1}{2}, \quad \beta_1 = \beta_3 = \beta_5 = 1, \quad \beta_2 = 2, \quad \beta_4 = 0, \quad \beta_6 = \beta_7 = \beta_8 = -1, \quad \beta_9 = -2, \quad (29.103)$$

which is used in Section 5.2.

It is not always necessary to do the complete eigenspectral analysis of the $\mathbf{A}_j$ s. It is sufficient to get a full-rank basis $\mathbf{V}_j$ for the range spaces. This is easily done by getting the null space through the appropriate function of *Mathematica* or *Maple*, and then forming $\mathbf{V}$ as the orthogonal complement by Gram-Schmidt. This alternative

path is useful for systems treated by exact arithmetic if the eigenvalues are complicated functions of the matrix coefficients.

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A TEMPLATE TUTORIAL

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Dedicated to Pål Bergan on his 60th birthday

Abstract. This article has a dual theme: historical and educational. It is a tutorial on finite element templates for two-dimensional structural problems. The exposition is aimed at readers with introductory level knowledge of finite element methods. It focuses on the four-node plane stress element of flat rectangular geometry, called the “rectangular panel” for brevity. This is one of the two oldest continuum structural elements. On the other hand the concept of finite element templates is a recent development, which had as key source the 1984–87 collaboration between Pål Bergan and the writer on the Free Formulation. Interweaving the old and the new throws historical perspective into the evolution of finite element methods. Templates provide a framework in which diverse element formulations can be fitted, compared and traced back to the sources. On the technical side templates facilitate the unified implementation of element families, as well as the construction of custom elements. To illustrate customization power beyond the rectangle, the Appendix presents the construction of a four-noded bending-optimal trapezoid. This model sidesteps MacNeal’s limitation theorem in that it passes the patch test for any geometry while staying bending-optimal along one direction and retaining full rank.

Key words: finite elements, history, templates, families, clones, quadrilateral membrane, Free Formulation, patch test

1 INTRODUCTION

This article interweaves historical and educational themes with the presentation of a new methodology for finite element development. It is expository in nature, and focuses on “FEM by its history.” It gently exposes the reader to the concept of templates, and explains why they emerge naturally from historical evolution. Much of the material is tutorial in nature, with some extracted from a FEM course offered by the writer.

Templates are parametrized algebraic forms that provide a continuum of consistent and stable finite element models of a given type and node/freedom configuration. Template instances produced by setting values to free parameters furnish specific elements. If the template embodies all possible consistent and stable elements of a given type and configuration, it is called *universal*.

Befitting the tutorial aim, the main body of this article focuses on the simplest two-dimensional element that possesses a nontrivial template: the four-node plane stress element of flat rectangular geometry. [The three-node linear triangle is simpler but its template is trivial.] This is called the *rectangular panel* for brevity.

The rectangular panel is interesting from both historical and instructional viewpoints because:

1. It is one of the two oldest continuum finite elements, the other being the linear triangle.¹
2. Along with its plane strain and axisymmetric cousins, it is the configuration treated by most new methods since the birth of finite elements. As such it represents an *in-vivo* specimen of FEM evolution over the past 50 years.
3. It is amenable to complete analytical development, even for anisotropic material behavior. This makes the element particularly suitable for homework and project assignments.
4. Analytical forms make the concept of signatures and clones highly visible to students.

The paper is organized as follows. Section 2 is a brief outline of element formulation approaches used from 1950 to date. Section 3 introduces the focus problem. Sections 4–6 follow up on the historical theme by presenting stress, strain and displacement-based models for the rectangular panel.

The concept of template is introduced in Section 7 by calling attention to a common structure that lurks behind the stiffness expressions of stress, strain and displacement models. Template terminology follows as consequence: families, signatures, instances and clones. The role of higher order patch tests in optimality is illustrated in Sections 8 and 9. SRI schemes are presented in Section 10 from the template standpoint to show that this approach naturally leads to correct splittings of the elasticity law. The concept of element families is illustrated in Section 11 using stress hybrid and displacement bubbles as examples. These clearly illuminate the futility of the “enrichment” approaches popular in the late sixties. Section 12 provides numerical examples and Section 13 offers conclusions.

The Appendix presents the construction of a four-noded bending-optimal trapezoid. Although this illustrates the customization power of templates beyond rectangles, the more advanced mathematical tools in use behind the scenes can make the results look like black magic to beginners. The optimal trapezoid partly circumvents MacNeal’s limitation theorem² in that it passes the patch test for arbitrary geometry and material and is bending optimal along one direction, while retaining a bounded condition number and thus fulfilling the inf-sup condition.

A sequel to this article³ covers the generalization to an optimal quadrilateral of arbitrary shape. The algebraic manipulations are beyond the power of any human to work out by hand over a lifetime, so in retrospect it is not surprising that this model has not been discovered sooner. Fortunately the symbolic work falls, although barely, within the grasp of computer algebra systems on a PC. The final results are surprisingly simple and elegant, even for arbitrary anisotropic material. The construction finishes a quest that has preoccupied FEM investigators over several decades.

2 HISTORICAL SKETCH

This section summarizes the history of structural finite elements since 1950 to date. It functions as a hub for dispersed historical references. Readers uninterested in these aspects should proceed directly to Section 3. For exposition convenience, structural “finiteelementology” may be divided into four generations that span 10 to 15 years each. There are no sharp intergenerational breaks, but noticeable changes of emphasis. The ensuing outline

does not cover the conjoint evolution of Matrix Structural Analysis into the Direct Stiffness Method from 1934 through 1970. This was the subject of a separate essay.⁴

2.1 G1: The Pioneers

The 1956 paper by Turner, Clough, Martin and Topp,¹ henceforth abbreviated to TCMT, is recognized as the start of the current FEM, as used in the overwhelming majority of commercial codes. Along with Argyris' serial⁵ they prototype the first generation, which spans 1950 through 1962. A panoramic picture of this period is available in two textbooks.^{6,7} Przemieniecki's text is still reprinted by Dover. The survey by Gallagher⁸ was influential but is now difficult to access.

The pioneers were structural engineers, schooled in classical mechanics. They followed a century of tradition in regarding structural elements as a device to transmit forces. This "element as force transducer" was the standard view in pre-computer structural analysis. It explains the use of flux assumptions to derive stiffness equations. Element developers worked in, or interacted closely with, the aircraft industry. [One reason is that only large aerospace companies were then able to afford mainframe computers.] Accordingly they focused on thin structures built up with bars, ribs, spars, stiffeners and panels. Although the Classical Force method dominated stress analysis during the fifties,⁴ stiffness methods were kept alive by use in dynamics and vibration.

2.2 G2: The Golden Age

The next period spans the golden age of FEM: 1962–1972. This is the "variational generation." Melosh⁹ showed that conforming displacement models are a form of Rayleigh-Ritz based on the minimum potential energy principle. This influential paper marks the confluence of three lines of research: Argyris' dual formulation of energy methods,⁵ the Direct Stiffness Method (DSM) of Turner,^{10–12} and early ideas of interelement compatibility as basis for error bounding and convergence.^{13,14} G1 workers thought of finite elements as idealizations of structural components. From 1962 onward a two-step interpretation emerges: discrete elements approximate continuum models, which in turn approximate real structures.

By the early 1960s FEM begins to expand into Civil Engineering through Clough's Boeing-Berkeley connection¹⁵ and had been named.^{16,17} Reading Fraeijs de Veubeke's celebrated article¹⁸ side by side with TCMT¹ one can sense the ongoing change in perspective opened up by the variational framework. The first book devoted to FEM appears in 1967.¹⁹ Applications to nonstructural problems start by 1965.²⁰

From 1962 onwards the displacement formulation dominates. This was given a big boost by the invention of the isoparametric formulation and related tools (numerical integration, fitted coordinates, shape functions, patch test) by Irons and coworkers.^{21–25} Low order displacement models often exhibit disappointing performance. Thus there was a frenzy to develop higher order elements. Other variational formulations, notably hybrids,^{26–29} mixed^{30,31} and equilibrium models¹⁸ emerged. G2 can be viewed as closed by the monograph of Strang and Fix,³² the first book to focus on the mathematical foundations.

2.3 G3: Consolidation

The post-Vietnam economic doldrums are mirrored during this post-1972 period. Gone is the youthful exuberance of the golden age. This is consolidation time. Substantial effort is put into improving the stock of G2 displacement elements by tools initially labeled "variational crimes" by Strang,³³ but later justified. Textbooks by Hughes³⁴ and Bathe³⁵ reflect the technology of this period. Hybrid and mixed formulations record steady progress.³⁶ Assumed strain formulations appear.³⁷ A booming activity in error estimation and mesh adaptivity is fostered by better understanding of the mathematical foundations.³⁸

Commercial FEM codes gradually gain importance. They provide a reality check on what works in the real world and what doesn't. By the mid-1980s there was gathering evidence that complex and high order elements were commercial flops. Exotic gadgetry interweaved amidst millions of lines of code easily breaks down in new releases. Complexity is particularly dangerous in nonlinear and dynamic analyses conducted by novice users. A trend back toward simplicity starts.^{39,40}

2.4 G4: Back to Basics

The fourth generation begins by the early 1980s. More approaches come on the scene, notably the Free Formulation of Bergan,^{41,42} which is further discussed below, orthogonal hourglass control,⁴³ Assumed Natural Strain methods,⁴⁴⁻⁴⁷ stress hybrid models in natural coordinates,⁴⁸⁻⁵⁰ as well as variants and derivatives of those approaches: ANDES,^{51,52} EAS^{53,54} and others. Although technically diverse the G4 approaches share two common objectives:

- (i) Elements must fit into DSM-based programs since that includes the vast majority of production codes, commercial or otherwise.
- (ii) Elements are kept simple but should provide answers of engineering accuracy with relatively coarse meshes. These were collectively labeled “high performance elements” in 1989.⁵⁵

“Things are always at their best in the beginning,” said Pascal. Indeed. By now FEM looks like an aggregate of largely disconnected methods and recipes. Sections 4-6 look at three disparate components of this edifice to anticipate the subsequent exhibition of common features by templates.

2.5 From the Free Formulation to Templates

In the early 1970s Pål Bergan — then a Professor at NTH-Trondheim — and L. Hanssen published a paper⁵⁶ in MAFELAP II, where a different approach to finite elements was advocated. The concept is well outlined in the Introduction of that paper:

“An important observation is that each element is, in fact, only represented by the numbers in its stiffness matrix during the analysis of the assembled system. The origin of these stiffness coefficients is unimportant to this part of the solution process ... The present approach is in a sense the opposite of that normally used in that the starting point is a generally formulated convergence condition and from there the stiffness matrix is derived ... The patch test is particularly attractive [as such a condition] for the present investigation in that it is a direct test on the element stiffness matrix and requires no prior knowledge of interpolation functions, variational principles, etc.”

This statement sets out what may be called the *direct algebraic approach* to finite elements: the element stiffness is to be derived directly from consistency conditions — provided by the Individual Element Test of Bergan and Hanssen^{56,57} — plus stability and accuracy considerations to determine algebraic redundancies if any.

This ambitious goal proved initially elusive because the direct algebraic construction of the stiffness matrix of most multidimensional elements becomes effectively a problem in constrained optimization. In the *symbolic form* necessitated by element design, such problem is much harder to tackle than the conventional element construction methods. A constructive step was provided by the Free Formulation, which emerged over the next decade.⁴² The foregoing difficulties were addressed using divide and conquer. The stiffness is decomposed into a basic part that takes care of consistency and mixability, and a higher order (HO) part that takes care of stability (rank sufficiency) and accuracy. Orthogonality conditions enforced between the two parts avoid pollution of the basic element response by the higher order component.

The writer's acquaintance with the FF began in 1984, while Pål Bergan was spending a 9-month sabbatical at Stanford. At the time the writer was in the staff of the Applied

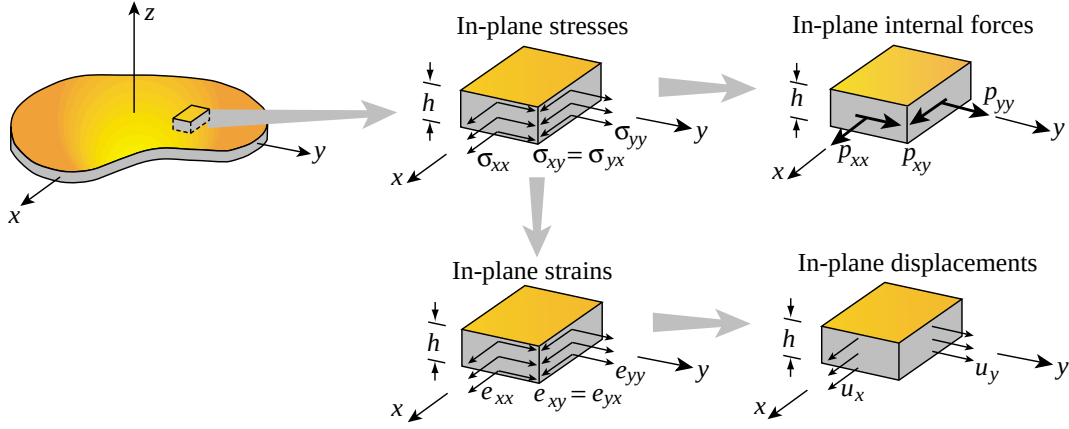


Figure 1. A thin plate in plane stress, illustrating notation.

Mechanics Laboratory of Lockheed Palo Alto Research Laboratories (LPARL, located in the Stanford Industrial Park), involved in gruelling software development supporting the Trident II system and underwater shock analysis codes. Our collaboration, supported by the LPARL Independent Research program, was the first element-related project undertaken by the writer since Berkeley days. It was a welcome respite from the software grind. It resulted in the first rank-sufficient triangular membrane element with drilling freedoms that passed the IET. Tom Hughes kindly speeded up publication.⁵⁸

The move to the University of Colorado in 1986 allowed the writer to pursue further these ideas within an academic environment. In the FF the HO stiffness was based on a displacement formulation, whereas the basic stiffness is method independent. Other techniques were tried for the HO stiffness. Particularly successful was a modification of the Assumed Natural Strain (ANS) method of Park and Stanley.^{46,47} In this variant only the deviatoric part of the assumed strains was carried forward, and the ANDES method emerged.^{51,52} Gradually the realization dawned that *all consistent and rank sufficient elements*, no matter how they are derived, must fit an algebraic form with free parameters, baptized as *template*. This “template as umbrella” concept emerged by 1990. It has since developed by fits and starts. The current status of the subject is summarized in Section 13.

3 PROBLEM DESCRIPTION

3.1 Governing Equations

Consider the thin homogeneous plate in plane stress shown in Figure 1. The inplane displacements are $\{u_x, u_y\}$, the associated strains are $\{e_{xx}, e_{yy}, e_{xy}\}$ and the inplane (membrane) stresses are $\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}$. Prescribed inplane body forces are $\{b_x, b_y\}$, but they will be set to zero in derivations of equilibrium elements. Prescribed displacements and surface tractions are denoted by $\{\hat{u}_x, \hat{u}_y\}$ and $\{\hat{t}_x, \hat{t}_y\}$ respectively. All fields are considered uniform through the thickness h . The governing plane-stress elasticity equations are

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix},$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (1)$$

The compact matrix version of (1) is

$$\mathbf{e} = \mathbf{D}\mathbf{u}, \quad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e}, \quad \mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \quad (2)$$

in which $\mathbf{E}$ is the plane stress elasticity matrix. Assuming this to be nonsingular, the inverse of $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$ is

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}, \quad \text{or} \quad \mathbf{e} = \mathbf{C}\boldsymbol{\sigma}, \quad (3)$$

where $\mathbf{C} = \mathbf{E}^{-1}$ is the matrix of elastic compliances.

3.2 The Rectangular Panel

The focus of this tutorial exposition is the *rectangular panel*. This is depicted in Figure 2. For an individual element the side-aligned local axes are also denoted as $\{x, y\}$ for brevity. The inplane dimensions are a and $b = a/\gamma$, where $\gamma = a/b$ is the aspect ratio. The thickness and elastic properties are constant over the element. The element has 4 corner nodes and 8 external (connective) degrees of freedom. The node displacement and force vectors are configured as

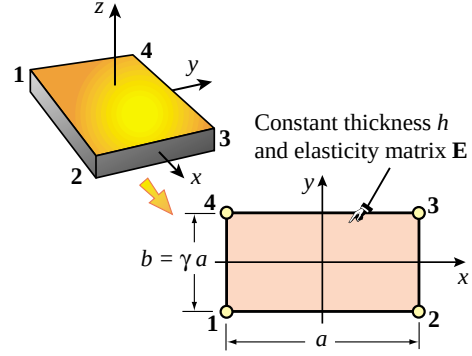


Figure 2. The rectangular panel.

$$\mathbf{u} = [u_{x1} \quad u_{y1} \quad u_{x2} \quad u_{y2} \quad u_{x3} \quad u_{y3} \quad u_{x4} \quad u_{y4}]^T, \quad (4)$$

$$\mathbf{f} = [f_{x1} \quad f_{y1} \quad f_{x2} \quad f_{y2} \quad f_{x3} \quad f_{y3} \quad f_{x4} \quad f_{y4}]^T. \quad (5)$$

As noted in the Introduction, most of the FEM formulation methods chronicled in Section 2 have been tried on this configuration as well as its plane strain and axisymmetric cousins. The reason for this popularity is that the rectangular panel is the *simplest multi-dimensional element that can be improved*. [The three-node linear triangle is simpler but cannot be improved.]

In keeping with the expository theme, the next three sections derive the rectangular panel stiffness from stress, strain and displacement assumptions, respectively. Mirroring history, the derivation of stress and strain models follows the matrix-based direct elasticity approach used by the first generation, as summarized in Gallagher's review.⁸

Ironically, the direct derivation from stress modes a la TCMT furnishes optimal or near-optimal elements with little toil, whereas the variationally derived displacement models will need tweaking to become useful.

4 THE STRESS ELEMENT

TCMT¹ is the starting point. In a historical summary Clough¹⁵ remarks that the paper belatedly reports work performed at Boeing's Commercial Airplane Division in 1952–53 (indeed a TCMT footnote states that the material was presented at the 22nd Annual Meeting of IAS, held on January 25–29, 1954.) Besides bars, beams and spars, TCMT presents two plane stress elements for modeling wing cover plates: the three-node triangle and the four-node flat rectangular panel. Quadrilateral panels of arbitrary geometry, not necessarily flat, are constructed as assemblies of four triangles.

Readers perusing that article for the first time have a surprise in store. The stiffness properties of both panel elements are derived from stress assumptions, rather than displacements, as became popular in the second generation. More precisely, simple patterns

of interelement boundary tractions (a.k.a. stress flux modes) that satisfy internal equilibrium are taken as starting point. Twenty years later Fraeijs de Veubeke⁵⁹ systematically extended the same idea in a variational setting, to produce what he called diffusive equilibrium elements. These are designed to weakly enforce interelement flux conservation. The comedy continues: mathematicians recently rediscovered flux elements, now renamed as “Discontinuous Galerkin Methods,” blissfully unaware of previous work.

The derivation below largely follows Chapter 3 of Gallagher,⁸ who presents a step by step procedure for what he calls the “equivalent force” approach. The main extension provided here is allowing for anisotropic material.

4.1 The 5-Parameter Stress Field

Since TCMT the appropriate stress field for the rectangular panel is known to be

$$\sigma_{xx} = \mu_1 + \mu_4 \frac{y}{b}, \quad \sigma_{yy} = \mu_2 + \mu_5 \frac{x}{a}, \quad \sigma_{xy} = \mu_3. \quad (6)$$

The five μ_i are stress-amplitude parameters with dimension of stress. They are collected in the 5-vector

$$\boldsymbol{\mu} = [\mu_1 \quad \mu_2 \quad \mu_3 \quad \mu_4 \quad \mu_5]^T. \quad (7)$$

The field (6) satisfies the internal equilibrium equations (1)₃ under zero body forces. Evaluation over element sides produces the traction patterns of Figure 3, transcribed verbatim from TCMT. Why five? On p. 813: “These load states are seen to represent uniform and linearly varying stresses plus constant shear, along the plate edges. Later it will be seen that the number of load states must be $2n - 3$, where n = number of nodes.”

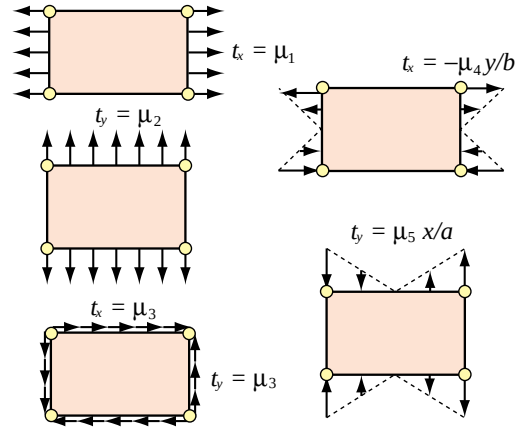


Figure 3. Interelement boundary tractions associated with the stress parameters μ_i in (7). After TCMT, p. 812, in which these five patterns are called “load states.”

To establish connection to node displacements, $\boldsymbol{\mu}$ is extended as

$$\boldsymbol{\mu}_+ = [\mu_1 \quad \mu_2 \quad \mu_3 \quad \mu_4 \quad \mu_5 \quad \mu_6 \quad \mu_7 \quad \mu_8]^T \quad (8)$$

This array contains three dimensionless coefficients: μ_6 , μ_7 and μ_8 , which define amplitudes of the three element rigid body modes (RBMs):

$$\text{RBM}_1: u_x = \mu_6 a, \quad u_y = 0, \quad \text{RBM}_2: u_x = 0, \quad u_y = \mu_7 b, \quad \text{RBM}_3: u_x = -\mu_8 y, \quad u_y = \mu_8 x. \quad (9)$$

These modes produce zero stress. The foregoing relations may be recast in matrix form:

$$\boldsymbol{\sigma} = \mathbf{N} \boldsymbol{\mu} = \mathbf{N}_+ \boldsymbol{\mu}_+, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & \frac{y}{b} & 0 \\ 0 & 1 & 0 & 0 & \frac{x}{a} \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_+ = \begin{bmatrix} 1 & 0 & 0 & \frac{y}{b} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{x}{a} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (10)$$

The boundary traction patterns of Figure 3 are converted to node forces by statics. This yields

$$\mathbf{f} = \mathbf{A} \boldsymbol{\mu}, \quad \mathbf{A}^T = \frac{1}{2} h \begin{bmatrix} -b & 0 & b & 0 & b & 0 & -b & 0 \\ 0 & -a & 0 & -a & 0 & a & 0 & a \\ -a & -b & -a & b & a & b & a & -b \\ \frac{1}{6}b & 0 & -\frac{1}{6}b & 0 & \frac{1}{6}b & 0 & -\frac{1}{6}b & 0 \\ 0 & \frac{1}{6}a & 0 & -\frac{1}{6}a & 0 & \frac{1}{6}a & 0 & -\frac{1}{6}a \end{bmatrix}. \quad (11)$$

Matrix $\mathbf{A}$ is the equilibrium matrix, also known as the leverage matrix in the FEM literature. When restricted to constant stress states (the first three columns of $\mathbf{A}$), it is called a force-lumping matrix and denoted by $\mathbf{L}$ in the Free Formulation of Bergan.^{41,42,58,60–65}

4.2 The Generalized Stiffness

Integrating the complementary energy density $\mathcal{U}^* = \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{C} \boldsymbol{\sigma}$ over the element volume V and identifying $U^* = \int_{V_e} \mathcal{U}^* dV$ with $\frac{1}{2} \boldsymbol{\mu}^T \mathbf{F}_\mu \boldsymbol{\mu}$ yields the 5×5 flexibility matrix $\mathbf{F}_\mu$ in terms of the stress parameters. Its inverse is the generalized stiffness matrix $\mathbf{S}_\mu = \mathbf{F}_\mu^{-1}$:

$$\mathbf{F}_\mu = V \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} C_{11} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} C_{22} \end{bmatrix}, \quad \mathbf{S}_\mu = \frac{1}{V} \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 \\ E_{13} & E_{23} & E_{33} & 0 & 0 \\ 0 & 0 & 0 & 12C_{11}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 12C_{22}^{-1} \end{bmatrix}, \quad (12)$$

in which $V = abh$ is the volume of the element.

4.3 The Physical Stiffness

Integration of the slave strain field $\mathbf{e} = \mathbf{E}^{-1} \boldsymbol{\sigma} = \mathbf{C} \mathbf{N}_+ \boldsymbol{\mu}_+$ produces the displacement field

$$\begin{aligned} u_x(x, y) &= \mu_6 a + \frac{1}{8} \omega_6 + (\mu_1 C_{11} + \mu_2 C_{12} + \mu_3 C_{13})x + \left(\frac{1}{2} (\mu_1 C_{13} + \mu_2 C_{23} + \mu_3 C_{33}) - \mu_8 \right) y \\ &\quad + \frac{1}{2} (\mu_5/a) C_{12} x^2 + (\mu_4/b) C_{11} xy + \frac{1}{2} ((\mu_4/b) C_{13} - (\mu_5/a) C_{22}) y^2, \\ u_y(x, y) &= \mu_7 b + \frac{1}{8} \omega_7 + \left(\frac{1}{2} (\mu_1 C_{13} + \mu_2 C_{23} + \mu_3 C_{33}) + \mu_8 \right) x + (\mu_1 C_{12} + \mu_2 C_{22} + \mu_3 C_{23}) y \\ &\quad + \frac{1}{2} ((\mu_5/a) C_{23} - (\mu_4/b) C_{11}) x^2 + (\mu_5/a) C_{22} xy + \frac{1}{2} (\mu_4/b) C_{12} y^2. \end{aligned} \quad (13)$$

with $\omega_6 = -b^2 C_{13} \mu_4/b + (b^2 C_{22} - a^2 C_{12})(\mu_5/a)$ and $\omega_7 = (a^2 C_{11} - b^2 C_{12})(\mu_4/b) - a^2 C_{23} \mu_5/a$. The constant terms in u_x and u_y , which do not affect strains and stresses, have been adjusted to get relatively simple terms in columns 4 through 8 of the matrix $\mathbf{T}_+$ below. Physically, (13) aligns the bending deformation patterns along the $\{x, y\}$ axes. Evaluating (13) at the nodes we obtain the matrix that connects node displacements to stress parameters: $\mathbf{u} = \mathbf{T}_+ \boldsymbol{\mu}_+$, where

$$\mathbf{T}_+ = \frac{1}{4} \begin{bmatrix} -2aC_{11} - bC_{13} & -2aC_{12} - bC_{23} & -2aC_{13} - bC_{33} & aC_{11} & 0 & 4a & 0 & 2b \\ -2bC_{12} - aC_{13} & -2bC_{22} - aC_{23} & -2bC_{23} - aC_{33} & 0 & bC_{22} & 0 & 4b & -2a \\ 2aC_{11} - bC_{13} & 2aC_{12} - bC_{23} & 2aC_{13} - bC_{33} & -aC_{11} & 0 & 4a & 0 & 2b \\ -2bC_{12} + aC_{13} & -2bC_{22} + aC_{23} & -2bC_{23} + aC_{33} & 0 & -bC_{22} & 0 & 4b & 2a \\ 2aC_{11} + bC_{13} & 2aC_{12} + bC_{23} & 2aC_{13} + bC_{33} & aC_{11} & 0 & 4a & 0 & -2b \\ 2bC_{12} + aC_{13} & 2bC_{22} + aC_{23} & 2bC_{23} + aC_{33} & 0 & bC_{22} & 0 & 4b & 2a \\ -2aC_{11} + bC_{13} & -2aC_{12} + bC_{23} & -2aC_{13} + bC_{33} & -aC_{11} & 0 & 4a & 0 & -2b \\ 2bC_{12} - aC_{13} & 2bC_{22} - aC_{23} & 2bC_{23} - aC_{33} & 0 & -bC_{22} & 0 & 4b & -2a \end{bmatrix} \quad (14)$$

The determinant of $\mathbf{T}_+$ is $a^4 b^4 C_{11} C_{22} \det(\mathbf{C})$, so $\mathbf{T}_+$ is invertible if $a \neq 0$, $b \neq 0$,

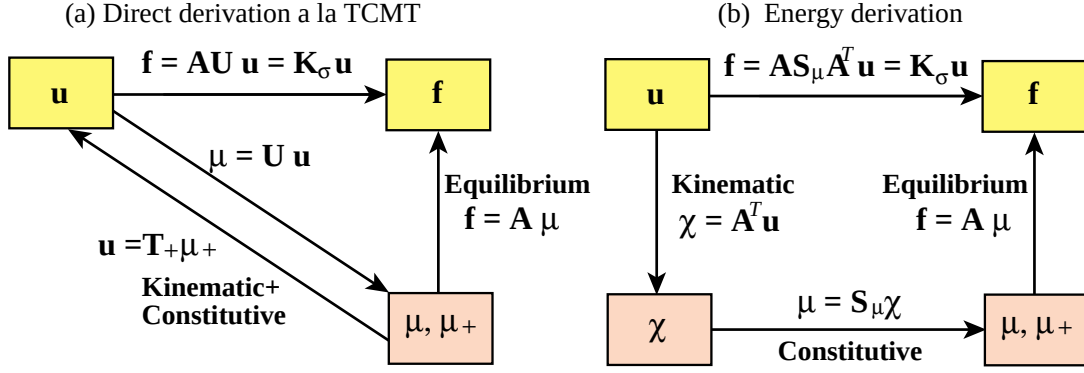


Figure 4. Derivation of the stress-assumed rectangular panel stiffness. Left side shows derivation bypassing energy methods.

$C_{11} \neq 0$, $C_{22} \neq 0$ and $\mathbf{C}$ is nonsingular. Inversion yields $\mu_+ = \mathbf{U}_+ \mathbf{u}$, where

$$\mathbf{U}_+ = \mathbf{T}_+^{-1} = \frac{1}{ab} \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} \\ U_{21} & U_{22} & U_{23} & U_{24} & U_{25} & U_{26} & U_{27} & U_{28} \\ U_{31} & U_{32} & U_{33} & U_{34} & U_{35} & U_{36} & U_{37} & U_{38} \\ bC_{11}^{-1} & 0 & -bC_{11}^{-1} & 0 & bC_{11}^{-1} & 0 & -bC_{11}^{-1} & 0 \\ 0 & aC_{22}^{-1} & 0 & -aC_{22}^{-1} & 0 & aC_{22}^{-1} & 0 & -aC_{22}^{-1} \\ \frac{1}{4}b & 0 & \frac{1}{4}b & 0 & \frac{1}{4}b & 0 & \frac{1}{4}b & 0 \\ 0 & \frac{1}{4}a & 0 & \frac{1}{4}a & 0 & \frac{1}{4}a & 0 & \frac{1}{4}a \\ \frac{1}{4}a & -\frac{1}{4}b & \frac{1}{4}a & \frac{1}{4}b & -\frac{1}{4}a & \frac{1}{4}b & -\frac{1}{4}a & -\frac{1}{4}b \end{bmatrix}, \quad (15)$$

in which $U_{11} = -\frac{1}{2}(bE_{11}+aE_{13})$, $U_{12} = -\frac{1}{2}(aE_{12}+bE_{13})$, $U_{13} = \frac{1}{2}(bE_{11}-aE_{13})$, $U_{14} = -\frac{1}{2}(aE_{12}-bE_{13})$, $U_{15} = \frac{1}{2}(bE_{11}+aE_{13})$, $U_{16} = \frac{1}{2}(aE_{12}+bE_{13})$, $U_{17} = -\frac{1}{2}(bE_{11}-aE_{13})$, $U_{18} = \frac{1}{2}(aE_{12}-bE_{13})$, $U_{21} = -\frac{1}{2}(bE_{12}+aE_{23})$, $U_{22} = -\frac{1}{2}(aE_{22}+bE_{23})$, $U_{23} = \frac{1}{2}(bE_{12}-aE_{23})$, $U_{24} = -\frac{1}{2}(aE_{22}-bE_{23})$, $U_{25} = \frac{1}{2}(bE_{12}+aE_{23})$, $U_{26} = \frac{1}{2}(aE_{22}+bE_{23})$, $U_{27} = -\frac{1}{2}(bE_{12}-aE_{23})$, $U_{28} = \frac{1}{2}(aE_{22}-bE_{23})$, $U_{31} = -\frac{1}{2}(bE_{13}+aE_{33})$, $U_{32} = -\frac{1}{2}(aE_{23}+bE_{33})$, $U_{33} = \frac{1}{2}(bE_{13}-aE_{33})$, $U_{34} = -\frac{1}{2}(aE_{23}-bE_{33})$, $U_{35} = \frac{1}{2}(bE_{13}+aE_{33})$, $U_{36} = \frac{1}{2}(aE_{23}+bE_{33})$, $U_{37} = -\frac{1}{2}(bE_{13}-aE_{33})$ and $U_{38} = \frac{1}{2}(aE_{23}-bE_{33})$. The stress-displacement matrix $\mathbf{U}$ that relates stress parameters to displacements: $\mu = \mathbf{U} \mathbf{u}$, is obtained by extracting the first five rows of $\mathbf{U}_+$:

$$\mathbf{U} = \frac{1}{ab} \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} \\ U_{21} & U_{22} & U_{23} & U_{24} & U_{25} & U_{26} & U_{27} & U_{28} \\ U_{31} & U_{32} & U_{33} & U_{34} & U_{35} & U_{36} & U_{37} & U_{38} \\ bC_{11}^{-1} & 0 & -bC_{11}^{-1} & 0 & bC_{11}^{-1} & 0 & -bC_{11}^{-1} & 0 \\ 0 & aC_{22}^{-1} & 0 & -aC_{22}^{-1} & 0 & aC_{22}^{-1} & 0 & -aC_{22}^{-1} \end{bmatrix} = \mathbf{S}_\mu \mathbf{A}^T. \quad (16)$$

The relation $\mathbf{U} = \mathbf{S}_\mu \mathbf{A}^T$ can be checked directly. For this element it can be proven to hold by energy methods, but that was not obvious in 1952. It must have been a relief when the element stiffness came out symmetric. As Gallagher⁸ remarks on p. 22, symmetry is the exception rather than the rule for more general geometric configurations. That complication proved a big boost for the energy and variational methods of the second generation.

The physical stiffness $\mathbf{K}_\sigma$ relates $\mathbf{f} = \mathbf{K}_\sigma \mathbf{u}$, where the σ subscript flags the stress

element. Combining $\mathbf{f} = \mathbf{A}\boldsymbol{\mu}$ and $\boldsymbol{\mu} = \mathbf{U}\mathbf{u} = \mathbf{S}_\mu \mathbf{A}^T \mathbf{u}$ yields

$$\mathbf{K}_\sigma = \mathbf{A}\mathbf{U} = \mathbf{A}\mathbf{S}_\mu \mathbf{A}^T. \quad (17)$$

Figure 4 summarizes the foregoing derivation steps. Note that one can bypass the calculation of the generalized stiffness $\mathbf{S}_\mu$ if so desired, as diagramed on the left of that figure. This is convenient for presentation to students without a background on energy methods.

Note that the displacement field (13) contains quadratic terms if μ_4 or μ_5 are nonzero. Hence the element is nonconforming. This is acknowledged but dismissed as innocuous on p. 814 of TCMT.

5 THE STRAIN ELEMENT

A strain-assumed element can be developed through an entirely analogous procedure. The counterpart of (6) is

$$e_{xx} = \chi_1 + \chi_4 \frac{y}{b}, \quad e_{yy} = \chi_2 + \chi_5 \frac{x}{a}, \quad 2e_{xy} = \chi_3. \quad (18)$$

where the χ_i are dimensionless strain-amplitude parameters. They are collected in the 5-vector

$$\boldsymbol{\chi} = [\chi_1 \quad \chi_2 \quad \chi_3 \quad \chi_4 \quad \chi_5]^T. \quad (19)$$

An extended vector is constructed by appending the RBM amplitudes

$$\boldsymbol{\chi}_+ = [\chi_1 \quad \chi_2 \quad \chi_3 \quad \chi_4 \quad \chi_5 \quad \chi_6 \quad \chi_7 \quad \chi_8]^T. \quad (20)$$

in which χ_6 , χ_7 and χ_8 are defined a a manner similar to (9). Note that $\mathbf{e} = \mathbf{N}\boldsymbol{\chi} = \mathbf{N}_+\boldsymbol{\chi}_+$ where $\mathbf{N}$ and $\mathbf{N}_+$ are defined in (10). Integrating the strains yields the displacement field

$$\begin{aligned} u_x(x, y) &= \chi_6 + \chi_8 y + (\chi_1 + \chi_4/b)xy - \frac{1}{2}(\chi_5/a)y^2, \\ u_y(x, y) &= \chi_7 + (\chi_3 - \chi_8)x + \chi_2 y - \frac{1}{2}(\chi_4/b)x^2 + (\chi_5/a)xy. \end{aligned} \quad (21)$$

Evaluating at the nodes and inverting yields $\boldsymbol{\chi}_+ = \mathbf{B}_+\mathbf{u}$ where

$$\mathbf{B}_+ = \frac{1}{8ab} \begin{bmatrix} -4b & 0 & 4b & 0 & 4b & 0 & -4b & 0 \\ 0 & -4a & 0 & -4a & 0 & 4a & 0 & 4a \\ -4a & -4b & -4a & 4b & 4a & 4b & 4a & -4b \\ 8b & 0 & -8b & 0 & 8b & 0 & -8b & 0 \\ 0 & 8a & 0 & -8a & 0 & 8a & 0 & -8a \\ 2ab & b^2 & 2ab & -b^2 & 2ab & b^2 & 2ab & -b^2 \\ a^2 & 2ab & -a^2 & 2ab & a^2 & 2ab & -a^2 & 2ab \\ -4a & 0 & -4a & 0 & 4a & 0 & 4a & 0 \end{bmatrix} \quad (22)$$

from which we extract the first five rows to get the strain-displacement matrix relating $\boldsymbol{\chi} = \mathbf{B}_\chi \mathbf{u}$:

$$\mathbf{B}_\chi = \frac{h}{2V} \begin{bmatrix} -b & 0 & b & 0 & b & 0 & -b & 0 \\ 0 & -a & 0 & -a & 0 & a & 0 & a \\ -a & -b & -a & b & a & b & a & -b \\ 2b & 0 & -2b & 0 & 2b & 0 & -2b & 0 \\ 0 & 2a & 0 & -2a & 0 & 2a & 0 & -2a \end{bmatrix} \quad (23)$$

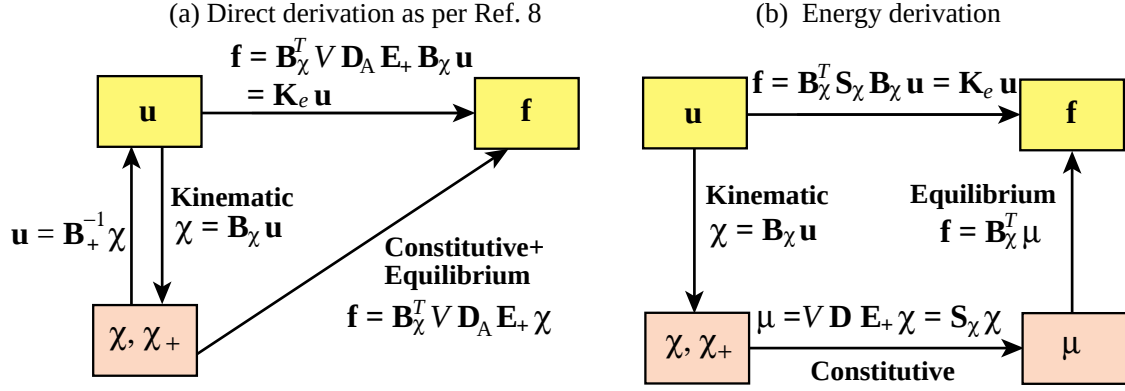


Figure 5. Derivation of the strain-assumed rectangular panel stiffness.
Left diagram shows derivation bypassing energy methods.

For use below we note the following relation between the transformation matrices of the stress and strain elements

$$\mathbf{A}^T = V \mathbf{D}_A \mathbf{B}_\chi, \quad \mathbf{B}_\chi = \frac{1}{V} \mathbf{D}_A^{-1} \mathbf{A}^T, \quad \mathbf{D}_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \frac{1}{12} \mathbf{I}_2 \end{bmatrix} = \mathbf{D}_A^T. \quad (24)$$

From (11) the lumping of the slave stress field $\mathbf{Ee} = \mathbf{E} \mathbf{N} \chi$ to node forces can be worked out to be

$$\mathbf{f} = \mathbf{A} \mathbf{E}_+ \chi = V \mathbf{B}_\chi^T \mathbf{D}_A \mathbf{E}_+ \chi, \quad \text{with} \quad \mathbf{E}_+ = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 \\ E_{13} & E_{23} & E_{33} & 0 & 0 \\ 0 & 0 & 0 & E_{11} & 0 \\ 0 & 0 & 0 & 0 & E_{22} \end{bmatrix} \quad (25)$$

Combining previous equations, the physical element stiffness is

$$\mathbf{K}_e = V \mathbf{B}_\chi^T \mathbf{D}_A \mathbf{E}_+ \mathbf{B}_\chi = \mathbf{B}_\chi^T \mathbf{K}_\chi \mathbf{B}_\chi, \quad \text{with} \quad \mathbf{K}_\chi = V \mathbf{D}_A \mathbf{E}_+. \quad (26)$$

Here $\mathbf{K}_\chi$ denotes the generalized stiffness in terms of χ . This matrix may be obtained also from standard energy arguments: the strain energy density is $\mathcal{U} = \frac{1}{2} \chi^T \mathbf{E} \chi$. Integrating over the element volume: $U = \int_{V_e} \mathcal{U} dV$ and identifying with $\frac{1}{2} \chi^T \mathbf{K}_\chi \chi$ gives

$$\mathbf{K}_\chi = V \mathbf{D}_A \mathbf{E}_+ = V \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 \\ E_{13} & E_{23} & E_{33} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} E_{11} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} E_{22} \end{bmatrix} \quad (27)$$

Figure 5 summarizes the foregoing derivation steps. The direct step from χ to $\mathbf{f}$ on the left is more difficult to explain to students than the step from $\mathbf{u}$ to μ in Figure 4. The energy based formulation shown on the right of Figure 5 tends to be more palatable.

6 THE CONFORMING DISPLACEMENT ELEMENT

This derivation of the assumed-displacement element starts from a conforming displacement field that enforces linear edge displacements. Using the matrix notation of Felippa and Clough⁶⁶ for Irons' isoparametric formulation²⁵ specialized to the rectangle, the displacement field is bilinearly interpolated as

$$\begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -a & 0 & a & 0 & a & 0 & -a & 0 \\ 0 & -b & 0 & -b & 0 & b & 0 & b \end{bmatrix} \begin{bmatrix} \frac{1}{4}(1-\xi)(1-\eta) \\ \frac{1}{4}(1+\xi)(1-\eta) \\ \frac{1}{4}(1+\xi)(1+\eta) \\ \frac{1}{4}(1-\xi)(1+\eta) \end{bmatrix}, \quad (28)$$

where $\xi = 2x/a$ and $\eta = 2y/b$ are the dimensionless quadrilateral coordinates running from -1 to 1 . The derivation based on the minimum potential energy principle is standard textbook material and only the final result is presented here:

$$\mathbf{K}_u = \mathbf{B}_u^T \mathbf{K}_q \mathbf{B}_u, \quad \text{with} \quad \mathbf{K}_q = \frac{1}{V} \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 \\ E_{13} & E_{23} & E_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{11} & Q_{12} \\ 0 & 0 & 0 & Q_{12} & Q_{22} \end{bmatrix}, \quad (29)$$

in which $\mathbf{B}_u = \mathbf{A}^T$ as given by (11) and

$$Q_{11} = 12 \frac{b^2 E_{11} + a^2 E_{33}}{ab^3 h}, \quad Q_{12} = 12 \left(\frac{E_{13}}{a^2 h} + \frac{E_{23}}{b^2 h} \right), \quad Q_{22} = 12 \frac{a^2 E_{22} + b^2 E_{33}}{a^3 b h} \quad (30)$$

This model has a checkered history. It was first derived as a rectangular panel with edge reinforcements (omitted here) by Argyris in his 1954 *Aircraft Engineering* series; see pp. 49–52 of the Butterworths reprint.⁵ Argyris used bilinear displacement interpolation in Cartesian coordinates. After much flailing, a conforming generalization to arbitrary geometry was published in 1964 by Taig and Kerr⁶⁷ using quadrilateral-fitted coordinates called $\{\xi, \eta\}$ but running from 0 to 1. This paper cites an 1961 English Electric Aircraft internal report as original source but Irons and Ahmad²⁵ comment in their reference [108] that the work goes back to 1957.) Irons, who was aware of Taig's work while at Rolls Royce, created the seminal isoparametric family as a far-reaching extension upon moving to Swansea.^{21–24}

7 TEMPLATES

7.1 Stiffness Decomposition

The stiffnesses $\mathbf{K}_\sigma$, $\mathbf{K}_e$ and $\mathbf{K}_u$ derived in the foregoing three Sections do not appear to have much in common. Indeed if one looks at just the matrix entries no pattern is readily seen. Closer examination reveals, however, that they are instances of the algebraic form

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_h = V \mathbf{H}_c^T \mathbf{E} \mathbf{H}_c + V \mathbf{H}_h^T \mathbf{W}^T \mathbf{R} \mathbf{W} \mathbf{H}_h, \quad (31)$$

where $V = abh$ is the element volume and

$$\begin{aligned} \mathbf{H}_c &= \frac{1}{2ab} \begin{bmatrix} -b & 0 & b & 0 & b & 0 & -b & 0 \\ 0 & -a & 0 & -a & 0 & a & 0 & a \\ -a & -b & -a & b & a & b & a & -b \end{bmatrix}, \\ \mathbf{H}_h &= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}, \\ \mathbf{W} &= \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix}. \end{aligned} \quad (32)$$

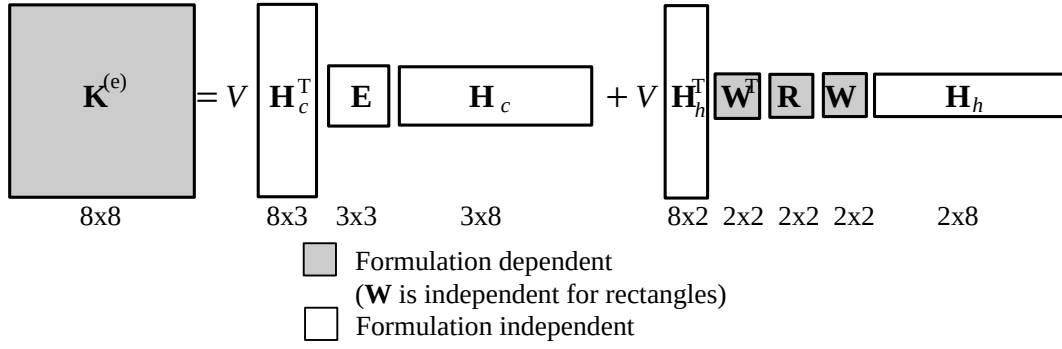


Figure 6. The template for the rectangular panel, illustrating formulation dependent and independent parts.

Matrices $\mathbf{H}_c$ and $\mathbf{H}_h$ are identical for the three models. The generalized bending rigidity $\mathbf{R}$ is formulation dependent. Matrix $\mathbf{W}$ is a “higher-order-mode weighting matrix,” hence the notation. For rectangular panels $\mathbf{W}$ is diagonal and formulation independent. For more complex geometries discussed in the Appendix and in the sequel³ $\mathbf{W}$ may be formulation-adjusted to make $\mathbf{R}$ simpler.

For the stress, strain and displacement models $\mathbf{R}$ becomes $\mathbf{R}_\sigma$, $\mathbf{R}_e$ and $\mathbf{R}_u$, respectively, where

$$\mathbf{R}_\sigma = \frac{1}{3} \begin{bmatrix} C_{11}^{-1} & 0 \\ 0 & C_{22}^{-1} \end{bmatrix}, \quad \mathbf{R}_e = \frac{1}{3} \begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix}, \quad \mathbf{R}_u = \frac{1}{3} \begin{bmatrix} E_{11} + \frac{a^2 E_{33}}{b^2} & \frac{b E_{13}}{a} + \frac{a E_{23}}{b} \\ \frac{b E_{13}}{a} + \frac{a E_{23}}{b} & E_{22} + \frac{b^2 E_{33}}{a^2} \end{bmatrix} \quad (33)$$

But we are not in fact restricted to these. Other expressions for $\mathbf{R}$ would yield other $\mathbf{K}$. These are possible, although not necessarily useful, stiffnesses for the rectangular panel if $\mathbf{R}$ is symmetric and positive definite, and if its entries have physical dimensions of elastic moduli. Further if $E_{13} = E_{23} = 0$ we set $R_{12} = 0$. The key discovery is that the element formulation affects only part of the stiffness expression, as highlighted in Figure 6.

7.2 Template Terminology

The algebraic form characterized by (31) and (32) is called a finite element stiffness template, or *template* for short.

Matrices $\mathbf{K}_b$ and $\mathbf{K}_h$ are called the basic and higher-order stiffness matrix, respectively, in accordance with the fundamental decomposition of the Free Formulation.^{41,42,58–65} These two matrices play different and complementary roles.

The basic stiffness $\mathbf{K}_b$ takes care of consistency and mixability. In the Free Formulation a restatement of (31) is preferred:

$$\mathbf{K}_b = V^{-1} \mathbf{L} \mathbf{E} \mathbf{L}^T, \quad (34)$$

where $\mathbf{L} = \mathbf{H}_c / V$ is called the force lumping matrix, or simply lumping matrix.

The higher order stiffness $\mathbf{K}_h$ is a *stabilization* term that provides the correct rank and may be adjusted for accuracy. This matrix is orthogonal to rigid body motions and constant strain states. To verify the claim for this particular template introduce the following 8×6

```

RectPanel4TemplateStiffness[{a_, b_}, Emat_, Cmat_, h_, name_, Rlist_] :=
Module[{V, found, Hc, Hh, W, Ke}, V=a*b*h;
{WRW, found}=RectPanel4TemplateWRW[{a, b}, Emat, Cmat, name, Rlist];
If [Not[found], Print["Illegal elem name: ", name]; Abort[]];
Hc={{-b, 0, b, 0, b, 0, -b, 0}, {0, -a, 0, -a, 0, a, 0, a},
{-a, -b, -a, b, a, b, a, -b}}/(2*a*b);
Hh={{1, 0, -1, 0, 1, 0, -1, 0}, {0, 1, 0, -1, 0, 1, 0, -1}}/2;
Ke=V*Transpose[Hc].Emat.Hc+V*Transpose[Hh].WRW.Hh;
Return[Simplify[Ke]];

RectPanel4TemplateWRW[{a_, b_}, Emat_, Cmat_, name_, Rlist_] :=
Module[{R11, R12, R22, Rmat, E11, E12, E13, E22, E23, E33,
found=False, C11, C22, C33, C12, C13, C23, Edet, Cdet, W, WRW},
{{E11, E12, E13}, {E12, E22, E23}, {E13, E23, E33}}=Emat;
If [Length[Cmat]<=0,
Edet=E11*E22*E33+2*E12*E13*E23-E11*E23^2-E22*E13^2-E33*E12^2;
C11=(E22*E33-E23^2)/Edet; C22=(E11*E33-E13^2)/Edet;
C33=(E11*E22-E12^2)/Edet; C12=(E13*E23-E12*E33)/Edet;
C13=(E12*E23-E13*E22)/Edet; C23=(E12*E13-E11*E23)/Edet,
{{C11, C12, C13}, {C12, C22, C23}, {C13, C23, C33}}=Cmat,
{{C11, C12, C13}, {C12, C22, C23}, {C13, C23, C33}}=Cmat];
If [name=="Stress" || name=="QM6" || name=="Q6",
R11=1/(3*C11); R22=1/(3*C22); R12=0; found=True];
If [name=="Strain", R11=E11/3; R22=E22/3; R12=0; found=True];
If [name=="Disp", R11=(E11+E33*a^2/b^2)/3;
R22=(E22+E33*b^2/a^2)/3; R12=(E13*b/a+E23*a/b)/3; found=True];
If [name=="Arbitrary", {R11, R12, R22}=Rlist; found=True];
W={{1/a, 0}, {0, 1/b}}; Rmat={{R11, R12}, {R12, R22}};
WRW=Transpose[W].Rmat.W; Return[{WRW, found}]];

```

Figure 7. A *Mathematica* implementation of the rectangular panel template (31).

matrix, called the basic-mode matrix in the Free Formulation:

$$\mathbf{G}_{rc} = \begin{bmatrix} 1 & 0 & y_1 & x_1 & 0 & y_1 \\ 0 & 1 & -x_1 & 0 & y_1 & x_1 \\ 1 & 0 & y_2 & x_2 & 0 & y_2 \\ 0 & 1 & -x_2 & 0 & y_2 & x_2 \\ 1 & 0 & y_3 & x_3 & 0 & y_3 \\ 0 & 1 & -x_3 & 0 & y_3 & x_3 \\ 1 & 0 & y_4 & x_4 & 0 & y_4 \\ 0 & 1 & -x_4 & 0 & y_4 & x_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & -b & -a & 0 & -b \\ 0 & 2 & a & 0 & -b & -a \\ 2 & 0 & -b & a & 0 & -b \\ 0 & 2 & -a & 0 & -b & a \\ 2 & 0 & b & a & 0 & b \\ 0 & 2 & -a & 0 & b & a \\ 2 & 0 & b & -a & 0 & b \\ 0 & 2 & a & 0 & b & -a \end{bmatrix}. \quad (35)$$

The six columns of $\mathbf{G}_{rc}$ span the rigid body modes and constant strain states evaluated at the nodes (these bases are not orthonormalized as that property is not required here). It is readily checked that $\mathbf{H}_h \mathbf{G}_{rc} = \mathbf{0}$. Therefore those modes, and any linear combination thereof, are orthogonal to the higher order stiffness: $\mathbf{K}_h \mathbf{G}_{rc} = \mathbf{0}$. So the role of $\mathbf{H}_h$ is essentially that of a *geometric projector*.

A *Mathematica* implementation of (31) as module `RectPanel4TemplateStiffness` is shown in Figure 7. The module arguments are the rectangle dimensions as list $\{a, b\}$, the elasticity matrix as list $\text{Emat} = \{\{E11, E12, E13\}, \{E12, E22, E23\}, \{E13, E23, E33\}\}$, the compliance matrix as $\text{Cmat} = \{\{C11, C12, C13\}, \{C12, C22, C23\}, \{C13, C23, C33\}\}$, the thickness h , the name as one of "Stress", "Strain", "Disp", "Q6", "QM6" or "Arbitrary", and finally the list $\text{Rlist} = \{R11, R12, R22\}$. The latter is used if the name is "Arbitrary". This comes handy for finding the signature of known elements leaving the entries of Rlist symbolic and using the `Solve` command to match existing or new

Table 1. A Clone Gallery

Name	Description	Clones and sources
StressRP (a.k.a. BORP)	5-stress-mode element of Section 4	Direct derivation: TCMT ¹ , Gallagher ⁸ Pian 5-mode stress hybrid ^{27,26,29} Wilson-Taylor-Doherty-Ghaboussi Q6 ⁶⁸ Taylor-Wilson-Beresford QM6 ⁶⁹ Belytschko-Liu-Engelmann QBI ⁷⁰ SRI of iso-P with E split as per (54)
StrainRP	5-strain-mode element of Section 5	MacNeal QUAD4 ^{39,71} SRI of iso-P with E split as per (56)
DispRP	Bilinear iso-P element of Section 6	Argyris ⁵ as edge stiffened rectangular panel Taig-Kerr ⁶⁷ as specialized quadrilateral
Note 1: Many plane stress models listed above were derived for quadrilateral geometries, and a few as membrane component of shells. The right-hand-column classification only pertains to the rectangular panel specialization. For example, Q6 and QM6 differ for non-parallelogram shapes. Note 2: Instances of the stress-hybrid and displacement-bubble-function “futile families” studied in Section 11 are omitted, as they lack practical value. Note 3: Post-1990 clones (e.g. EAS⁵⁴) omitted to save space. See Lautersztajn and Samuelsson⁷² for a recent survey.		

elements. If $\mathbf{Cmat}$ is supplied as the empty list $\{ \}$, the compliance matrix is calculated internally as inverse of $\mathbf{Emat}$.

The module returns the 8×8 stiffness matrix $\mathbf{Ke}$ as function value. To get the basic stiffness $\mathbf{K}_b$ only, call with `name = "Arbitrary"` and `Rlist={0,0,0}`.

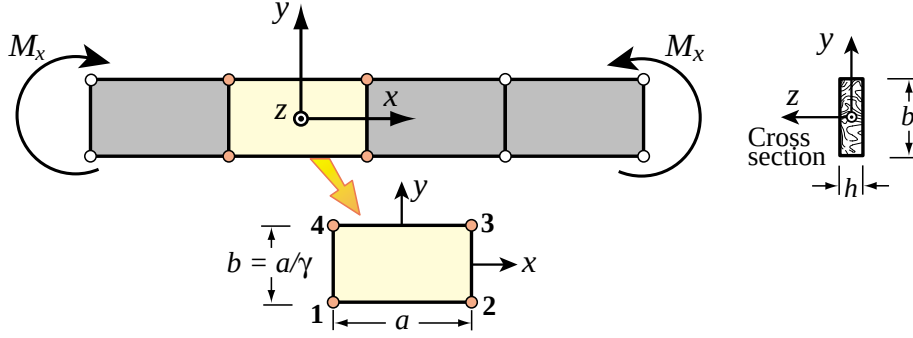
7.3 Requirements

An acceptable template fulfills four conditions: (C) consistency, (S) stability (correct rank), (I) observer invariance and (P) parametrization. These are discussed at length in other papers.^{73–79} Conditions (C) and (S) are imposed to ensure convergence as the mesh size is reduced by enforcing *a priori* satisfaction of the Individual Element Test (IET) of Bergan and Hanssen.^{56,57} Condition (P) means that the template contains free parameters or free matrix entries. In the case of (31), the simplest choice of parameters are the entries R_{11} , R_{12} , R_{22} themselves. To fulfill stability, $R_{11} > 0$, $R_{22} > 0$ and $R_{11}R_{22} - R_{12}^2 > 0$. Parametrization facilitates performance optimization as well as tuning elements, or combinations of elements, to fulfill specific needs.

Using the IET as departure point it is not difficult to show⁸⁰ that (31), under the stated restrictions on $\mathbf{R}$, includes all stiffnesses that satisfy the IET and stability. Observer invariance is a moot point for this element since $\{x, y\}$ are side aligned. Thus (31) is in fact a universal template for the rectangular panel.

7.4 Instances, Signatures, Clones

Setting the free parameters to specific values yields element instances. The set of free parameters is called the template *signature*, a term introduced in previous papers.^{77,78} Borrowing terminology from biogenetics, the signature may be viewed as an “element DNA” that uniquely characterizes it as an individual entity. Elements derived by different techniques that share the same signature are called *clones*.


 Figure 8. Constant-moment inplane-bending test along the x side dimension.

One of the “template services” is automatic identification of clones. If two elements fitting the template (31) share R_{11} , R_{12} and R_{22} , they are clones. Inasmuch as most FEM formulation schemes have been tried on the rectangular panel, it should come as no surprise that there are many clones, particularly of the stress element. Those published before 1990 are collected in Table 1. For example, the incompatible mode element Q6 of Wilson et al.⁶⁸ is a clone of StressRP. The version QM6 of Taylor et al.⁶⁹ which passes the patch test for arbitrary geometries, reduces to Q6 for rectangular and parallelogram shapes. Even for this simple geometry recognition of some of the coalescences took some time, as recently narrated by Pian.²⁹

8 FINDING THE BEST

An universal template is nice to have. The obvious question arises: among the infinity of elements that it can generate, is there a best one? By construction all instances verify exactly the IET for rigid body modes and uniform strain states. Hence the optimality criterion must rely on higher order patch tests.

8.1 The Bending Tests

The obvious tests involve response to in-plane bending along the side directions. This leads to comparisons in the form of energy ratios. These have been used since 1984 to tune up the higher order stiffness of triangular elements.^{58–62,81} An extension introduced in this article is consideration of arbitrary anisotropic material. All symbolic calculations were carried out with *Mathematica*.

The x bending test is depicted in Figure 8. A Bernoulli-Euler plane beam of thin rectangular cross-section with height b and thickness h (normal to the plane of the figure) is bent under applied end moments M_x . The beam is fabricated of anisotropic material with the stress-strain law $\sigma = \mathbf{E}\mathbf{e}$ of $(2)_2$. Except for possible end effects the exact solution of the beam problem (from both the theory-of-elasticity and beam-theory standpoints) is a constant bending moment $M(x) = M_x$ along the span. The associated stress field is $\sigma_{xx} = -M_x y/I_b$, $\sigma_{yy} = \sigma_{xy} = 0$, where $I_b = \frac{1}{12}hb^3$.

For the y bending test, depicted in Figure 9, the beam cross section has height a and thickness h , and is subjected to end moments M_y . The exact solution is $M(y) = M_y$. The associated stress field is $\sigma_{yy} = M_y x/I_a$ and $\sigma_{xx} = \sigma_{xy} = 0$, where $I_a = \frac{1}{12}ha^3$. For comparing with the FEM discretizations below, the internal (complementary) energies taken up by beam segments of lengths a and b in the configurations of Figures 8 and 9, respectively, are

$$U_x^{\text{beam}} = \frac{6aC_{11}M_x^2}{b^3h}, \quad U_y^{\text{beam}} = \frac{6bC_{22}M_y^2}{a^3h} \quad (36)$$

For the 2D element tests, each beam is modeled with one layer of identical 4-node rectangular panels dimensioned $a \times b$ as shown in Figures 8 and 9. The aspect ratio b/a is denoted by γ . By analogy with the exact solution, all rectangles in the finite element model will undergo the same deformations and stresses. We can therefore consider a typical element. For x bending the exact stress distribution is represented by (7) on taking $\mu_4 = -M_x b/I_b = -12M_x/(b^2h)$ and $\mu_1 = \mu_2 = \mu_3 = \mu_5 = 0$. The rigid body mode amplitudes are chosen to be zero for convenience: $\mu_6 = \mu_7 = \mu_8 = 0$. Inserting these μ_i into (14) we get the node displacement vector

$$\mathbf{u}_{bx} = \frac{12M_x C_{11}a}{b^2h} [-1 \ 0 \ 1 \ 0 \ -1 \ 0 \ 1 \ 0]^T. \quad (37)$$

Likewise, for the y bending test the element stress field is obtained by taking $\mu_5 = M_y a/I_a = 12M_y/(a^2h)$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = \mu_7 = \mu_8 = 0$. The node displacement vector given by (14) is

$$\mathbf{u}_{by} = \frac{12M_y C_{22}b}{a^2h} [0 \ 1 \ 0 \ -1 \ 0 \ 1 \ 0 \ -1]^T. \quad (38)$$

The strain energies absorbed by the panel element under these applied node displacements are $U_x^{\text{panel}} = \frac{1}{2}\mathbf{u}_{bx}^T \mathbf{K} \mathbf{u}_{bx}$ and $U_y^{\text{panel}} = \frac{1}{2}\mathbf{u}_{by}^T \mathbf{K} \mathbf{u}_{by}$, respectively. Define the bending energy ratios as

$$r_x = \frac{U_x^{\text{panel}}}{U_x^{\text{beam}}}, \quad r_y = \frac{U_y^{\text{panel}}}{U_y^{\text{beam}}}. \quad (39)$$

These happen to be the ratios of the exact (beam) displacement solution to that of the rectangular panel solution. Hence $r_x = 1$ means that we get the exact answer under M_x , that is, the panel is x -bending exact. If $r_x > 1$ or $r_x < 1$ the panel is overstiff or overflexible in x bending, respectively. Likewise for y bending.

If $r_x = 1$ and $r_y = 1$ for any aspect ratio $\gamma = b/a$ and arbitrary material properties the element is called *bending optimal*. If $r_x \gg 1$ if $a \gg b$ and/or $r_y \gg 1$ if $a \ll b$ the element is said to experience *aspect ratio locking* along the x or y direction, respectively. This is called *shear locking* in the FEM literature because it is traceable to spurious shear energy, as shown in Section 8.4.

8.2 The Optimal Panel

Applying the tests to the template (31) yields

$$r_x = 3C_{11}R_{11}, \quad r_y = 3C_{22}R_{22}. \quad (40)$$

Clearly to get $r_x = r_y = 1$ for any aspect ratio we must take

$$R_{11} = \frac{1}{3}C_{11}^{-1}, \quad R_{22} = \frac{1}{3}C_{22}^{-1} \quad (41)$$

Since R_{12} is absent from (41) one can set $R_{12} = 0$ for convenience. Comparing to the $\mathbf{R}_\sigma$ of (33) shows that the 5-parameter stress model of TCMT and its clones is the *bending-optimal rectangular panel*. For isotropic material $R_{11} = R_{22} = \frac{1}{3}E$. The StressRP template instance will be henceforth also identified by the acronym BORP, for Bending Optimal Rectangular Panel.

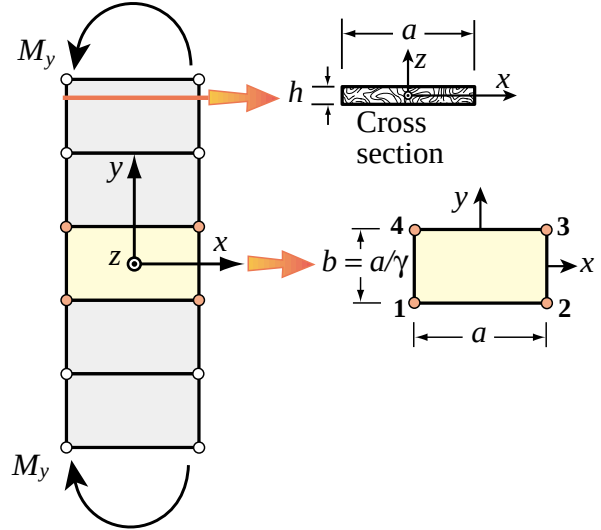


Figure 9. Constant-moment inplane-bending test along the y side dimension.

8.3 The Strain Element Does Not Lock

It is interesting to apply the result (40) to other template instances. The StrainRP element generated by the $\mathbf{R}_e$ of (33) gives

$$r_x = C_{11}E_{11}, \quad r_y = C_{22}E_{22}. \quad (42)$$

If the material is isotropic, $C_{11} = C_{22} = 1/E$ and $E_{11} = E_{22} = E/(1 - \nu^2)$. This yields $r_x = r_y = 1/(1 - \nu^2)$, which varies between 1 and 4/3. For an orthotropic body with principal material axes aligned with the rectangle sides, $E_{11} = E_1/(1 - \nu_{12}\nu_{21})$, $E_{22} = E_2/(1 - \nu_{12}\nu_{21})$, $C_{11} = 1/E_1$, $C_{22} = 1/E_2$, and $r_x = r_y = 1/(1 - \nu_{12}\nu_{21})$. These are independent of the aspect ratio γ . Consequently StrainRP and its clones *do not lock*, although the element is not generally optimal. Note that if $C_{11}E_{11}$ and/or $C_{22}E_{22}$ differ widely from 1, as may happen in highly anisotropic materials, the bending performance will be poor. The example of Section 12.2 displays this vividly.

8.4 But the Displacement Element Does

Instance DispRP is generated by the $\mathbf{R}_u$ of (33). Inserting its entries into (40) we get

$$\begin{aligned} r_x &= C_{11}(E_{11} + E_{33}\gamma^2) = \frac{(E_{22}E_{33} - E_{23}^2)(E_{11} + E_{33}\gamma^2)}{\det(\mathbf{E})}, \\ r_y &= C_{22}(E_{22} + E_{33}\gamma^{-2}) = \frac{(E_{11}E_{33} - E_{13}^2)(E_{22} + E_{33}\gamma^{-2})}{\det(\mathbf{E})}. \end{aligned} \quad (43)$$

in which $\det(\mathbf{E}) = E_{11}E_{22}E_{33} + 2E_{12}E_{13}E_{23} - E_{11}E_{23}^2 - E_{22}E_{13}^2 - E_{33}E_{12}^2$. For an isotropic material

$$r_x = \frac{2 + \gamma^2(1 - \nu)}{2(1 - \nu^2)}, \quad r_y = \frac{1 + 2\gamma^2 - \nu}{2\gamma^2(1 - \nu^2)}. \quad (44)$$

These relations clearly indicate aspect ratio locking for bending along the longest side dimension. For instance if $\nu = 0$ and $a = 10b$, whence $\gamma = a/b = 10$, then $r_x = 51$ and DispRP is over 50 times stiffer in x bending than the Bernoulli-Euler beam element. The expression (43) makes clear that *locking happens for any material law* as long as $E_{33} \neq 0$. Since this is the shear modulus, the name *shear locking* used in the FEM literature is justified.

8.5 Multiple Element Layers

Results of the energy bending test can be readily extended to predict the behavior of $2n$ ($n = 1, 2, \dots$) identical layers of elements symmetrically placed through the beam height. If $2n$ layers are placed along the y direction in the configuration of Figure 8 and γ stays the same, the energy ratio becomes

$$r_x^{(2n)} = \frac{2^{2n} - 1 + r_x}{2^{2n}}, \quad (45)$$

where r_x is the ratio (40) for one layer. If $r_x \equiv 1$, $r_x^{(2n)} \equiv 1$ so bending exactness is maintained, as can be expected. For example, if $n = 1$ (two element layers), $r_x^{(2)} = (3 + r_x)/4$. The same result holds for r_y if $2n$ layers are placed along the x direction in the configuration of Figure 9.

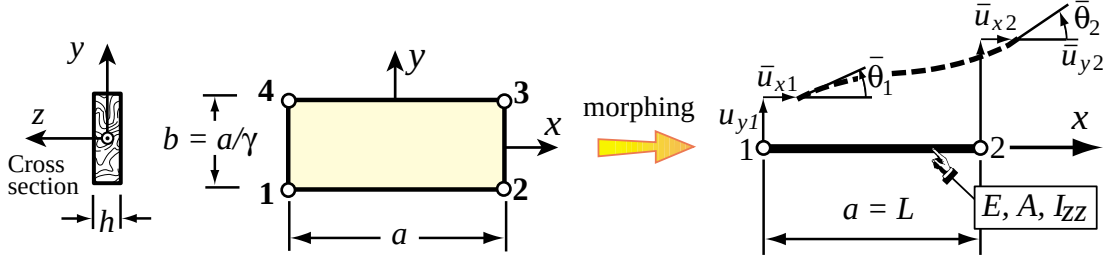


Figure 10. Morphing a 8-DOF rectangular panel unit to a 6-DOF beam-column element in the x direction.

9 MORPHING TO BEAM-COLUMN

Morphing means transforming an individual element or macroelement into a simpler model using kinematic constraints. Often the simpler element has lower dimensionality. For example a plate bending macroelement may be morphed to a Bernoulli-Euler beam or to a torqued shaft.⁷⁹ To illustrate the idea consider morphing the rectangular panel of Figure 10 into the two-node beam-column element shown on the right of that Figure. The length, cross sectional area and moment of inertia of the beam-column element, respectively, are denoted by $L = a$, $A = bh$ and $I_{zz} = b^3h/12 = a^3h/(12\gamma^3)$, respectively.

The transformation between the freedoms of the panel and those of the beam-column is

$$\mathbf{u}_R = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2}b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2}b \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2}b \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{2}b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{\theta}_1 \\ \bar{u}_{x2} \\ \bar{u}_{y2} \\ \bar{\theta}_2 \end{bmatrix} = \mathbf{T}_m \bar{\mathbf{u}}_m. \quad (46)$$

where a superposed bar distinguishes the beam-column freedoms grouped in array $\bar{\mathbf{u}}_m$. As source select StressRP fabricated of isotropic material. The morphed beam-column element stiffness is

$$\mathbf{K}_m = \mathbf{T}_m^T \mathbf{K}_\sigma \mathbf{T}_m = \frac{E}{L} \begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & 12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & -12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L \\ 0 & 6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & -6c_{23}I_{zz}/L & 4c_{33}I_{zz} \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & 12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & -12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L \\ 0 & 6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & -6c_{23}I_{zz}/L & 4c_{33}I_{zz} \end{bmatrix} \quad (47)$$

in which $c_{22} = c_{23} = \frac{1}{2}\gamma^2/(1 + \nu)$, and $c_{33} = \frac{1}{4}(1 + 3c_{22})$. The entries in rows/columns 1 and 4 form the well known two-node bar stiffness. Those in rows and columns 2, 3, 5 and 6 are dimensionally homogeneous to those of a plane beam, and may be grouped into the following matrix configuration:

$$\mathbf{K}_m^{beam} = \frac{EI_{zz}}{L} \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \beta_m \begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 3 & -6/L & 3 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 3 & -6/L & 3 \end{bmatrix} \right) \quad (48)$$

where $\beta_m = c_{22} = c_{23} = \frac{1}{2}\gamma^2/(1+\nu)$. But (48), with β_m replaced by a free parameter β , happens to be the universal template of a prismatic plane beam.⁷³ Mass-stiffness template combinations have been studied for dynamics and vibration using Fourier methods.^{82,83}

The basic stiffness on the left characterizes the pure-bending symmetric response to a uniform moment, whereas the higher-order stiffness on the right characterizes the antisymmetric response to a linearly-varying, bending moment of zero mean. For the Bernoulli-Euler beam constructed with cubic shape functions, $\beta = 1$. For the Timoshenko beam, the exact equilibrium model developed in §5.6 of Przemieniecki⁷ is matched by $\beta = \beta_{C0} = 1/(1+\phi)$, $\phi = 12EI_z/(GA_sL^2)$, in which $A_s = 5bh/6$ is the shear area and $G = \frac{1}{2}E/(1+\nu)$ the shear modulus. The morphed β_m is always higher than β_{C0} for all $0 \leq \nu \leq \frac{1}{2}$ and aspect ratios $\gamma > 0$. This indicates that in beam-like problems involving transverse shear the rectangular panel will be stiffer than the exact C^0 beam model. For example if $\nu = 1/4$

$$\frac{\beta_{C0}}{\beta_m} = \frac{5}{2(3+\gamma^2)}. \quad (49)$$

This never exceeds 5/6 and goes to zero as $\gamma \rightarrow \infty$. This reflects the fact that a 4-node panel can only respond to such antisymmetric nodal motions by deforming in pure shear. The symmetric response, however, is exact for any aspect ratio γ , confirming the optimality of StressRP (= BORP). Observe also that what was a higher order patch test on the two-triangle mesh unit becomes a basic (constant-moment) patch test on the morphed element. This is typical of morphing transformations that reduce spatial dimensionality.

For nonoptimal elements, one finds that the basic stiffness of the morphed beam is wrong except under very special circumstances. For example isotropic StrainRP with zero ν , or one of the SRI elements studied next.

10 A G3 DEVICE: SELECTIVE REDUCED INTEGRATION

The three canonical models of Sections 4-6 were known by the end of Generation 2. Next a third generation tool will be studied in the context of templates. Full Reduced Integration (FRI) and Selective Reduced Integration (SRI) emerged during 1969–72^{84–87} as tools to “unlock” isoparametric displacement models. Initially labeled as “variational crimes” by Strang³³ they were eventually justified through lawful association with mixed variational methods.^{88–90} Both FRI and SRI turned out to be particularly useful for legacy and nonlinear codes because they allow shape function and numerical integration modules to be reused. For the 4-node panel only SRI is considered because FRI leads to rank deficiency: $R_{11} = R_{12} = R_{22} = 0$. Two questions can be posed:

- (i) Can the template (31)–(32) be reproduced for any material law by a SRI scheme?
- (ii) Can BORP be cloned for any material law by a SRI scheme that is independent of the aspect ratio?

As shown below, the answers are (i): yes if $R_{12} = 0$; (ii): yes.

10.1 Concept and Notation

In the FEM literature, SRI identifies a scheme for forming $\mathbf{K}$ as the sum of two or more matrices computed with different integration rules and different constitutive properties, within the framework of the isoparametric (iso-P) displacement model. We will focus here on a two-way constitutive decomposition. Split the plane stress constitutive matrix $\mathbf{E}$ into

$$\mathbf{E} = \mathbf{E}_I + \mathbf{E}_{II}. \quad (50)$$

The iso-P displacement formulation leads to the expression $\mathbf{K} = \int_{A^e} h \mathbf{B}_u^T \mathbf{E} \mathbf{B}_u d\Omega$ where A^e is the element area and $\mathbf{B}_u$ the iso-P strain-displacement matrix. To apply SRI

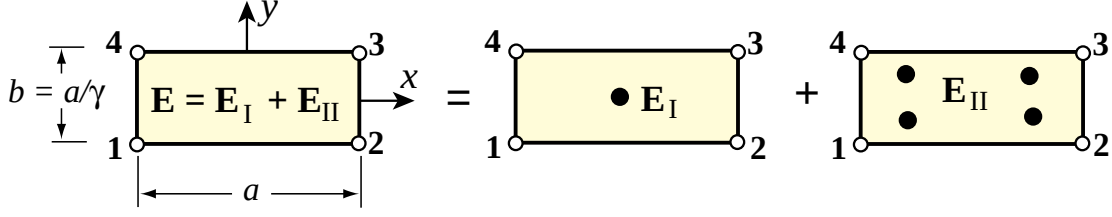


Figure 11. Matrix split for two-way SRI.

insert the splitting (50) into $\mathbf{E}$ to get two integrals:

$$\mathbf{K} = \int_{A^e} h \mathbf{B}_u^T \mathbf{E}_I \mathbf{B}_u d\Omega + \int_{A^e} h \mathbf{B}_u^T \mathbf{E}_{II} \mathbf{B}_u d\Omega = \mathbf{K}_I + \mathbf{K}_{II}. \quad (51)$$

The two matrices in (51) are done through different numerical quadrature schemes: rule (I) for the first integral and rule (II) for the second. For the rectangular panel the isoparametric model is the 4-node bilinear element. Rules (I) and (II) will be the 1×1 (one point) and 2×2 (4-point) Gauss product rules, respectively. A general split of the elasticity matrix is

$$\mathbf{E} = \mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} E_{11} \rho_1 & E_{12} \rho_3 & E_{13} \tau_2 \\ E_{12} \rho_3 & E_{22} \rho_2 & E_{23} \tau_2 \\ E_{13} \tau_2 & E_{23} \tau_2 & E_{33} \tau_1 \end{bmatrix} + \begin{bmatrix} E_{11}(1 - \rho_1) & E_{12}(1 - \rho_3) & E_{13}(1 - \tau_2) \\ E_{12}(1 - \rho_3) & E_{22}(1 - \rho_2) & E_{23}(1 - \tau_3) \\ E_{13}(1 - \tau_2) & E_{23}(1 - \tau_3) & E_{33}(1 - \tau_1) \end{bmatrix}, \quad (52)$$

in which $\rho_1, \rho_2, \rho_3, \tau_1, \tau_2$ and τ_3 are dimensionless coefficients to be chosen.

10.2 The Case $R_{12} = 0$

A template with $R_{12} = 0$ and arbitrary $\{R_{11}, R_{22}\}$ is matched by taking

$$\rho_1 = \frac{1 - 3R_{11}}{E_{11}}, \quad \rho_2 = \frac{1 - 3R_{22}}{E_{22}}, \quad \tau_1 = \tau_2 = \tau_3 = 1. \quad (53)$$

Since ρ_3 does not appear, it is convenient to set it to one to get a diagonal $\mathbf{E}_{II}$. The resulting split is

$$\mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} E_{11} - 3R_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} - 3R_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} 3R_{11} & 0 & 0 \\ 0 & 3R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (54)$$

To get the optimal element (BORP) set $R_{11} = \frac{1}{3}C_{11}^{-1}$ and $R_{22} = \frac{1}{3}C_{22}^{-1}$:

$$\mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} E_{11} - C_{11}^{-1} & E_{12} & E_{13} \\ E_{12} & E_{22} - C_{22}^{-1} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} C_{11}^{-1} & 0 & 0 \\ 0 & C_{22}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (55)$$

For isotropic material this becomes

$$\mathbf{E}_I + \mathbf{E}_{II} = \frac{E}{1 - \nu^2} \begin{bmatrix} \nu^2 & \nu & 0 \\ \nu & \nu^2 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{bmatrix} + E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (56)$$

To match the (suboptimal) StrainRP, in which $R_{11} = \frac{1}{3}E_{11}$ and $R_{22} = \frac{1}{3}E_{22}$ the appropriate split is

$$\mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} 0 & E_{12} & E_{13} \\ E_{12} & 0 & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (57)$$

For isotropic material this becomes

$$\mathbf{E}_I + \mathbf{E}_{II} = E \begin{bmatrix} 0 & \nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{bmatrix} + E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (58)$$

Some FEM books suggest using the dilatational (a.k.a. volumetric, bulk) elasticity law for $\mathbf{E}_I$. As can be seen, the recommendation is incorrect for this element.

10.3 The Case $R_{12} \neq 0$

The case $R_{12} \neq 0$ arises in anisotropic displacement models in which $E_{13} \neq 0$ and/or $E_{23} \neq 0$. Now τ_2 and τ_3 must verify $E_{13}\gamma^{-1}\tau_2 + E_{23}\gamma\tau_3 = E_{13}\gamma^{-1} + E_{23}\gamma - 3R_{12}$. Solve for that τ_i ($i = 2, 3$) that has an associated nonzero modulus. Note that the aspect ratio γ will generally appear in the SRI rule.

This case lacks practical interest because optimality can be achieved with $R_{12} = 0$. But for DispRP an obvious solution that eliminates all aspect ratio dependent is $\rho_1 = \rho_2 = \rho_3 = \tau_1 = \tau_2 = \tau_3 = 0$, whence $\mathbf{E}_I = \mathbf{0}$, $\mathbf{E}_{II} = \mathbf{E}$ and the fully integrated isoP element, which locks, is recovered.

10.4 Selective Directional Integration

The template can also be generated by non-Gaussian rules. For example, the following three-way directional split

$$\mathbf{E}_I + \mathbf{E}_{II} + \mathbf{E}_{III} = \begin{bmatrix} E_{11} - C_{11}^{-1} & E_{12} & E_{13} \\ E_{12} & E_{22} - C_{22}^{-1} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} C_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{22}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (59)$$

generates the optimal panel in conjunction with three rules. Rule (I) is one-point Gauss with $\{\xi, \eta\} = \{0, 0\}$ and weight 4; Rule (II) has two points on the $y = 0$ median: $\{\xi, \eta\} = \{0, \pm 1/\sqrt{3}\}$ with weight 2; rule (III) has two points on the $x = 0$ median: $\{\xi, \eta\} = \{\pm 1/\sqrt{3}, 0\}$ with weight 2. This selective directional integration is difficult to extend to arbitrary quadrilaterals while preserving observer invariance.

11 FUTILE FAMILIES

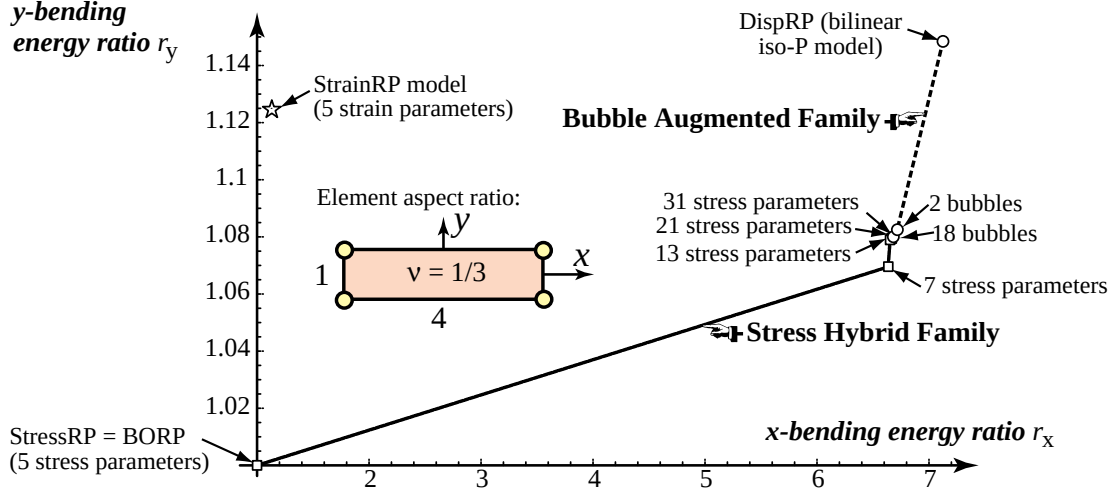
Families are template subsets that arise naturally from specific methods as function of discrete or continuous decision parameters. To render the concept more concrete two historically important, albeit practically useless, families for the rectangular panel are considered next.

11.1 Equilibrium Stress Hybrids

This family was studied in the late 1960s by hapless authors with the not unreasonable belief that “more is better.” It is obtained by generalizing the 5-parameter stress form of Section 4 with a polynomial series in $\{x, y\}$. An obvious choice is to make σ_{xx} , σ_{yy} and σ_{xy} complete polynomials in $\{x, y\}$:

$$\sigma_{xx} = \sum_{i,j} a_{ij} x^i y^j, \quad \sigma_{yy} = \sum_{i,j} b_{ij} x^i y^j, \quad \sigma_{xy} = \sum_{i,j} c_{ij} x^i y^j, \quad i \geq 0, \quad j \geq 0, \quad i + j \leq n. \quad (60)$$

For a complete expansion of order $n \geq 0$ one gets $3(n+1)(n+2)/2$ coefficients. Imposing strongly the two internal equilibrium equations $(1)_3$ for zero body forces reduces the set to $n_\sigma = 3 + 3n + n^2$ independent coefficients. For $n = 0, 1, 3, 5$ and 7 this gives $n_\sigma = 3, 7, 13, 21$ and 31 coefficients, respectively. (Only odd n is of interest beyond


 Figure 12. Representation of template families on the $\{r_x, r_y\}$ plane.

$n = 0$, since terms with $i + j = 2, 4, \dots$ etc., cancel out on integrating strains over the rectangle and have no effect on the element stiffness.)

The stiffness equations of this family can be obtained by the hybrid stress method of Pian and Tong.^{28,49} To display the effect of n_σ , the signature of the template (31)–(32) and the associated bending energy ratios were calculated for aspect ratio $\gamma = a/b = 4$, isotropic material with modulus E and Poisson's ratio $\nu = 1/3$.

Table 2. Signatures and Bending Ratios for Stress Hybrid Family

n_σ	5	7	13	21	31
R_{11}/E	0.33333	2.21173	2.21762	2.22125	2.22235
R_{22}/E	0.33333	0.35650	0.35967	0.35979	0.35981
R_{12}	0	0	0	0	0
r_x	1.00000	6.63518	6.65386	6.66375	6.66705
r_y	1.00000	1.06949	1.07900	1.07938	1.07944

The results are collected in Table 2. The bending energy ratios are displayed in Figure 12. Increasing the number of stress terms rapidly stiffens the element in x -bending. This is an instance of what may be called *equilibrium stress futility*: adding more stress terms makes things worse. (The phenomenon is well known but a representation such as that in Figure 12 is new.) As $n_\sigma \rightarrow \infty$ the template signature approaches the limit $R_{11}/E \approx 0.2224$ and $R_{22}/E \approx 0.3599$ to 4 places.

11.2 Bubble-Augmented Isoparametrics

A second family can be generated by starting from the conforming iso-P element DispRP of Section 6, and injecting n_b displacement bubble functions. [Bubble are shape functions that vanish over the element boundaries.] The idea is also a late-G2 curiosity but has resurfaced recently. Results for 2 and 18 bubbles (associated with 1 and 9 internal nodes, respectively) are collected in Table 3 and displayed also in Figure 12.

As can be expected injecting bubbles makes the element more flexible but the improvement is marginal. If $n_b \rightarrow \infty$ the signature approaches that of the $n_\sigma \rightarrow \infty$ hybrid-stress model of the previous subsection. For all this extra work (these models rapidly become expensive on account of high order Gauss integration rules and DOF condensation), r_x decreases from 7.12 to 6.67. This is a convincing illustration of *bubble futility*.

Table 3. Signatures and Bending Ratios for Bubble-Augmented Family

n_b	0	2	18
R_{11}/E	2.37501	2.23894	2.22546
R_{22}/E	0.38281	0.36088	0.35998
R_{12}	0.	0.	0
r_x	7.12505	6.71683	6.67637
r_y	1.14844	1.08265	1.07994

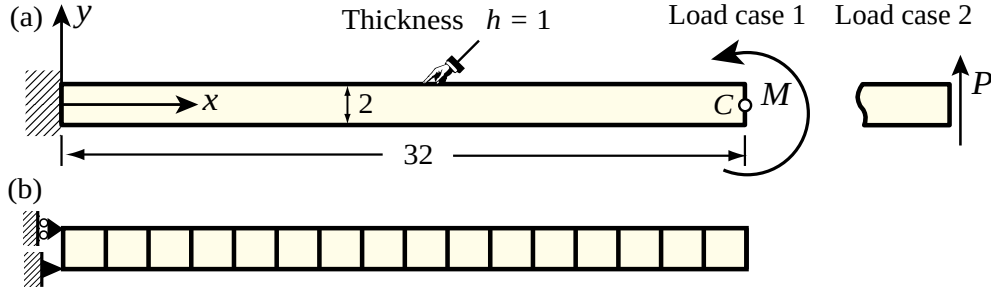


Figure 13. Slender cantilever beam for Examples 1 and 2.
A 16×1 FEM mesh with $\gamma = 1$ is shown in (b).

Figure 12 also marks the energy ratios of the StrainRP element. For this instance $R_{11}/E = R_{22}/E = 3/8 = 0.375$ and $r_x = r_y = 1.125$. Consequently the element is only slightly overstiff. Increasing the number of strain terms, however, would lead to another “futile family.”

12 NUMERICAL EXAMPLES

Three benchmark examples involving cantilever beams are studied below. The sequel³ presents benchmarks involving general quadrilateral shapes and thin-wall shell structures.

12.1 Example 1: Slender Isotropic Cantilever

The slender 16:1 cantilever beam of Figure 13(a) is fabricated of isotropic material, with $E = 7680$, $\nu = 1/4$ and $G = (2/5)E = 3072$. The dimensions are shown in the Figure. Two end load cases are considered: an end moment $M = 1000$ and a transverse end shear $P = 48000/1027 = 46.7381$. Both tip deflections $\delta_C = u_{yC}$ from beam theory: $ML^2/(2EI_z)$ and $PL^3/(3EI_z) + PL/(GA_s)$, in which $I_z = b^3h/12$ and $A_s = 5A/6 = 5bh/6$, are exactly 100. For the second load case the shear deflection is only 0.293% of u_{yC} ; thus the particular expression used for A_s is not very important.

Regular meshes with only one element ($N_y = 1$) through the beam height are considered. The number N_x of elements along the span is varied from 1 to 64, giving elements with aspect ratios from $\gamma = 16$ through $\gamma = \frac{1}{4}$. The root clamping condition is imposed by setting u_x to zero at both root nodes, but u_y is only fixed at the lower one thus allowing for Poisson’s contraction at the root.

Tables 4 and 5 report computed tip deflections u_{yC} for several element types. The first three rows list results for the 3 rectangular panel models of Sections 4–6. The last three rows give results for selected triangular elements. BODT is the Bending Optimal Drilling Triangle: a 3-node membrane element with drilling freedoms studied in previous papers.^{52,81,91,92} ALL-EX is the exactly integrated 1988 Allman triangle with drilling freedoms.⁹³ CST is the Constant Strain Triangle, also called linear triangle and Turner triangle.¹ Both ALL-EX and BODT have three freedoms per node whereas all others

have two. To get exactly 100.00% from BODT under an end-moment requires particular attention to the end load consistent lumping.⁹²

BORP is exact for all γ under end-moment and converges rapidly under end-shear. The performance of BODT is similar, inasmuch as this triangle is constructed to be bending exact in rectangular-mesh units. (In the end-shear load case BORP and BODT, which morph to different beam templates, converge to slightly different limits as $\gamma \rightarrow 0$.) StrainRP is about 6% stiffer than BORP, which can be expected since $1/(1 - \nu^2) = 16/15$. DispRP, as well as the triangles ALL-EX and CST, lock as γ increases.

The response for more element layers through the height can be readily estimated from equation (45). Consequently those results are omitted to save space. For example, to predict the DispRP answer on a 8×4 mesh under end-moment, proceed as follows. The aspect ratio is $\gamma = 8$. From the $\gamma = 8$ column of Table 4 read off $r_x = 100/3.75 = 26.667$. Set $n = 2$ in (45) to get $r_x^{(4)} = (15 + r_x)/16 = 2.60417$. The estimated tip deflection is $100/2.60417 = 38.40$. Running the program gives $\delta_C = 38.3913$ as average of the y displacement of the two end nodes. Predictions for the end-shear-load case will be less accurate but sufficient for quick estimation.

12.2 Example 2: Slender Anisotropic Cantilever

Next assume that the beam of Figure 13(a) is fabricated of anisotropic material with the elasticity properties

$$\mathbf{E} = \begin{bmatrix} 880 & 600 & 250 \\ 600 & 420 & 150 \\ 250 & 150 & 480 \end{bmatrix}, \quad \mathbf{C} = \mathbf{E}^{-1} = \frac{1}{35580} \begin{bmatrix} 1791 & -2505 & -150 \\ -2505 & 3599 & 180 \\ -150 & 180 & 96 \end{bmatrix}. \quad (61)$$

That these are physically realizable can be checked by getting the eigenvalues of $\mathbf{E}$: $\{1386.1, 387.3, 6.63\}$, whence both $\mathbf{E}$ and $\mathbf{C}$ are positive definite. The load magnitudes are adjusted to get beam-theory tip deflections of 100: $M = 2.58672$ and $P = 0.121153$. Since

$$E_{11}C_{11} = 44.297 \quad (62)$$

the energy ratio analysis of Sections 8.3–8.4, through equations (42) and (43), predicts that the strain and displacement models will be big losers, because $r_x \geq 44.297$. This is verified in Tables 6 and 7, which report computed tip deflections u_{yC} for the three rectangular panel models. While BORP shines, the strain and displacement models are way off, regardless of how many elements one puts along x .

Putting more element layers through the height will help StrainRP and DispRP but too slowly to be practical. To give an example, a 128×8 mesh of StrainRP (or clones) under end moment will have $r_x^{(8)} = (63 + 44.297)/64 = 1.676$ and estimated deflection of $100/1.676 = 59.647$. Running that mesh gives $u_{yC} = 59.65$. So using over 2000 freedoms in this fairly trivial problem the results are still off by about 40%.

12.3 Example 3: Short Cantilever Under End Shear

The shear-loaded cantilever beam defined in Figure 14 has been selected as a test problem for plane stress elements by many investigators since proposed in the writer's thesis.⁹⁴ A full root-clamping condition is implemented by constraining both displacement components to zero at nodes located on at the root section $x = 0$. The applied shear load varies parabolically over the end section and is consistently lumped at the nodes. The main comparison value is the tip deflection $\delta_C = u_{yC}$ at the center of the end cross section. Reference⁸¹ recommends $\delta_C = 0.35601$, which is also adopted here. The converged value of digits 4-5 is clouded by the mild singularity developing at the root section. This singularity is displayed for σ_{xy} in the form of an intensity contour plot in Figure 15.

Table 4 Tip Deflections (exact=100) for Slender Isotropic Cantilever under End Moment

Element	Mesh: x -subdivisions $\times$ y -subdivisions ($N_x \times N_y$)						
	1×1 ($\gamma = 16$)	2×1 ($\gamma = 8$)	4×1 ($\gamma = 4$)	8×1 ($\gamma = 2$)	16×1 ($\gamma = 1$)	32×1 ($\gamma = \frac{1}{2}$)	64×1 ($\gamma = \frac{1}{4}$)
StressRP (BORP)	100.00	100.00	100.00	100.00	100.00	100.00	100.00
StrainRP	93.75	93.75	93.75	93.75	93.75	93.75	93.75
DispRP	0.97	3.75	13.39	37.49	68.18	85.71	91.60
ALL-EX	0.04	0.63	7.40	35.83	58.44	64.89	66.45
CST	0.32	1.25	4.46	12.50	22.73	28.57	30.53
BODT	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 5 Tip Deflections (exact=100) for Slender Isotropic Cantilever under End Shear

Element	Mesh: x -subdivisions $\times$ y -subdivisions ($N_x \times N_y$)						
	1×1 ($\gamma = 16$)	2×1 ($\gamma = 8$)	4×1 ($\gamma = 4$)	8×1 ($\gamma = 2$)	16×1 ($\gamma = 1$)	32×1 ($\gamma = \frac{1}{2}$)	64×1 ($\gamma = \frac{1}{4}$)
StressRP (BORP)	75.02	93.72	98.39	99.56	99.86	99.94	99.97
StrainRP	70.35	87.88	92.26	93.35	93.63	93.71	93.73
DispRP	0.97	3.75	13.39	37.49	68.16	85.69	91.58
ALL-EX	0.24	0.69	6.36	35.18	59.59	65.70	67.03
CST	0.48	1.41	4.62	12.66	22.88	28.73	30.69
BODT	75.20	93.37	98.20	99.55	99.93	100.12	100.15

Table 6 Tip Deflections (exact=100) for Slender Anisotropic Cantilever under End Moment

Element	Mesh: x -subdivisions $\times$ y -subdivisions ($N_x \times N_y$)						
	1×1 ($\gamma = 16$)	2×1 ($\gamma = 8$)	4×1 ($\gamma = 4$)	8×1 ($\gamma = 2$)	16×1 ($\gamma = 1$)	32×1 ($\gamma = \frac{1}{2}$)	64×1 ($\gamma = \frac{1}{4}$)
StressRP (BORP)	100.00	100.00	100.00	100.00	100.00	100.00	100.00
StrainRP	2.26	2.26	2.26	2.26	2.26	2.26	2.26
DispRP	0.02	0.07	0.25	0.76	1.53	2.08	2.25

Table 7 Tip Deflections (exact=100) for Slender Anisotropic Cantilever under End Shear

Element	Mesh: x -subdivisions $\times$ y -subdivisions ($N_x \times N_y$)						
	1×1 ($\gamma = 16$)	2×1 ($\gamma = 8$)	4×1 ($\gamma = 4$)	8×1 ($\gamma = 2$)	16×1 ($\gamma = 1$)	32×1 ($\gamma = \frac{1}{2}$)	64×1 ($\gamma = \frac{1}{4}$)
StressRP (BORP)	74.95	93.68	98.37	99.54	99.84	99.92	99.96
StrainRP	1.70	2.12	2.22	2.26	2.26	2.26	2.26
DispRP	0.02	0.07	0.25	0.75	1.52	2.06	2.23

Table 8 gives computed deflections for rectangular mesh units with aspect ratios of 1, 2 and 4, using the three canonical rectangular panel models and the three triangles identified in Example 1. For end deflection reporting the load was scaled by $(100/0.35601)$ so that the “theoretical solution” becomes 100.00. (In comparing stress values the unscaled load of $P = 40$ was used.) There are no drastically small deflections because element aspect

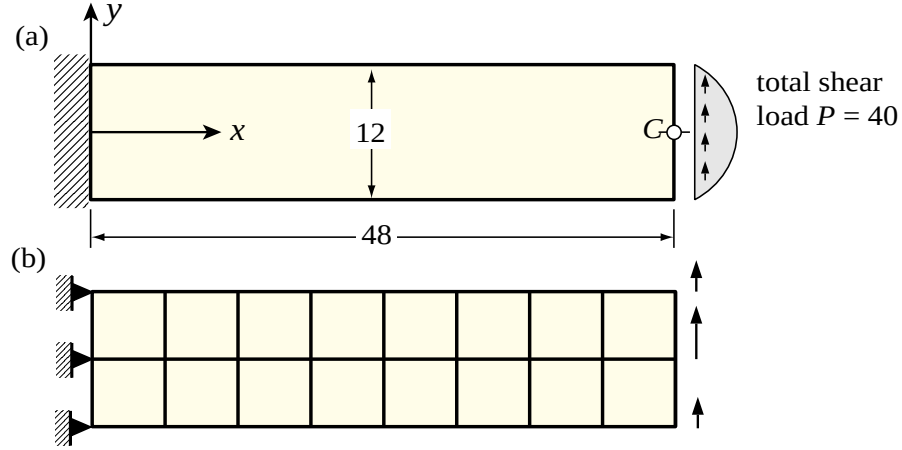


Figure 14. Short cantilever under end-shear benchmark: $E = 30000$, $\nu = 1/4$, $h = 1$; root contraction not allowed, a 8×2 mesh is shown in (b).



Figure 15. Intensity contour plot of σ_{xy} given by the 64×16 BORP mesh. Produced by *Mathematica* and Gaussian filtered by Adobe Photoshop. Stress node values averaged between adjacent elements. The root singularity pattern is clearly visible.

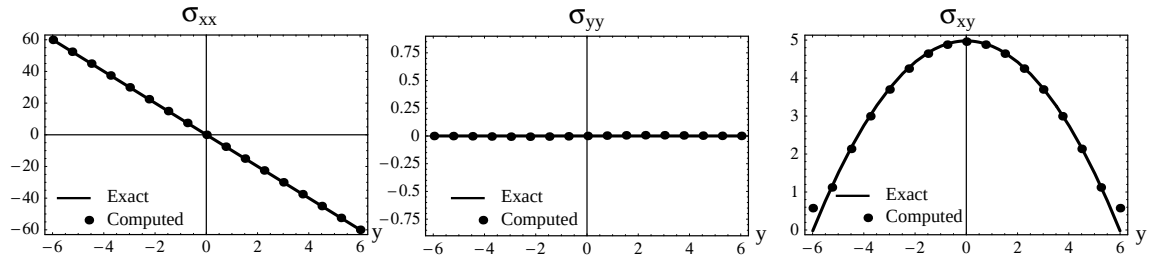


Figure 16. Distributions of σ_{xx} , σ_{yy} and σ_{xy} at $x = 12$ given by the 64×16 BORP mesh. Stress node values averaged between adjacent elements. Note different stress scales. Deviations at $y = \pm 6$ (free edges) due to “upwinded” y averaging.

ratios only go up to 4:1. Elements StressRP (BORP), StrainRP and BODT outperformed the others. There is little to choose between these 3 models, which is typical of isotropic materials. The BODT triangle is more versatile but carries one more freedom per node.

Figure 16 plots averaged node stress values at section $x = 12$ computed from the 64×16 BORP mesh. The agreement with the standard beam stress distribution (that section being sufficiently away from the root) is very good except for σ_{xy} near the free edges $y = \pm 6$, at which the interelement averaging process becomes biased.

13 DISCUSSION AND CONCLUSIONS

What can templates contribute to FEM technology? Advantages in two areas are clear: *Synthesis*. Only one procedure (module, function, subroutine) is written to do many elements. This simplifies comparison and verification benchmarking, as well as streamlining

Table 8 Tip Deflections (exact = 100) for Short Cantilever under End Shear

Element	Mesh: x -subdivisions $\times$ y -subdivisions ($N_x \times N_y$)				
	8×2	16×4	32×8	64×16	128×32
StressRP (BORP)	98.80	99.59	99.88	99.97	100.00
StrainRP	97.24	99.19	99.77	99.94	99.99
DispRP	88.83	96.83	99.16	99.78	99.95
ALL-EX	89.43	96.88	99.16	99.79	99.96
CST	55.09	82.59	94.90	98.65	99.66
BODT	101.68	100.30	100.03	100.00	100.00
Element	Mesh: x -subdivisions $\times$ y -subdivisions ($N_x \times N_y$)				
	4×2	8×4	16×8	32×16	64×32
StressRP (BORP)	97.22	99.08	99.71	99.92	99.99
StrainRP	95.67	98.67	99.61	99.89	99.98
DispRP	69.88	90.05	97.24	99.28	99.82
ALL-EX	70.71	89.63	96.93	99.15	99.77
CST	37.85	69.86	90.04	97.25	99.28
BODT	96.68	98.44	99.37	99.78	99.93
Element	Mesh: x -subdivisions $\times$ y -subdivisions ($N_x \times N_y$)				
	2×2	4×4	8×8	16×16	32×32
StressRP (BORP)	91.94	97.41	99.19	99.75	99.93
StrainRP	90.47	97.03	99.07	99.72	99.92
DispRP	37.84	70.57	90.39	97.35	99.31
ALL-EX	26.16	56.93	83.54	95.14	98.69
CST	17.83	43.84	75.01	92.13	97.86
BODT	92.24	96.99	98.70	99.48	99.81

maintenance. A unified implementation automatically weeds out clones.

Customability. Templates can produce optimal and custom elements not obtainable (or hard to obtain) through conventional methods.

A striking example of the latter is the UBOTP macroelement presented in Section A.3 of the Appendix. This concludes a three decade search for a four noded trapezoid which is insensitive to distortion, passes the patch test and retains rank sufficiency. To the writer's knowledge, this model, as well as its generalization to an arbitrary quadrilateral presented in the sequel³ cannot be obtained with conventional formulations.

Will the synthesis power translate into teaching changes in finite element courses? This is not presently likely. Two reasons can be cited.

First, advantages may show up only in advanced or seminar-level courses. Beginning calculus students are not taught Lebesgue integration and distribution theory despite their wider scope. Likewise, introductory FEM courses are best organized around a few specific methods. Students must be exposed to a range of formulations and hands-on work before they can appreciate the advantages of unified implementation.

Second, the theory has not progressed to the point where the configuration of a template can be written down from first principles in front of an audience. Only two general rules are presently known: the fundamental decomposition into basic and higher order components, and the procedure to get the matrix structure of the basic component. No general rules to construct the higher order component can be stated aside from orthogonality and definiteness constraints.

How far can templates go? As of this writing templates are only known for a few elements in one and two dimensions, such as beams and flat plates of simple geometry. What is the major technical obstacle to go beyond those? Symbolic power. One must rely on computer-aided symbolic manipulation because geometric, constitutive and fabrication

properties must be carried along as variables. This can lead, and does, to a combinatorial tarpit as elements become more complicated.

The good news is that computer algebra programs are gradually becoming more powerful, and are now routinely available on laptops and personal computers. Over the next 5 years PCs are expected to migrate to 64-bit multiple-CPU's capable of addressing hundreds of GBs of memory at over 10GHz cycle speeds. As that happens the development of templates for 3D solid and shell elements in reasonable time will become possible.

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Appendix A. MORE GENERAL PANEL GEOMETRIES

The template framework of four noded membrane elements can be extended to more general geometries, at the cost of increased complexity in symbolic computations. As an aperitif for the sequel³ this appendix present templates for parallelogram and trapezoidal geometries.

The first-generation (pre-1962) direct elasticity methods recalled in Sections 4–5 do not work properly beyond the parallelogram. The resulting “node collocation” elements fail the patch test and thus cannot fit in the template framework.

Variational methods are required to get stress-assumed and strain-assumed elements that work. For stress elements the Hellinger-Reissner (HR) principle is used. For strain elements, a strain-fit method⁹⁵ in conjunction with de Veubeke’s strain-displacement mixed functional is used.

A.1 Parallelogram (Swept) Panel

The geometry of the parallelogram panel shown in Figure 17 is defined by the dimensions a , b and the skewangle ω , positive counterclockwise. The template again has the configuration (31) displayed in Figure 6.

With $s = \tan \omega$ the matrices to be adjusted are

$$\mathbf{W} = \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{s}{b} & \frac{1}{b} \end{bmatrix},$$

$$\mathbf{H}_c = \frac{1}{2ab} \begin{bmatrix} -b & 0 & b & 0 & b & 0 & -b & 0 \\ 0 & -a-bs & 0 & -a+bs & 0 & a+bs & 0 & a-bs \\ -a-bs & -b & -a+bs & b & a+bs & b & a-bs & -b \end{bmatrix}, \quad (63)$$

The higher order projector $\mathbf{H}_h$ is exactly as in (32), whereas $\mathbf{R}$ depends on the formulation, as explained below. For future use the compliance and elasticity along the median direction y_ω (see Figure 17) are denoted by

$$C_{22}^\omega = C_{22} \cos^4 \omega - 2C_{23} \cos^3 \omega \sin \omega + (2C_{12} + C_{33}) \cos^2 \omega \sin^2 \omega - 2C_{13} \cos \omega \sin^3 \omega$$

$$+ C_{11} \sin^4 \omega = \frac{C_{22} - 2C_{23}s + (2C_{12} + C_{33})s^2 - 2C_{13}s^3 + C_{11}s^4}{(1 + s^2)^2},$$

$$E_{22}^\omega = E_{22} \cos^4 \omega - 4E_{23} \cos^3 \omega \sin \omega + (2E_{12} + 4E_{33}) \cos^2 \omega \sin^2 \omega - 4E_{13} \cos \omega \sin^3 \omega$$

$$+ E_{11} \sin^4 \omega = \frac{E_{22} - 4E_{23}s + (2E_{12} + 4E_{33})s^2 - 4E_{13}s^3 + E_{11}s^4}{(1 + s^2)^2}. \quad (64)$$

Stress element. A 5-parameter stress element StressPP can be constructed either directly, as done by Gallagher⁸ or by the HR principle, starting from the energy-orthogonal stress field

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & y/b & \sin^2 \omega x_\omega \\ 0 & 1 & 0 & 0 & \cos^2 \omega x_\omega \\ 0 & 0 & 1 & 0 & -\sin \omega \cos \omega x_\omega \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix}, \quad (65)$$

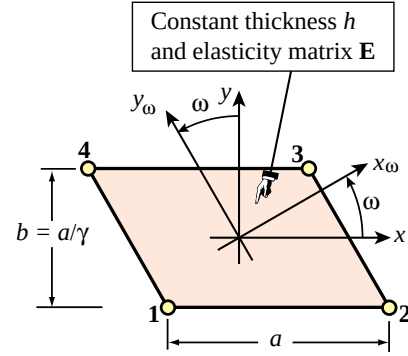


Figure 17. The 4-node swept panel.

in which $\sin \omega = s/\sqrt{1+s^2}$, $\cos \omega = 1/\sqrt{1+s^2}$, and $x_\omega = (x \cos \omega + y \sin \omega)/(a \cos \omega) = (x + ys)/a$. Both methods give the same stiffness. [Because (65) is an equilibrium field, an equilibrium stress hybrid formulation gives the same answer.] The stiffness is matched by the template with

$$R_{11} = \frac{1}{3C_{11}}, \quad R_{12} = 0, \quad R_{22} = \frac{1}{3C_{22}^\omega (1+s^2)^2}. \quad (66)$$

If the material is isotropic the diagonal entries are $R_{11} = \frac{1}{3}E$ and $R_{22} = \frac{1}{3}E/(1+s^2)^2$. The Q6 and QM6 elements continue to be clones of StressPP.

Strain element. A 5-parameter strain element StrainPP can be constructed by the direct elasticity method of Section 5, or by a variational strain-fitting method⁹⁵, starting from the companion of (65):

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & y/b & \sin^2 \omega x_\omega \\ 0 & 1 & 0 & 0 & \cos^2 \omega x_\omega \\ 0 & 0 & 1 & 0 & -2 \sin \omega \cos \omega x_\omega \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{bmatrix}. \quad (67)$$

Both methods give the same result. The stiffness is matched by setting

$$R_{11} = \frac{1}{3}E_{11}, \quad R_{12} = 0, \quad R_{22} = \frac{E_{22}^\omega}{3(1+s^2)^2}. \quad (68)$$

Displacement element. The conforming, exactly integrated isoparametric element DispPP is matched by setting

$$\begin{aligned} R_{11} &= \frac{1}{3} (E_{11} + 4E_{13}s + (2E_{12} + 4E_{33})s^2 + 4E_{23}s^3 + E_{22}s^4 + (E_{33} + 2E_{23}s + E_{22}s^2)\gamma^2), \\ R_{12} &= \frac{1}{3\gamma} (E_{13} + (E_{12} + 2E_{33})s + 3E_{23}s^2 + E_{22}s^3 + (E_{23} + E_{22}s)\gamma^2), \\ R_{22} &= \frac{1}{3\gamma^2} (E_{33} + 2E_{23}s + E_{22}(s^2 + \gamma^2)). \end{aligned} \quad (69)$$

The StressPP element (as well as its clones Q6 and QM6) is again bending optimal along both x and y_ω (median) directions. The symbolic verification is far more involved than for the rectangular element because it requires the use of free-free flexibility methods⁹⁶ and is omitted.

A.1 Trapezoidal Panel

The geometry of the trapezoidal panel shown in Figure 18 is defined by the dimensions $a, b = a/\gamma$ and the two angles ω_1 and ω_2 , both positive counterclockwise. Define

$$s_1 = \tan \omega_1, \quad s_2 = \tan \omega_2, \quad s = \frac{1}{2}(s_1 + s_2), \quad d = \frac{1}{2}(s_1 - s_2), \quad \phi = bd/a = d/\gamma. \quad (70)$$

The template is again given by the matrix form (31). Matrices $\mathbf{H}_c$ and $\mathbf{W}$ are as in (63), except that s has the new definition (70). The higher order projector matrix is

$$\mathbf{H}_h = \frac{1}{2} \begin{bmatrix} 1-\phi & 0 & -1+\phi & 0 & 1+\phi & 0 & -1-\phi & 0 \\ 0 & 1-\phi & 0 & -1+\phi & 0 & 1+\phi & 0 & -1-\phi \end{bmatrix}, \quad (71)$$

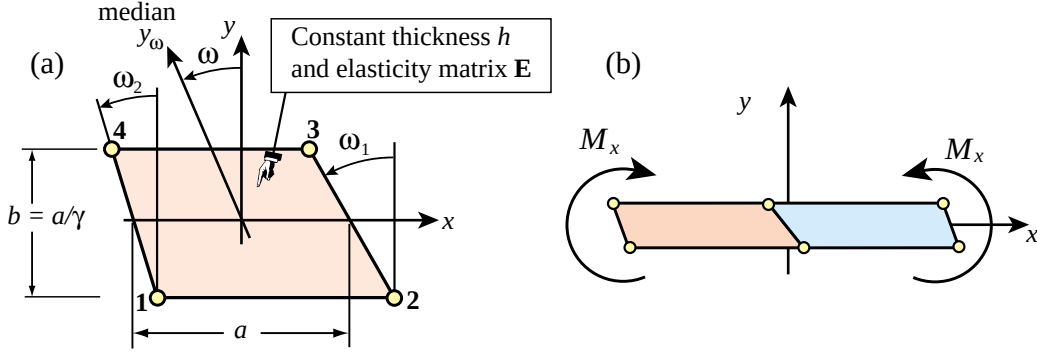


Figure 18. The four noded trapezoidal panel and a two-trapezoid repeatable macroelement.

whereas $\mathbf{R}$ depends on the formulation, as detailed next.

Stress element. Element StressTP is generated by the 5-parameter stress assumption (65), with one change: the (1,4) entry y/b is replaced by $(y - y_C)/b$. If $y_C = -b^2(s_2 - s_1)/(12a) = -\frac{1}{6}ad/\gamma^2$ the bending stresses are energy orthogonal to constant stress fields. The stiffness matrix derived with the HR principle is matched by

$$R_{11} = \frac{1}{C_{11}(3 - d^2/\gamma^2)}, \quad R_{12} = 0, \quad R_{22} = \frac{1}{3C_{22}^\omega(1 + d^2/\gamma^2)(1 + s^2)^2}, \quad (72)$$

in which C_{22}^ω is the compliance along the median y_ω (cf. Figure 18), given by (64).

QM6 element. The incompatible-mode element QM6⁶⁹ is no longer a clone of the stress element unless $d = 0$. Its stiffness is matched by

$$R_{11} = \frac{1}{C_{11}(3 - d^2/\gamma^2)}, \quad R_{12} = 0, \quad R_{22} = \frac{1}{C_{22}^\omega(3 - d^2/\gamma^2)(1 + s^2)^2}. \quad (73)$$

The only change is in R_{22} . The original incompatible-mode element Q6 of⁶⁸ fails the patch test if $d \neq 0$ and consequently cannot be matched by the template (31).

Strain element. Element StrainTP is generated by the 5-parameter strain assumption (67), with one change: the (1,4) entry y/b is replaced by $(y - y_C)/b$. Energy orthogonality is again obtained if $y_C = -b^2(s_2 - s_1)/(12a) = -\frac{1}{6}ad/\gamma^2$. A strain-fitting variational formulation⁹⁵ yields a stiffness matched by

$$R_{11} = \frac{E_{11}}{3 - d^2/\gamma^2}, \quad R_{12} = 0, \quad R_{22} = \frac{E_{22}^\omega}{3(1 + d^2/\gamma^2)(1 + s^2)^2}, \quad (74)$$

in which E_{22}^ω is the direct elasticity along the median y_ω direction, as given by (64).

Displacement element. The conforming isoparametric displacement element DispTP with 2×2 Gauss integration is matched by

$$\begin{aligned} R_{11} &= \frac{E_{11} + 4E_{13}s + s^2(2E_{12} + 4E_{33} + 4E_{23}s + E_{22}s^2) + (E_{33} + 2E_{23}s + E_{22}s^2)\gamma^2}{3 - d^2/\gamma^2}, \\ R_{12} &= \frac{E_{13} + s(E_{12} + 2E_{33} + 3E_{23}s + E_{22}s^2) + (E_{23} + E_{22}s)\gamma^2}{\gamma(3 - d^2/\gamma^2)}, \\ R_{22} &= \frac{E_{33} + 2E_{23}s + E_{22}(s^2 + \gamma^2)}{\gamma^2(3 - d^2/\gamma^2)} \end{aligned} \quad (75)$$

If $a \gg b$ and $s_1 \neq s_2$ the StressTP-fabricated macroelement rapidly becomes overstiff and overflexible in x - and y -bending, respectively. For example if $a/b = \gamma = 8$, $s_1 = 0$, $s_2 = 1/2$ and isotropic material with $\nu = 1/4$ the bending ratios are $r_x = 11.97$ and $r_y = 0.1414$. For the anisotropic elasticity matrix (61), $r_x = 6.93$ and $r_y = 0.0792$. If an elongated macroelement is supposed to model unidirectional x -bending correctly, the overstiffness caused by $s_1 \neq s_2$ is called *distortion locking*. This phenomenon has been widely studied since the MacNeal-Harder test suite gained popularity.⁹⁷

$$\mathbf{W} = \begin{bmatrix} 1/a & 0 \\ -\frac{(C_{11}(3\gamma^2 - ds) + C_{13})(s - d) - C_{12}d}{C_{11}(3\gamma^2 - d^2)b} & 1/b \end{bmatrix} \quad (76)$$

Figure 1 consists of a schematic diagram and a graph. The schematic shows a beam of length $2a = 10$ and height $b = 2$, fixed at the left end and subjected to a bending moment M_x/b at the right end. A rectangular element of width e is highlighted. The material properties are $E = 1500$, $\nu = 1/4$, and $h = 1$. The graph plots the 'Percentage of exact answer' (0 to 100) against the 'Distortion parameter $2e/b$ ' (0 to 5). Five curves are shown: UBOTP (dashed purple), Q6 (dashed red), StressTP (solid red), QM6 (solid blue), and DispTP (solid light blue). The UBOTP curve is the highest, followed by Q6, StressTP, QM6, and DispTP. The StressTP and QM6 curves are very close to each other.

Figure 19 presents results for a widely used mesh distortion test, which involves one macroelement of the type discussed. Results for six element types: UBOTP, StressTP, StrainTP, DispTP, Q6 and QM6 are shown. The percentage of the correct answer is of course $100/r_x$.

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cantilever is changed to, say $2a/b = 10$, the differences between elements become more dramatic.

For distortion performance results on other elements such as Pian-Sumihara and Enhanced Assumed Strain, see Lautersztajn and Samuelsson.⁷² A penalty-augmented modification of the Pian-Sumihara quadrilateral constructed by Wu and Cheung⁹⁸ that achieves distortion insensitivity at the cost of rank deficiency (and hence fails the inf-sup condition) is discussed in the sequel.³

At first sight the existence of UBOTP contradicts a theorem by MacNeal,² which says that four noded quadrilaterals cannot simultaneously pass the patch test and be insensitive to distortion. The escape hatch is that y -bending optimality (along the skew angular direction ω_1 of the macroelement) is not attempted. If one tries imposing $r_x = r_y = 1$, the solutions for $\{R_{11}, R_{12}, R_{22}\}$ become complex if $\gamma \gg 1$ as soon as d deviates slightly from zero.

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Shell Structures: Basic Concepts

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§29.1. GENERAL REMARKS

The word *shell* is an old one and is commonly used to describe the hard covering of eggs, crustacea, tortoises, etc. The dictionary says that the word shell is derived from the Latin *scalus*, as in fish scale. But to us now there is a clear difference between the tough but flexible scaly covering of a fish and the tough but rigid shell of, say, a turtle.

In this course we shall be concerned with man-made shell structures as used in various branches of engineering. There are many interesting aspects of the use of shells in engineering, but one alone stands out as being of paramount importance: the *structural* aspect.

The *theory of structures* tends to deal with a class of idealized mathematical models, stripped of many of the features that make them recognizable as useful object in engineering. Thus a *beam* is often idealized as a line endowed with certain mechanical properties, irrespective of whether it is a large bridge, an aircraft wing, or a flat spring inside a machine. In a similar way, the theory of shell structures deals, for example, with the “cylindrical shell” as an idealized entity: it is a cylindrical surface endowed with certain mechanical properties. The treatment is the same whether the actual structure under study is a gas-transmission pipeline, a grain storage silo, or a steam boiler.

Before entering this realm of geometry, in which shells are classified by their geometry (cylindrical, spherical, etc.) rather than their function, it is desirable to give a glimpse of the wide range of applications of shell structures in engineering practice. Indeed, a list of familiar examples will be useful in enabling us to pick up some structural features in a qualitative way to provide an introduction to the theory.

§29.2. SHELL STRUCTURES OVERVIEW

§29.2.1. Applications

It is instructive to assemble a list of applications from a historical point of view, and to take as a connecting theme the way in which the introduction of the *thin shell* as a structural form made an important contribution to the development of several branches of engineering. The following is a brief list, which is by no means complete.

Architecture and building. The development of masonry domes and vaults in the Middle Ages made possible the construction of more spacious buildings. In more recent times the availability of reinforced concrete has stimulated interest in the use of shells for roofing purposes.

Power and chemical engineering. The development of steam power during the Industrial Revolution depended to some extent on the construction of suitable boilers. These thin shells were constructed from plates suitable formed and joined by riveting. More recently the use of welding in pressure vessel construction has led to more efficient designs. Pressure vessels and associated pipework are key components in thermal and nuclear power plants, and in all branches of the chemical and petroleum industries.

Structural engineering. An important problem in the early development of steel for structural purposes was to design compression members against buckling. A striking advance was the use of tubular members in the construction of the Forth railway bridge in 1889: steel plates were riveted together to form reinforced tubes as large as 12 feet in diameter, and having a radius/thickness ratio of between 60 and 180.

Vehicle body structures. The construction of vehicle bodies in the early days of road transport involved a system of structural ribs and non-structural paneling or sheeting. The modern form of vehicle construction, in which the skin plays an important structural part, followed the introduction of sheet-metal

components, preformed into thin doubly curved shells by large power presses, and firmly connected to each other by welds along the boundaries. The use of the curved skin of vehicles as a load bearing member has similarly revolutionized the construction of railway carriages and aircraft. In the construction of all kind of spacecraft the idea of a thin but strong skin has been used from the beginning.

Composite construction. The introduction of fiberglass and similar lightweight composite materials has impacted the construction of vehicles ranging from boats, racing cars, fighter and stealth aircraft, and so on. The exterior skin can be used as a strong structural shell.

Miscellaneous Examples. Other examples of the impact of shell structures include water cooling towers for power stations, grain silos, armour, arch dams, tunnels, submarines, and so forth.

§29.2.2. Continuity and Curvature

The essential ingredients of a shell structure in all of the foregoing examples are *continuity* and *curvature*. Thus, a fiberglass hull of a boat is continuous in a way that the overlapping planks of clinker construction are not. A pressure vessel must be obviously constructed to hold a fluid at pressure, although the physical components may be joined to each other by riveting, bolting or welding. On the other hand, an ancient masonry dome or vault is not obviously continuous in the sense that it may be composed of of separate stone subunits or voussoirs not necessarily cemented to each other. But in general domes are in a state of compression throughout, and the subunits are thus held in compressive contact with each other. The important point here is that shells are structurally continuous in the sense that they can transmit forces in a number of different directions in the surface of the shell, as required. These structures have quite a different mode of action from *skeletal structures*, of which simple examples are trusses, frameworks, and trees. These structures are only capable of transmitting forces along their discrete structural members.

The fundamental effect of *curvature* and its effect on the strength and stiffness of a shell is discussed in §29.4.

§29.2.3. The Empirical Approach

Many of the structures listed in §29.2.1 were constructed long before there was anything like a textbook on the subject of shell structures. The early engineers had a strongly empirical outlook. They could see the advantages of shell construction from simple small-scale models, and clearly understood the practical benefits of doing “overload tests” on prototypes or scale models. Much the same brand of empiricism is practice successfully today in the design of motor vehicles, where the geometry of the structure is so complicated as to defy even simple description, let alone calculation. But in other areas of engineering, where precision is needed in the interest of economical design and where the geometry is more straightforward, the theory of shell structures is an important design tool.

§29.2.4. Closed and Open Shells

Before describing the main body of the theory it is useful to discuss quantitatively an important practical point.

Anyone who has built children’s toys from thick paper or thin cardboard will be familiar with the fact that a *closed* box is rigid, whereas an *open* box is easily deformable. Similarly, a chocolate box with the lid open can easily be twisted, yet it is effectively rigid when the lid is closed. The same sort of thing applies to an aluminum can, which may be squashed far more easily after an end has been removed. Again, it is noticeable that a boiled egg will not normally fit snugly into a rigid egg cup until the top of the shell has been removed: the closed egg is so rigid that small deviations from circularity are

noticeable; but it becomes flexible enough to adapt to the shape of the egg cup once an opening has been made.

There seems to be a principle here that *closed surfaces are rigid*. This is used in many areas of engineering construction. For example, the deck of a ship is not merely a horizontal surface to walk on: it also closes the hull, making a box-like structure. It is easy to think of many other examples of this form of construction, including aircraft wings, suspension bridge roadway girders, rocket skins and unibody cars.

Conversely, a boat without a deck, such as a small fishing or rowing boat, has little rigidity by virtue of form. It must rely on the provision of ribs and struts for what little rigidity it has.

In practice, of course, it is not usually possible to make completely closed structural boxes. In a ship, for example, there will be various cutouts in the deck for things such as hatches and stairways. It is sometimes possible to close such openings with doors and hatch covers that provide structural continuity. Submarines and aircraft are obvious examples. But this is often not possible and compromise solutions must be adopted. The usual plan is to reinforce the edge of the hole in such a way as to compensate, to a certain extent, for the presence of the hole. The amount of reinforcement that is required depends on the size of the hole, and to what extent the presence of the hole makes the structure an open one. Large openings are essential in some forms of construction, such as cooling towers. A more extreme example is provided by shell roofs in general. Here the shell is usually very open, being merely a “cap” of a shell, and the provision of adequate edge ribs, together with suitable supports, is of crucial importance. A main objective in the design of shell roofs is to eliminate those aspects of behavior that spring from the open nature of the shell.

From the foregoing discussion it is obvious that although the ideas of open and closed shells, respectively, are fairly clear, it is difficult to quantify intermediate cases. The majority of actual shell structures fall into such a grey area. While the effect of a small cutout on the overall rigidity of a shell structure may be trivial, the effect of a large cutout can be serious. The crux of this problem is to quantify the ideas of “small” and “large” in this context. Unfortunately there is no simple way to do this, because the problem involves the interaction between “global” and “local” effects. It is largely for this reason that the subject of shell structures generally is a difficult one.

§29.2.5. A Simple Geometric Approach

The notion that a closed surface is rigid is well known in the field of pure Euclidean geometry. There is a theorem of Cauchy which states that a convex polyhedron is rigid. The concept of rigidity is, of course, hedged around with suitable restrictions, but will be an obvious one to anybody who has made cardboard cutout models of polyhedra. It is significant that the qualifier *convex* appears in the theorem. Although it is possible to demonstrate by means of simple examples that some non-convex polyhedra (that is, polyhedra with regions of non-convexity) are rigid, it is also possible to demonstrate special cases of non-convex polyhedra which are not rigid, and are capable of undergoing infinitesimal distortions at least. This is a difficult area of pure mathematics. For the present purposes we note that convexity guarantees rigidity whereas non-convexity may produce deformability.

§29.2.6. A Disadvantage of Rigidity

While rigidity and strength are in many cases desirable attributes of shell structures, there are some important difficulties that can occur precisely on account of unavoidable rigidity. As an example of this consider a chemical plant where two large pressure vessels, firmly mounted on separate foundations, are connected by a length of straight pipe. Thermal expansion of the vessels can only be accommodated

without distortion if the pipe contracts in length. If it also expands thermally very large forces can be set up as a result of the rigidity of the vessels. In cases like this it is often convenient to accommodate expansion by a device such as a *bellows* unit. Alternatively, when the interconnecting pipework has bends, it is sometimes possible to make use of the fact that the bends can be relatively flexible. In the case of bellows and bends the flexibility is to a large extent related to the geometry of the respective surfaces. It is significant that both are *non-convex*. Nevertheless this of itself does not constitute a proper explanation of their flexibility.

§29.2.7. Catastrophic Failures

The property of closed shell structures being rigid and strong is of great practical value. But it should not be in ignorance of a well known design principle: *efficient structures may fail catastrophically*. Here the term “efficient” describes the consequences of using the closed shell principle. By designing a shell structure as a closed box rather than an open one we may be able to use thinner sheet material and hence produce a more efficient design.

On the other hand, thin shells under compressive membrane forces are prone to *buckling* of a particularly unstable kind. The rapid change in geometry after buckling and consequent decrease of load capacity leads to catastrophic collapse. This is illustrated by the well known experience of “crumpling” of thin wall cylinders like soda cans, under axial compression. The crumpling of a thin convex shell is accompanied by a loss of convexity, which partly explains why the post-buckling rigidity is so low.

§29.3. MATHEMATICAL MODELS OF SHELLS

§29.3.1. The Governing Equations

Mathematical models of shells are constructed in principle following the same general idea used for flat plates. The actual shell, which is a three dimensional object, is replaced by a surface endowed with certain mechanical properties. The transfer of such properties onto the surface is done by kinematic and statical assumptions similar to those made in the theory of plates.

As a result of these assumptions one obtains a set of mechanical properties expressed, in the case of an elastic material, in the form of a generalized Hooke’s law relating the deformation of a small material element to the stresses applied to it. These are called the *constitutive equations*. Subsequent steps in the theory of shells are similar to those followed for beams and plates: obtain *equilibrium equations* relating the stress resultants in the structure to the applied external forces, and *kinematic relations*, also called *compatibility equations*, that connect strains and displacements. Where dynamic effects are important, as in vibration problems, the equilibrium equations include inertial and possibly damping terms.

In *linear* shell theory the equations of equilibrium and compatibility are written in terms of the initial geometry of the structure. This assumption necessarily restricts the displacements to be very small. The buckling analysis of shells requires consideration of first-order nonlinear effects, and thus are beyond the scope of this course.

The three sets of field equations: kinematic, constitutive and equilibrium, along with appropriate boundary conditions, comprise the governing equations of the mathematical model. What makes the shell problem more complex is the fact that the equations have to be set up with respect to a generally curved surface in three dimensional space. By comparison, in the theory of flat plates the governing equations are set up merely over a planar surface.

§29.3.2. The Dominating Effect of Geometry

In the mechanics of deformable solids the character of the resulting mathematical problem (a system of partial differential equations) is usually determined by the material properties only. For example, one major difference between the theories of elasticity and plasticity is while in elasticity the governing differential equations are elliptic, in plasticity they are sometimes hyperbolic and thus demand different approaches for their solution. In the linear theory of shells the governing equations may be rendered hyperbolic as a consequence of *geometrical* properties of the shell surface.

§29.3.3. Interaction of Bending and Stretching Effects

The mechanical properties of a shell element describe its resistance to deformation in terms of separable stretching and bending effects. Loads applied to the shell are carried in general through a combination of bending and stretching actions, which generally vary from point to point. One of the major difficulties in the theory of shells is to find a relatively simple way of describing the interaction between the two effects. This aspect of the theory has been troublesome from the beginning. Rayleigh¹ argued that the deformation of a thin hemispherical bowl would be primarily inextensional, and accordingly he developed a special method of analysis which took into account only the bending energy of the shell. On the other hand Love² argued that for thin shells stretching was the dominant effect. At that time Love had not grasped the strong contrast between open and closed shells. The controversy was resolved by Lamb³ and Basset⁴, who solve Love's general equations for a cylindrical shell and demonstrated the possibility of a narrow *boundary layer* in which there was a rapid transition between bending and stretching effects. The width of the layer was determined by the interaction between those effects.

§29.3.4. Classification in Terms of Thickness Ratio

As in the case of plates, one can classify shell mathematical models in terms of the ratio of the thickness to a characteristic dimension:

- Very thick: 3D effects
- Thick: stretching, bending and higher order transverse shear
- Moderately thick: stretching, bending and first order transverse shear
- Thin shells: stretching and bending energy considered but transverse shear neglected
- Very thin shells: dominated by stretching effects. Also called membranes.

The main difference from flat plates is that the determination of characteristic dimensions is more complex.

¹ Lord Rayleigh, On the infinitesimal bending of surfaces of revolution, *Proc. London Math. Soc.*, **13**, 4–16, 1881. Also *The Theory of Sound*, Vol. 1, Ch. 10, MacMillan, London, 1884.

² A. E. H. Love, On the small free vibrations and deformations of thin elastic shells, *Phil. Trans. Royal Soc. London*, series A, **179**, 491–546, 1888.

³ H. Lamb, On the determination of an elastic shell, *Proc. London Math. Soc.*, **21**, 119–146, 1890.

⁴ A. B. Basset, On the extension and flexure of cylindrical and spherical thin shells, *Phil. Trans. Royal Soc. London*, Series A, **181**, 433–480, 1890.

32

A Solid Shell Element

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§32.1. Introduction

Solid-shell elements form a class of finite element models intermediate between thin shell and conventional solid elements. They have the same node and freedom configuration of solid elements but account for shell-like behavior in the thickness direction. They are useful for modeling shell-like portions of a 3D structure without the need to connect solid element nodes to shell nodes. See Figure 1(a).

This report presents in detail the derivation and computer implementation of an 8-node solid shell element called SS8. Most of the derivation follows the Assumed Natural Strain (ANS) method introduced by Park and Stanley [21] for doubly curved thin-shell elements and by Bathe and Dvorkin [2] for Mindlin-Reissner plates.

The derivation of solid-shell elements is more complicated than that of standard solid elements since they are prone to the following problems:

1. Shear and membrane locking. These have been addressed with the hybrid strain formulation [1,13,20], hybrid stress [24], and the Assumed and Enhanced Natural Strain formulations [7,8,14,19,23].
2. Trapezoidal locking caused by deviation of midplane planform from rectangular shape [8,16,17].
3. Thickness locking due to Poisson's ratio coupling of the inplane and transverse normal stresses [1,7,14,19,22,20,25,26].

As noted, the formulation that follows uses primarily the *Assumed Natural Strain* (ANS) method with adjustments to make the element kinematics work correctly for laminate wall constructions such as that depicted in Figure 1(b). This is done by assuming a uniform thickness stress instead of a uniform strain, which results in modified thickness-integrated constitutive equations.

A *Mathematica* implementation of the SS8 element is also presented in this report. This implementation was used for rapid prototyping and experimentation. This was particularly useful for some element components, such as the inplane strains, which were frequently updated during the course of the project. A Fortran 77 implementation is presented in a separate document [11].

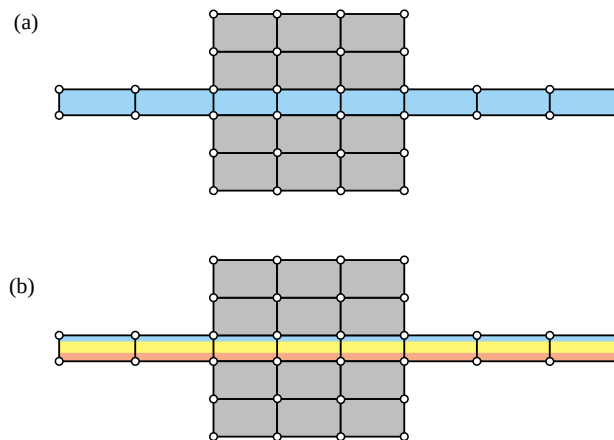


Figure 1. Solid shell elements (in color) connected to standard solid brick elements (in gray). Figure (b) shows a laminate wall configuration.

§32.2. Geometric Description

The point of departure is the well known 8-node brick (hexahedron) element shown in Figure 2(a). The

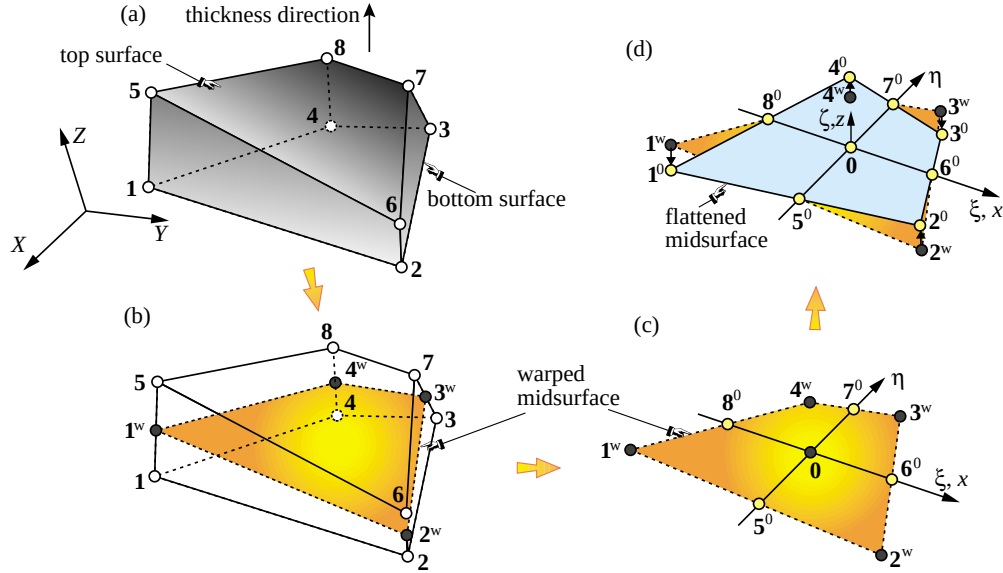


Figure 2. Initial steps in morphing a conventional 8-node brick (hexahedron) into the SS8 solid shell element: (a) the source brick, (b) the warped midsurface, (c) isolated warped midsurface and medians, (d) the flattened midsurface.

element is referred to a global Cartesian system $\{X, Y, Z\}$. The local node numbers are 1 through 8, arranged as illustrated in that figure. This *hexahedron numbering* is used to define preferred directions, as explained next. The coordinates of these 8 nodes may be grouped in a table:

$$\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 \\ Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 & Z_8 \end{bmatrix} \quad (32.1)$$

Anticipating the transition to a thick shell element, one of the brick dimensions is identified as the *shell thickness direction* also called *wall direction*. This has edges going from nodes 1,2,3,4 to 5,6,7,8, respectively. Segments 1-5, 2-6, 3-7, and 4-8 are called the *thickness edges*. Face 1-2-3-4 defines the *bottom surface* and face 5-6-7-8 the *top surface*. Surfaces extending along the other two directions are called *inplane surfaces*. Directions contained in an inplane surface are called *inplane directions*.

The four thickness edge midpoints are denoted by 1^w , 2^w , 3^w and 4^w , where superscript w stands for “warped.” See Figure 2(b). These four points are the corners of a quadrilateral known as the *warped midsurface*. It is qualified as warped since the four corners do not generally lie in a plane. Figure 2(c) shows the isolated warped midsurface. The midpoints between its corners are called 5^w , 6^w , 7^w and 8^w . The segments 5^w - 7^w and 8^w - 6^w that join the midpoints are called the *midsurface medians*. No matter how warped the surface is, the midsurface medians cross at a point 0 and hence define a plane.

§32.2.1. The Flattened Brick

The plane defined by the midsurface medians is called the *midsurface plane*. Using this fact, the local Cartesian axes are defined as follows:

- x along median 6^0 - 8^0
- z normal to the midsurface plane at the intersection of the medians
- y normal to x and z and hence located in the midsurface plane, forming a right-handed system.

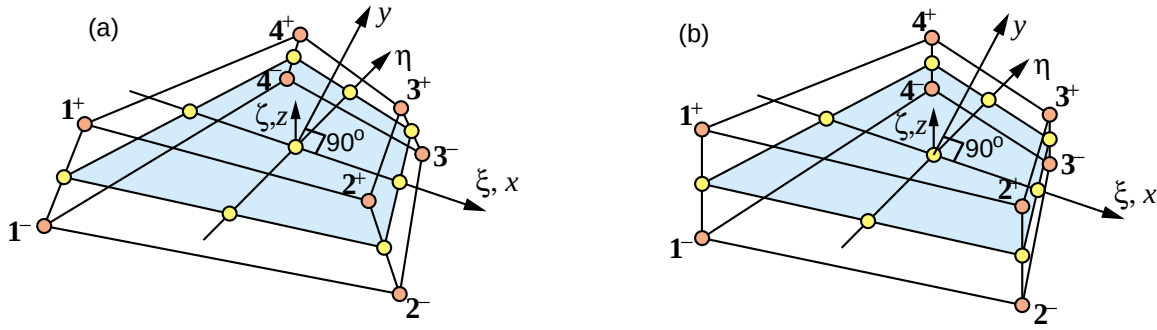


Figure 3. The flattened brick geometry: (a) thickness tapered (TT) element, (b) thickness-prismatic (TP) element

Points 1^w through 4^w are projected on that plane to locations labeled 1^0 , 2^0 , 3^0 and 4^0 , respectively. Since the projection is along the z direction this is done by equating their $\{x, y\}$ coordinates:

$$\begin{bmatrix} x_1^0 & x_2^0 & x_3^0 & x_4^0 \\ y_1^0 & y_2^0 & y_3^0 & y_4^0 \end{bmatrix} = \begin{bmatrix} x_1^w & x_2^w & x_3^w & x_4^w \\ y_1^w & y_2^w & y_3^w & y_4^w \end{bmatrix} \quad (32.2)$$

Points 1^0 through 4^0 are the corners of a flat quadrilateral, shown in Figure 2(d). Other points in the original brick are moved by the appropriate amount along z to get a *flattened brick*, as shown in Figure 3. This is the geometry on which the solid-shell element development will be based. Nodes 1 through 8 of the original brick map to 1^- through 4^+ of the flattened brick. The local coordinates of these eight nodes can be expressed as in array form as

$$\begin{bmatrix} x_1^- & x_2^- & x_3^- & x_4^- & x_1^+ & x_2^+ & x_3^+ & x_4^+ \\ y_1^- & y_2^- & y_3^- & y_4^- & y_1^+ & y_2^+ & y_3^+ & y_4^+ \\ z_1^- & z_2^- & z_3^- & z_4^- & z_1^+ & z_2^+ & z_3^+ & z_4^+ \end{bmatrix} = \begin{bmatrix} x_1^0 - \frac{1}{2}s_{x1}h_1 & x_2^0 - \frac{1}{2}s_{x2}h_2 & x_3^0 - \frac{1}{2}s_{x3}h_3 & x_4^0 - \frac{1}{2}s_{x4}h_4 \\ y_1^0 - \frac{1}{2}s_{y1}h_1 & y_2^0 - \frac{1}{2}s_{y2}h_2 & y_3^0 - \frac{1}{2}s_{y3}h_3 & y_4^0 - \frac{1}{2}s_{y4}h_4 \\ -\frac{1}{2}h_1 & -\frac{1}{2}h_2 & -\frac{1}{2}h_3 & -\frac{1}{2}h_4 \\ x_1^0 + \frac{1}{2}s_{x1}h_1 & x_2^0 + \frac{1}{2}s_{x2}h_2 & x_3^0 + \frac{1}{2}s_{x3}h_3 & x_4^0 + \frac{1}{2}s_{x4}h_4 \\ y_1^0 + \frac{1}{2}s_{y1}h_1 & y_2^0 + \frac{1}{2}s_{y2}h_2 & y_3^0 + \frac{1}{2}s_{y3}h_3 & y_4^0 + \frac{1}{2}s_{y4}h_4 \\ \frac{1}{2}h_1 & \frac{1}{2}h_2 & \frac{1}{2}h_3 & \frac{1}{2}h_4 \end{bmatrix} \quad (32.3)$$

Here $h_n = z_n^+ - z_n^-$ ($n = 1, 2, 3, 4$) are the corner thicknesses, as depicted in Figure 4, and $\{s_{xn}, s_{yn}\}$ are the $\{x, y\}$ slopes of the n^{th} thickness edge with respect to the midsurface normal z .

If $h_1 = h_2 = h_3 = h_4 = h$ the flattened brick is said to be of *constant thickness*. It is of *variable thickness* otherwise. If $s_{x1} = s_{y1} = s_{x2} = s_{y2} = s_{x3} = s_{y3} = s_{x4} = s_{y4} = 0$ the flattened brick is said to be *thickness prismatic* or TP for short. In this case $x_n^- = x_n^+$, $n = 1, 2, 3, 4$. This situation happens in modeling thick flat plates. If at least one slope is nonzero, the flattened brick is said to be *thickness tapered* or TT for short. This occurs in modeling curved thick shells. Figures 3 and 4 illustrate these geometric concepts.

The distinction between TP and TT geometries and constant versus variable thickness is important in the evaluation of the solid shell element. The element will pass the patch test under certain geometric constraints but only approximately otherwise.

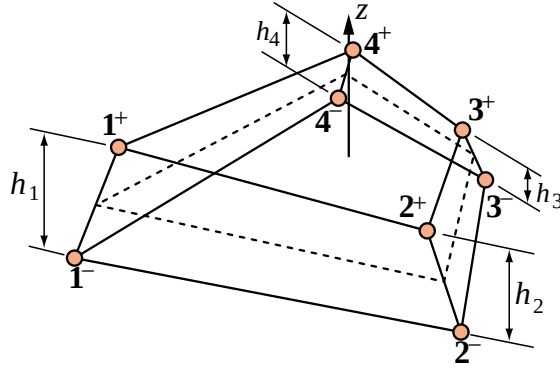


Figure 4. Corner thicknesses in a flattened brick.

§32.2.2. Natural Coordinates

On first impression it appears as if two sets of natural hexahedron coordinates need to be defined, one for the actual (warped) brick, and the other for the flattened brick. After some thought, however, one realizes the the same set: $\{\xi, \eta, \zeta\}$ can be used for both geometries. It is only necessary to state for which of the two configurations that set is being used. The relation between the two geometries is mathematically stated in §2.4.

Coordinates ξ and η are directed along the midsurface medians 6^w-8^w (same as 6^0-8^0) and 5^w-7^w (same as 5^0-7^0), respectively, while ζ parametrizes the thickness direction. Equations $\zeta = -1$, $\zeta = 0$ and $\zeta = 1$ characterize the points of the bottom surface, midsurface and top surfaces, respectively, in both warped and flattened geometries.

§32.2.3. Coordinate Transformations

The relation between local and global coordinates is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} T_{d11} & T_{d12} & T_{d13} \\ T_{d21} & T_{d22} & T_{d23} \\ T_{d31} & T_{d32} & T_{d33} \end{bmatrix} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} = \mathbf{T}_d (\mathbf{X} - \mathbf{X}_0) \quad (32.4)$$

where matrix $\mathbf{T}_d$ stores the direction cosines of $\{x, y, z\}$ with respect to $\{X, Y, Z\}$, and

$$X_0 = \frac{1}{4}(X_1^w + X_2^w + X_3^w + X_4^w), \quad Y_0 = \frac{1}{4}(Y_1^w + Y_2^w + Y_3^w + Y_4^w), \quad Z_0 = \frac{1}{4}(Z_1^w + Z_2^w + Z_3^w + Z_4^w), \quad (32.5)$$

are the global coordinates of the median intersection point, labeled as 0 in Figure 2.

Inserting the coordinates of the original brick into (32.4) produces the local coordinates of its 8 corners, which are arranged as the 3×8 array

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \end{bmatrix} \quad (32.6)$$

Array (32.6) should not be confused with the first one in (32.3) since brick flattening has not been done yet. Next one derives the x, y coordinates of the warped midsurface nodes and the corner thicknesses.

```

SS8LocalSystem[XYZcoor_] := Module[{
  XYZ, XYZ1, XYZ2, XYZ3, XYZ4, XYZ5, XYZ6, XYZ7, XYZ8, XYZ0,
  XYZ1w, XYZ2w, XYZ3w, XYZ4w, XYZ50, XYZ60, XYZ70, XYZ80,
  dx1, dx2, dx3, dy1, dy2, dy3, dz1, dz2, dz3, xlr, ylr, zlr,
  x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8,
  z1, z2, z3, z4, z5, z6, z7, z8, xw, yw, hw, x, y, z, dx, dy, dz,
  x10, x20, x30, x40, y10, y20, y30, y40, xyz, xyzwh, A0},
{XYZ1, XYZ2, XYZ3, XYZ4, XYZ5, XYZ6, XYZ7, XYZ8} = XYZcoor;
XYZ1w = (XYZ5 + XYZ1)/2; XYZ2w = (XYZ6 + XYZ2)/2;
XYZ3w = (XYZ7 + XYZ3)/2; XYZ4w = (XYZ8 + XYZ4)/2;
XYZ50 = (XYZ1w + XYZ2w)/2; XYZ60 = (XYZ2w + XYZ3w)/2;
XYZ70 = (XYZ3w + XYZ4w)/2; XYZ80 = (XYZ4w + XYZ1w)/2;
XYZ0 = (XYZ1w + XYZ2w + XYZ3w + XYZ4w)/4;
XYZ = Transpose[XYZcoor] - XYZ0;
{dx1, dx2, dx3} = XYZ60 - XYZ80; {dy1, dy2, dy3} = XYZ70 - XYZ50;
xlr = Sqrt[dx1^2 + dx2^2 + dx3^2];
If[xlr <= 0, Print["SS8LocalSystem: midnodes 6-8 coincide"];
Return[Null]];
dx1 = dx1/xlr; dx2 = dx2/xlr; dx3 = dx3/xlr;
dz1 = dx2*dy3 - dx3*dy2; dz2 = dx3*dy1 - dx1*dy3;
dz3 = dx1*dy2 - dx2*dy1; zlr = Sqrt[dz1^2 + dz2^2 + dz3^2];
If[zlr <= 0, Print["SS8LocalSystem: colinear medians"];
Return[Null]];
dz1 = dz1/zlr; dz2 = dz2/zlr; dz3 = dz3/zlr;
dy1 = dz2*dx3 - dz3*dx2; dy2 = dz3*dx1 - dz1*dx3;
dy3 = dz1*dx2 - dz2*dx1; ylr = Sqrt[dy1^2 + dy2^2 + dy3^2];
dx = {dx1, dx2, dx3}; dy = {dy1, dy2, dy3}; dz = {dz1, dz2, dz3};
{dx, dy, dz} = Simplify[{dx, dy, dz}]; Td = {dx, dy, dz};
xyz = Td.XYZ; {x, y, z} = xyz;
{x1, x2, x3, x4, x5, x6, x7, x8} = x;
{y1, y2, y3, y4, y5, y6, y7, y8} = y;
{z1, z2, z3, z4, z5, z6, z7, z8} = z;
xw = {x5 + x1, x6 + x2, x7 + x3, x8 + x4}/2;
yw = {y5 + y1, y6 + y2, y7 + y3, y8 + y4}/2;
zw = {z5 + z1, z6 + z2, z7 + z3, z8 + z4}/2;
hw = {z5 - z1, z6 - z2, z7 - z3, z8 - z4}; xyzwh = {xw, yw, zw, hw};
{x10, x20, x30, x40} = xw; {y10, y20, y30, y40} = yw;
A0 = (1/2)*((x30 - x10)*(y40 - y20) - (x40 - x20)*(y30 - y10));
Return[{xyz, xyzwh, Td, A0}]]];

```

Figure 5. *Mathematica* module for the SS8 element localization.

The elements is flattened by subtracting out the z_n^w coordinates. The operation is:

$$\begin{bmatrix} x_1^w & x_2^w & x_3^w & x_4^w \\ y_1^w & y_2^w & y_3^w & y_4^w \\ z_1^w & z_2^w & z_3^w & z_4^w \\ h_1 & h_2 & h_3 & h_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x_5 + x_1) & \frac{1}{2}(x_6 + x_2) & \frac{1}{2}(x_7 + x_3) & \frac{1}{2}(x_8 + x_4) \\ \frac{1}{2}(y_5 + y_1) & \frac{1}{2}(y_6 + y_2) & \frac{1}{2}(y_7 + y_3) & \frac{1}{2}(y_8 + y_4) \\ \frac{1}{2}(z_5 + z_1) & \frac{1}{2}(z_6 + z_2) & \frac{1}{2}(z_7 + z_3) & \frac{1}{2}(z_8 + z_4) \\ z_5 - z_1 & z_6 - z_2 & z_7 - z_3 & z_8 - z_4 \end{bmatrix} \xrightarrow{\text{flatten}} \begin{bmatrix} x_1^0 & x_2^0 & x_3^0 & x_4^0 \\ y_1^0 & y_2^0 & y_3^0 & y_4^0 \\ 0 & 0 & 0 & 0 \\ h_1 & h_2 & h_3 & h_4 \end{bmatrix} \quad (32.7)$$

These element localization operations are implemented in the *Mathematica* module SS8LocalSystem listed in Figure 5. The module is referenced as

$$\{xyz, xyzwh, Td, A0\} = \text{SS8LocalSystem}[XYZcoor] \quad (32.8)$$

The only argument is

XYZcoor The global coordinates of the 8 corner nodes of the original brick, arranged as $\{\{X1, Y1, Z1\}, \{X2, Y2, Z2\}, \dots, \{X8, Y8, Z8\}\}$.

The module returns a list containing four items:

xyz The local coordinates of the 8 corner nodes of the original brick, arranged as $\{\{x1, x2, \dots, x8\}, \{y1, y2, \dots, y8\}, \{z1, z2, \dots, z8\}\}$

Note: to get the node by node arrangement

xyzcoor={ { x1,y1,z1 }, { x2,y2,z2 }, . . . { x8,y8,z8 } }

transpose this array: xyzcoor=Transpose[xyz].

xyzwh The local coordinates of the warped midsurface corners and the corner thicknesses of the flattened element, arranged as
 { { x1w,x2w,x3w,x4w }, { y1w,y2w,y3w,y4w }, { z1w,z2w,z3w,z4w }, { h1,h2,h3,h4 } }
 Note that x10=x1w, x20=x2w, etc., so the flat midsurface coordinates are immediately available from the first two rows.

Td The 3×3 direction cosine matrix $\mathbf{T}_d$ defined in (32.4).

A0 The signed area of the flat midsurface. A negative or zero area would flag an input error.

If an error is detected during processing, the module returns Null.

§32.2.4. Warped to Flat Geometric Transformation

For further use in the transformation of flat to actual (warped) geometry of Section 3, the “flattening” coordinate transformations are recorded here. The warped geometry is described by the isoparametric transformations

$$\begin{bmatrix} x^w \\ y^w \\ z^w \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_8 \end{bmatrix} \quad (32.9)$$

where N_i are the trilinear shape functions

$$\begin{aligned} N_1 &= \frac{1}{8}(1 - \xi)(1 - \eta)(1 - \zeta), & N_2 &= \frac{1}{8}(1 + \xi)(1 - \eta)(1 - \zeta), \\ N_3 &= \frac{1}{8}(1 + \xi)(1 + \eta)(1 - \zeta), & N_4 &= \frac{1}{8}(1 - \xi)(1 + \eta)(1 - \zeta), \\ N_5 &= \frac{1}{8}(1 - \xi)(1 - \eta)(1 + \zeta), & N_6 &= \frac{1}{8}(1 + \xi)(1 - \eta)(1 + \zeta), \\ N_7 &= \frac{1}{8}(1 + \xi)(1 + \eta)(1 + \zeta), & N_8 &= \frac{1}{8}(1 - \xi)(1 + \eta)(1 + \zeta). \end{aligned} \quad (32.10)$$

The flattening transformation can be mathematically stated as

$$\begin{bmatrix} x^0 \\ y^0 \\ z^0 \end{bmatrix} = \begin{bmatrix} x^w \\ y^w \\ z^w \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ z_1^w N_1^Q + z_2^w N_2^Q + z_3^w N_3^Q + z_4^w N_4^Q \end{bmatrix} \quad (32.11)$$

Here $z_1^w = \frac{1}{2}(z_5 + z_1)$, $z_2^w = \frac{1}{2}(z_6 + z_2)$, $z_3^w = \frac{1}{2}(z_7 + z_3)$ and $z_4^w = \frac{1}{2}(z_8 + z_4)$, whereas $N_1^Q = \frac{1}{4}(1 - \xi)(1 - \eta)$, $N_2^Q = \frac{1}{4}(1 + \xi)(1 - \eta)$, $N_3^Q = \frac{1}{4}(1 + \xi)(1 + \eta)$, $N_4^Q = \frac{1}{4}(1 - \xi)(1 + \eta)$ are the bilinear shape functions of the flat midsurface, which is a quadrilateral.¹

§32.2.5. Warping Check

The derivation of the solid shell stiffness matrix is carried out on the flattened (unwarped) brick geometry, and globalized to the actual geometry. If the element is warped, certain tests, notably patch tests for constant strain states, are only approximately satisfied. Poor results can be expected if the element is excessively warped. Consequently it is convenient to have a “deviation from flatness” measure to be checked during model preprocessing.

¹ Since the coordinate directions are not treated equally, the flattened element is no longer isoparametric.

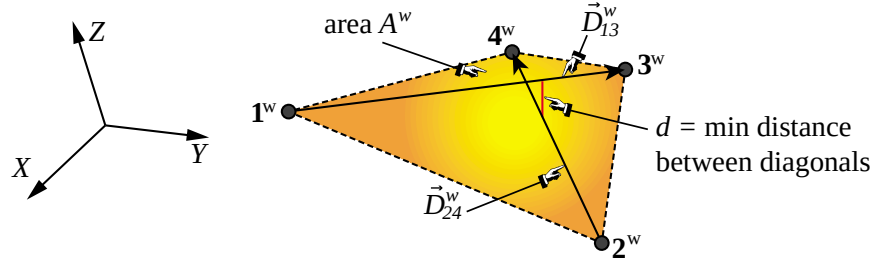


Figure 6. Ingredients for computing a warping measure Ψ : minimum distance d between diagonals, and area A^w of warped midsurface.

Several warping measures for shell elements have been proposed over the years. Here we shall use a minor variation of the oldest one, invented at Boeing by 1965 for the SAMECS code, which has the advantages of easy physical interpretation. The geometric ingredients are shown in Figure 6. Denote by d the shortest distance between the diagonals $\vec{D}_{13}^w$ and $\vec{D}_{24}^w$ of the warped midsurface, and A^w the area of the midsurface. The dimensionless warping measure is defined as

$$\Psi = \frac{d}{\sqrt{2|A^w|}}, \quad \text{where} \quad |A^w| = \left| \vec{D}_{13}^w \times \vec{D}_{24}^w \right| \quad (32.12)$$

(Boeing's measure is $d/\sqrt{A^w}$; the factor change makes (32.12) identical to $\frac{1}{4}(|z_1^w| + |z_2^w| + |z_3^w| + |z_4^w|)/\sqrt{A}$ for slightly warped elements.) Using the abbreviations $X_{ij} = X_i - X_j$, $Y_{ij} = Y_i - Y_j$, $Z_{ij} = Z_i - Z_j$, the expression of Ψ in terms of the global node coordinates can be shown to be

$$\Psi = \frac{|X_{21}p + Y_{21}q + Z_{21}r|}{(p^2 + q^2 + r^2)^{3/4}}, \quad (32.13)$$

in which $p = Y_{41}Z_{32} - Z_{41}Y_{32}$, $q = Z_{41}X_{32} - X_{41}Z_{32}$ and $r = X_{41}Y_{32} - Y_{41}X_{32}$. In the numerator of (32.13) $\{X_{21}, Y_{21}, Z_{21}\}$ may be actually replaced by $\{X_{ji}, Y_{ji}, Z_{ji}\}$ where $j = 2, 3, 4, 1$ is the cyclic permutation of $i = 1, 2, 3, 4$.

If $\Psi = 0$ the original brick is flat in the sense that its midsurface is flat. If $\Psi > \Psi_{max}$ warping is excessive and the preprocessing should be aborted. If $\Psi_{warn} < \Psi < \Psi_{max}$ an error warning should be given. Boeing manuals recommend $\Psi_{max} = 0.075$ and $\Psi_{warn} = \frac{1}{4}\Psi_{max}$.²

A *Mathematica* implementation of the computation of Ψ is shown in Figure 7. Module SS8Warping receives the original brick coordinates XYZcoord ordered $\{\{X1, Y1, Z1\}, \{X2, Y2, Z2\}, \dots, \{X8, Y8, Z8\}\}$, and returns Psi.

§32.2.6. Volume Computation

The computation of the signed volume of the original or flattened brick is useful for several tasks. For example, preprocessing checks for positive volume, as well as mass and gravity load computations.

Consider an arbitrary 8-node brick with corner coordinates (32.2). and natural coordinates $\{\xi, \eta, \zeta\}$.

² For the SS8 element a much lower threshold, say $\Psi_{max} = 0.01$, is recommended until its ability to handle warped configurations is further ascertained.


```

SS8Warping[XYZcoor_] := Module[
  {X1, Y1, Z1, X2, Y2, Z2, X3, Y3, Z3, X4, Y4, Z4,
   X5, Y5, Z5, X6, Y6, Z6, X7, Y7, Z7, X8, Y8, Z8,
   X1w, Y1w, Z1w, X2w, Y2w, Z2w, X3w, Y3w, Z3w, X4w, Y4w, Z4w,
   p, q, r, Psi},
  {{X1, Y1, Z1}, {X2, Y2, Z2}, {X3, Y3, Z3}, {X4, Y4, Z4},
   {X5, Y5, Z5}, {X6, Y6, Z6}, {X7, Y7, Z7}, {X8, Y8, Z8}} = XYZcoor;
  {X1w, Y1w, Z1w} = ({X1, Y1, Z1} + {X5, Y5, Z5})/2;
  {X2w, Y2w, Z2w} = ({X2, Y2, Z2} + {X6, Y6, Z6})/2;
  {X3w, Y3w, Z3w} = ({X3, Y3, Z3} + {X7, Y7, Z7})/2;
  {X4w, Y4w, Z4w} = ({X4, Y4, Z4} + {X8, Y8, Z8})/2;
  p = (Y3w - Y1w) * (Z4w - Z2w) - (Z3w - Z1w) * (Y4w - Y2w);
  q = (Z3w - Z1w) * (X4w - X2w) - (X3w - X1w) * (Z4w - Z2w);
  r = (X3w - X1w) * (Y4w - Y2w) - (Y3w - Y1w) * (X4w - X2w);
  Psi = Abs[(X2w - X1w) * p + (Y2w - Y1w) * q + (Z2w - Z1w) * r] /
    (p^2 + q^2 + r^2)^(3/4) / Sqrt[2];
  Return[Psi];

```

Figure 7. *Mathematica* module to compute the warping measure Ψ .

The entries of the 3×3 Jacobian $\mathbf{J}$ of $\{X, Y, Z\}$ with respect to $\{\xi, \eta, \zeta\}$ are given by

$$\begin{aligned}
 \mathbf{J} &= \frac{\partial(X, Y, Z)}{\partial(\xi, \eta, \zeta)} = \begin{bmatrix} \partial X / \partial \xi & \partial Y / \partial \xi & \partial Z / \partial \xi \\ \partial X / \partial \eta & \partial Y / \partial \eta & \partial Z / \partial \eta \\ \partial X / \partial \zeta & \partial Y / \partial \zeta & \partial Z / \partial \zeta \end{bmatrix} = \begin{bmatrix} J_{X\xi} & J_{Y\xi} & J_{Z\xi} \\ J_{X\eta} & J_{Y\eta} & J_{Z\eta} \\ J_{X\zeta} & J_{Y\zeta} & J_{Z\zeta} \end{bmatrix}, \\
 J_{X\xi} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \xi} X_i, & J_{X\eta} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \eta} X_i, & J_{X\zeta} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \zeta} X_i, \\
 J_{Y\xi} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \xi} Y_i, & J_{Y\eta} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \eta} Y_i, & J_{Y\zeta} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \zeta} Y_i, \\
 J_{Z\xi} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \xi} Z_i, & J_{Z\eta} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \eta} Z_i, & J_{Z\zeta} &= \sum_{i=1}^8 \frac{\partial N_i}{\partial \zeta} Z_i,
 \end{aligned} \tag{32.14}$$

in which $N_i(\xi, \eta, \zeta)$, are the isoparametric shape functions (32.10). The brick volume is given by

$$V = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 J d\xi d\eta d\zeta, \quad \text{in which} \quad J = \det(\mathbf{J}). \tag{32.15}$$

This integral is evaluated by a product Gauss quadrature rule. It can be shown that³ a $2 \times 2 \times 2$ rule exactly integrates (32.15) because the variation of each natural coordinate: ξ , η and ζ in J is at most quadratic. Consequently only the $1 \times 1 \times 1$ and $2 \times 2 \times 2$ rules need to be considered.

A *Mathematica* implementation of the computation of V is listed in Figure 8. Module SS8Volume is referenced as

$$\text{vol} = \text{SS8Volume}[\text{XYZcoor}, n] \tag{32.16}$$

The arguments are

XYZcoor The Cartesian coordinates of the 8 corner nodes of the brick. For the original brick use the global coordinates arranged as $\{\{X1, Y1, Z1\}, \{X2, Y2, Z2\}, \dots, \{X8, Y8, Z8\}\}$.

³ Some finite element books still erroneously state that a 1-point Gauss rule gives the exact volume of the 8-node brick. That statement is correct in two dimensions for 4-node quadrilateral elements, but not in 3D.

```

SS8Volume[XYZcoor_, n_] := Module[
  {X1, Y1, Z1, X2, Y2, Z2, X3, Y3, Z3, X4, Y4, Z4,
   X5, Y5, Z5, X6, Y6, Z6, X7, Y7, Z7, X8, Y8, Z8,
   J11, J12, J13, J21, J22, J23, J31, J32, J33,
   dNhex, Xhex, Yhex, Zhex, g=Sqrt[1/3], gtab, vol=0},
  {{X1, Y1, Z1}, {X2, Y2, Z2}, {X3, Y3, Z3}, {X4, Y4, Z4},
   {X5, Y5, Z5}, {X6, Y6, Z6}, {X7, Y7, Z7}, {X8, Y8, Z8}}=XYZcoor;
  If [n!=1&& n!=2, Print["SS8Volume: n not 1 or 2"];
   Return[Null]];
  dNhex[{ξ_, η_, ζ_}]:=
  {{-(1-η)*(1-ξ), (1-η)*(1-ξ), (1+η)*(1-ξ), -(1+η)*(1-ξ),
    -(1-η)*(1+ξ), (1-η)*(1+ξ), (1+η)*(1+ξ), -(1+η)*(1+ξ)},
   {-(1-ξ)*(1-ξ), -(1+ξ)*(1-ξ), (1+ξ)*(1-ξ), (1-ξ)*(1-ξ),
    -(1-ξ)*(1+ξ), -(1+ξ)*(1+ξ), (1+ξ)*(1+ξ), (1-ξ)*(1+ξ)},
   {-(1-ξ)*(1-η), -(1+ξ)*(1-η), -(1+ξ)*(1+η), -(1-ξ)*(1+η),
    (1-ξ)*(1-η), (1+ξ)*(1-η), (1+ξ)*(1+η), (1-ξ)*(1+η)}}/8;
  Xhex={X1, X2, X3, X4, X5, X6, X7, X8};
  Yhex={Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8};
  Zhex={Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8};
  If [n==1, {dNξ, dNη, dNζ}=dNhex[{0, 0, 0}];
   J11=dNξ.Xhex; J21=dNη.Xhex; J31=dNζ.Xhex;
   J12=dNξ.Yhex; J22=dNη.Yhex; J32=dNζ.Yhex;
   J13=dNξ.Zhex; J23=dNη.Zhex; J33=dNζ.Zhex;
   vol=Simplify[J11*J22*J33+J21*J32*J13+J31*J12*J23-
    J31*J22*J13-J11*J32*J23-J21*J12*J33]*8;
   Return[vol]];
  gtab={{-g, -g, -g}, {g, -g, -g}, {g, g, -g}, {-g, g, -g},
   {-g, -g, g}, {g, -g, g}, {g, g, g}, {-g, g, g}};
  For [i=1, i<=8, i++, {dNξ, dNη, dNζ}=dNhex[gtab[[i]]];
   J11=dNξ.Xhex; J21=dNη.Xhex; J31=dNζ.Xhex;
   J12=dNξ.Yhex; J22=dNη.Yhex; J32=dNζ.Yhex;
   J13=dNξ.Zhex; J23=dNη.Zhex; J33=dNζ.Zhex;
   vol+=Simplify[J11*J22*J33+J21*J32*J13+J31*J12*J23-
    J31*J22*J13-J11*J32*J23-J21*J12*J33]];
  Return[Simplify[vol]];

```

Figure 8. *Mathematica* module to compute brick volume.

Same result should be obtained using the local coordinates arranged as

$\{\{x1, y1, z1\}, \{x2, y2, z2\}, \dots, \{x8, y8, z8\}\}$.

For the volume of the flattened brick⁴ use

$\{\{x1, y1, -h1/2\}, \{x2, y2, -h2/2\}, \dots, \{x8, y8, h4/2\}\}$.

- n Gauss rule integration index. Use n=1 or n=2 to specify the $1 \times 1 \times 1$ or the $2 \times 2 \times 2$ rules, respectively.

The module returns the volume $vol=V$ as function value. A negative volume would flag an input error.

§32.3. SS8 Element Description

The SS8 solid-shell element has eight nodes numbered 1 through 8, which are located at the corners of the actual (warped) brick. The geometry is defined with reference to a global $\{X, Y, Z\}$ system; see Figure 2(a). As noted in §2.1, nodes 1-2-3-4 define the bottom surface and 5-6-7-8 the top surface. The four edges 1-5, 2-6, 3-7 and 4-8 that connect the bottom and top surfaces collectively define the thickness direction.

§32.3.1. Global Displacements

⁴ An interesting result is that the volumes of the warped and flattened bricks are identical if the $1 \times 1 \times 1$ Gauss rule is used. This comes from the fact that the Jacobian at the sample point $\xi = \eta = \zeta = 0$ is the same; a property verifiable from the transformation (32.11). On the other hand, the exact volumes furnished by the $2 \times 2 \times 2$ rule generally differs.

The displacement components of an arbitrary point in the global system $\{X, Y, Z\}$ are $\{U_X, U_Y, U_Z\}$. These three functions define the displacement field of the element. For some developments they are collected into one 3-vector:

$$\vec{\mathbf{U}} = [U_X \quad U_Y \quad U_Z]^T. \quad (32.17)$$

The element has three translational degrees of freedom (DOFs): $\{U_{Xn}, U_{Yn}, U_{Zn}\}$ at each node $n = 1, 2, \dots, 8$. This makes a total of 24 DOFs. They are collected into the global node displacement vector in a node by node arrangement:

$$\mathbf{U}^{(e)} = [U_{X1} \quad U_{Y1} \quad U_{Z1} \quad U_{X2} \quad U_{Y2} \quad U_{Z2} \quad \dots \quad U_{X8} \quad U_{Y8} \quad U_{Z8}]^T. \quad (32.18)$$

Since the element has 6 rigid body modes, it possesses 18 deformational modes. These are allocated to mechanical response effects: membrane, bending, thickness and transverse shear, as described in §4.1.

§32.3.2. Local Displacements

The displacement components of an arbitrary point in the local system $\{x, y, z\}$ are $\{u_x, u_y, u_z\}$. These three functions define the displacement field of the element. For some developments they are collected into one 3-vector:

$$\vec{\mathbf{u}} = [u_x \quad u_y \quad u_z]^T. \quad (32.19)$$

The 24 DOFs are collected into the local node displacement vector in a node by node arrangement:

$$\hat{\mathbf{u}}^{(e)} = [u_{x1} \quad u_{y1} \quad u_{z1} \quad u_{x2} \quad u_{y2} \quad u_{z2} \quad \dots \quad u_{x8} \quad u_{y8} \quad u_{z8}]^T. \quad (32.20)$$

Here $\hat{\mathbf{u}}^{(e)}$ is used instead of simply $\mathbf{u}^{(e)}$ to distinguish (32.20) from the node displacements vector of the flattened brick introduced below. The local and global fields are connected by the transformation matrix $\mathbf{T}_d$ introduced in §2.3:

$$\vec{\mathbf{u}} = \mathbf{T}_d \vec{\mathbf{U}}, \quad \vec{\mathbf{U}} = \mathbf{T}_d^T \vec{\mathbf{u}}. \quad (32.21)$$

Applying this relation to the node displacement vectors (32.18) and (32.20) gives

$$\hat{\mathbf{u}}^{(e)} = \mathbf{T}_d^{(e)} \mathbf{U}^{(e)}, \quad \mathbf{U}^{(e)} = (\mathbf{T}_d^{(e)})^T \hat{\mathbf{u}}^{(e)}, \quad (32.22)$$

where $\mathbf{T}_d^{(e)}$ is the 24×24 block diagonal matrix formed by stacking the 3×3 matrix $\mathbf{T}_d$ eight times along its diagonal.

§32.3.3. Flattening

Flattening the brick in the local system through (32.11) in general alters the position of nodes 1, 2, $\dots$ 8. These become $1^-, 2^-, \dots, 4^+$, as illustrated in Figure 3. Consequently the node displacement values change. They are collected into

$$\mathbf{u}^{(e)} = [u_{x1}^- \quad u_{y1}^- \quad u_{z1}^- \quad u_{x2}^- \quad u_{y2}^- \quad u_{z2}^- \quad \dots \quad u_{x4}^+ \quad u_{y4}^+ \quad u_{z4}^+]^T. \quad (32.23)$$

To connect $\mathbf{u}^{(e)}$ and $\hat{\mathbf{u}}^{(e)}$ it is noted that the flattening coordinate mapping (32.11) only affects z . Assuming that the local displacement varies linearly in z (a reasonable assumption since there are only two node layers) we have the following interpolations along edges 1-5, 2-6, 3-7 and 4-8:

$$\begin{aligned} h_1 u_{x15}(z) &= u_{x5}(z - z_1) + u_{x1}(z_5 - z), & h_2 u_{x26}(z) &= u_{x6}(z - z_2) + u_{x2}(z_6 - z), \\ h_3 u_{x37}(z) &= u_{x7}(z - z_3) + u_{x3}(z_7 - z), & h_4 u_{x48}(z) &= u_{x8}(z - z_4) + u_{x4}(z_8 - z). \end{aligned} \quad (32.24)$$

with similar interpolations for u_y and u_z . Setting $z = \pm \frac{1}{2}h_i$ we obtain

$$\begin{aligned} h_1 u_{x1}^- &= u_{x5}(-\frac{1}{2}h_1 - z_1) + u_{x1}(z_5 + \frac{1}{2}h_1), & h_2 u_{x2}^- &= u_{x6}(-\frac{1}{2}h_2 - z_2) + u_{x2}(z_6 + \frac{1}{2}h_2), \\ h_3 u_{x3}^- &= u_{x7}(-\frac{1}{2}h_3 - z_3) + u_{x3}(z_7 + \frac{1}{2}h_3), & h_4 u_{x4}^- &= u_{x8}(-\frac{1}{2}h_4 - z_4) + u_{x4}(z_8 + \frac{1}{2}h_4), \\ h_1 u_{x1}^+ &= u_{x5}(\frac{1}{2}h_1 - z_1) + u_{x1}(z_5 - \frac{1}{2}h_1), & h_2 u_{x2}^+ &= u_{x6}(\frac{1}{2}h_2 - z_2) + u_{x2}(z_6 - \frac{1}{2}h_2), \\ h_3 u_{x3}^+ &= u_{x7}(\frac{1}{2}h_3 - z_3) + u_{x3}(z_7 - \frac{1}{2}h_3), & h_4 u_{x4}^+ &= u_{x8}(\frac{1}{2}h_4 - z_4) + u_{x4}(z_8 - \frac{1}{2}h_4), \end{aligned} \quad (32.25)$$

The u_y and u_z components are related in the same manner. From these a 24×24 transformation matrix T_h relating

$$\mathbf{u}^{(e)} = T_h \hat{\mathbf{u}}^{(e)} \quad (32.26)$$

can be built:

$$T_h = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 \\ \dots & & & & & & & & & & & & & & & & & & \end{bmatrix} \quad (32.27)$$

where $a_1 = \frac{1}{2} + z_5/h_1$, $b_1 = -\frac{1}{2} - z_1/h_1$, $a_2 = \frac{1}{2} + z_6/h_2$, $b_2 = -\frac{1}{2} - z_2/h_2$ etc. This matrix has the block pattern

$$T_h = \begin{bmatrix} a_1 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & b_1 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & a_2 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & b_2 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & a_3 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & b_3 \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a_4 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & b_4 \mathbf{I}_3 \\ b_5 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_1 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & b_6 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_2 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & b_7 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_3 \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & b_8 \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_4 \mathbf{I}_3 \end{bmatrix} \quad (32.28)$$

where $\mathbf{I}_3$ is the 3×3 identity matrix and a_i and b_i are coefficients. If the element is originally flat: $z_1 = -\frac{1}{2}h_1$, $z_5 = \frac{1}{2}h_1$, etc., all $a_i = 1$ and $b_i = 0$ and T_h reduces to the 24×24 identity matrix, as may be expected.

§32.4. SS8 Stiffness Matrix

This section gives an overall picture of the derivation of the stiffness matrix for the SS8 solid shell element. Figure 9 gives the schematics of the top level calculation sequence, along with the name of the *Mathematica* modules that perform them.

§32.4.1. Behavioral Assumptions

The solid shell element stiffness accounts for the following mechanical actions. All of them are assumed to act in the flattened brick geometry.

Inplane response. This is controlled by strains and stresses in the $\zeta = \text{const}$ planes. These are further decoupled into *membrane* and *bending* actions. Membrane strains are constant in ζ whereas bending strains vary linearly in ζ .⁵

⁵ Since the constitutive properties may vary along ζ in the case of laminate wall construction, the statement does not extend to stresses.

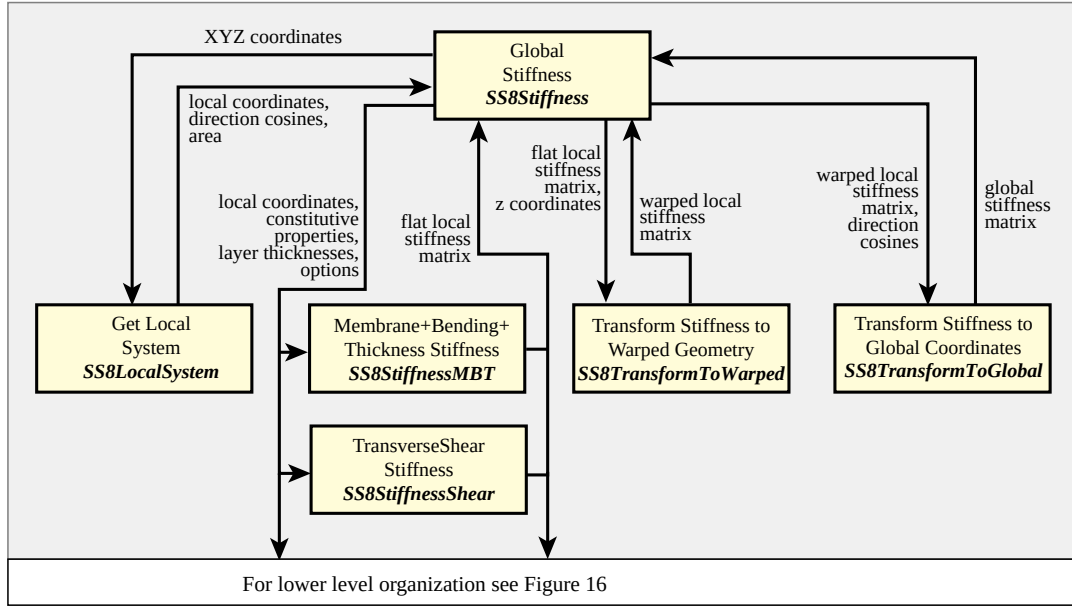


Figure 9. Organization of top-level SS8 stiffness computations.

Thickness response. This is controlled by extensional strains and normal stresses in the ζ direction.

Transverse shear response. This is controlled by transverse shear strains and stresses, where “transverse” is to be understood in the sense of normal to the midsurface.

The fundamental assumption made in the subsequent development is:

In the flattened local geometry, transverse shear actions are decoupled from inplane and thickness actions through the constitutive equations.

Mathematically, the stress-strain constitutive equations in the flattened element local system must be of the form

$$\begin{bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \sigma_{xy}^k \\ \sigma_{xy}^k \\ \sigma_{yz}^k \\ \sigma_{zx}^k \end{bmatrix} = \begin{bmatrix} E_{11}^k & E_{12}^k & E_{13}^k & E_{14}^k & 0 & 0 \\ E_{12}^k & E_{22}^k & E_{23}^k & E_{24}^k & 0 & 0 \\ E_{13}^k & E_{23}^k & E_{33}^k & E_{34}^k & 0 & 0 \\ E_{14}^k & E_{24}^k & E_{34}^k & E_{44}^k & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55}^k & E_{56}^k \\ 0 & 0 & 0 & 0 & E_{56}^k & E_{66}^k \end{bmatrix} \begin{bmatrix} e_{xx}^k \\ e_{yy}^k \\ e_{zz}^k \\ 2e_{xy}^k \\ 2e_{yz}^k \\ 2e_{zx}^k \end{bmatrix} \quad (32.29)$$

for each k , where k is the layer index in a laminate composite construction. In other words, the material of each layer is monoclinic with z as principal material axis. If this condition is not met, this element should not be used. A general solid element would be more appropriate. The assumption covers, however, many important cases of wall fabrication.

Note that coupling between inplane and thickness actions is permitted. In fact, even for isotropic materials such coupling exists in the form of Poisson’s ratio effects.

A consequence of this assumption is that the element stiffness in the flattened geometry is the superposition of a membrane+bending+thickness stiffness (a grouping often abbreviated to MBT or mbt) and a shear stiffness:

$$\mathbf{K}_{mbs}^{(e)} = \mathbf{K}_{mbt}^{(e)} + \mathbf{K}_s^{(e)}. \quad (32.30)$$

```

SS8Stiffness[XYZcoor_, {ElayerMBT_, ElayerS_}, layers_, options_] :=
Module[{xyz, xyzcoor, xyzwh, Td, A0, Kmbt, Ks, Kmbts, Kw, Ke},
{xyz, xyzwh, Td, A0} = SS8LocalSystem[XYZcoor];
xyzcoor = Transpose[xyz];
Kmbt = SS8StiffnessMBT[xyzcoor, ElayerMBT, layers, {0, 0}, options];
Ks = SS8StiffnessShear[xyzcoor, ElayerS, layers, options];
Kmbts = Kmbt + Ks;
Kw = SS8TransformToWarped[Kmbts, xyz[[3]]];
Ke = SS8TransformToGlobal[Kw, Td];
Return[Ke];

```

Figure 10. Top level module for calculation of global stiffness of the SS8 solid shell element.

As noted the element has a total of 24 degrees of freedom. Since it possesses 6 rigid body modes, it has 18 deformational modes. These are allocated to the four actions noted above: membrane, bending, thickness and transverse shear, in the numbers indicated in Table 1.

Table 1. Element Mode Count For Mechanical Actions

Action	Mode count
rigid body motions	6
membrane	5
bending	5
thickness	4
transverse shear	4
Total	24

§32.4.2. Top Level Implementation

The *Mathematica* implementation of the top level module SS8Stiffness is shown in Figure 10. This module is referenced as

$$Ke = SS8Stiffness[XYZcoor, \{ElayerMBT, ElayerS\}, layers, options] \quad (32.31)$$

The arguments are:

XYZcoor	The global Cartesian coordinates of the 8 corner nodes, arranged as $\{\{X1, Y1, Z1\}, \{X2, Y2, Z2\}, \dots, \{X8, Y8, Z8\}\}$.
ElayerMBT	Stress-strain constitutive matrices for MBT actions stacked in layer by layer sequence, as further described in §5.4.
ElayerS	Stress-strain onstitutive matrices for shear actions stacked in layer by layer sequence, as further described in §5.6.
layers	A list that specifies layer thicknesses, as described in §5.2ff.
options	Optional inputs, described later.

The module returns the 24×24 element stiffness matrix Ke in global coordinates.

§32.4.3. Calculation Steps

The top-level calculations steps schematized in Figure 9 are as follows.

Localization. This is done by module SS8LocalSystem, which was described in §2.3.

Local Stiffness Computation. Carried out by SS8StiffnessMBT, which returns the stiffness $\mathbf{K}_{mbt}^{(e)}$ due to membrane, bending and thickness (MBT) effects, and SS8StiffnessShear, which returns the stiffness $\mathbf{K}_s^{(e)}$ due to transverse shear effects. The two stiffnesses are added to form the local stiffness matrix as per (32.30).

Both sets of stiffness computations are carried out in the flattened brick geometry and are described in Sections 6 and 7, respectively.

Transformation to Warped Geometry. The stiffness matrix $\mathbf{K}_{mbts}^{(e)}$ of the flattened brick is transformed to the actual warped geometry by the congruential transformation

$$\mathbf{K}_w^{(e)} = \mathbf{T}_h^T \mathbf{K}_{mbts}^{(e)} \mathbf{T}_h, \quad (32.32)$$

where $\mathbf{T}_h$ is the transformation matrix introduced in Section 3.3.

This operation is implemented in module SS8TransformToWarped, which is listed in Figure 11. The implementation takes advantage of the sparse nature of matrix $\mathbf{T}_h$ to reduce the volume of operations.

```
SS8TransformToWarped[Ke0_, z_] := Module[{h1, h2, h3, h4,
  z1, z2, z3, z4, z5, z6, z7, z8, a, b, c1, c2, c3, c4, d1, d2, d3, d4,
  cc13, cc24, cc32, cd14, cd23, cd24, cd31, cd32, cd41, dd14, dd24,
  dd31, i, j, n, m, ni, nij, mj, mjj, K11, K12, K21, K22, Ke},
  {z1, z2, z3, z4, z5, z6, z7, z8} = z;
  {h1, h2, h3, h4} = {z5 - z1, z6 - z2, z7 - z3, z8 - z4};
  a = 1/2 + {z5/h1, z6/h2, z7/h3, z8/h4, -z1/h1, -z2/h2, -z3/h3, -z4/h4};
  b = -1/2 + {z5/h1, z6/h2, z7/h3, z8/h4, -z1/h1, -z2/h2, -z3/h3, -z4/h4};
  a = Simplify[a]; b = Simplify[b]; Ke = Table[0, {24}, {24}];
  For [n = 1, n <= 4, n++, For [m = 1, m <= 4, m++,
    c1 = a[[n]]; c2 = a[[n + 4]]; c3 = a[[m]]; c4 = a[[m + 4]];
    d1 = b[[n]]; d2 = b[[n + 4]]; d3 = b[[m]]; d4 = b[[m + 4]];
    cc13 = c1 * c3; cc24 = c2 * c4; cc32 = c3 * c2; cd14 = c1 * d4;
    cd24 = c2 * d4; cd23 = c2 * d3; cd31 = c3 * d1; cd32 = c3 * d2;
    cd41 = c4 * d1; dd14 = d1 * d4; dd24 = d2 * d4; dd31 = d3 * d1;
    For [i = 1, i <= 3, i++, ni = 3 * n - 3 + i; nij = ni + 12;
      For [j = 1, j <= 3, j++, mj = 3 * m - 3 + j; mjj = mj + 12;
        K11 = Ke0[[ni, mj]]; K12 = Ke0[[ni, mjj]];
        K21 = Ke0[[nij, mj]]; K22 = Ke0[[nij, mjj]];
        Ke[[ni, mj]] = cc13 * K11 + cd14 * K12 + cd32 * K21 + dd24 * K22;
        Ke[[nij, mj]] = cd31 * K11 + dd14 * K12 + cc32 * K21 + cd24 * K22;
        Ke[[mj, nij]] = Ke[[nij, mj]];
        Ke[[nij, mjj]] = dd31 * K11 + cd41 * K12 + cd23 * K21 + cc24 * K22;
      ]];
    ]];
  Return[Ke];
];
```

Figure 11. Module to transform the flattened brick stiffness to the warped geometry.

The module is referenced as

$$\mathbf{K}_w = \text{SS8TransformToWarped}[\mathbf{K}_{mbts}, \mathbf{z}_c] \quad (32.33)$$

where the arguments are

$\mathbf{K}_{mbts}$ The flat brick stiffness matrix obtained from the addition (32.30).

zc The local z coordinates $\{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$ of the warped brick corners arranged as list $\{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$. This list is available as third item of array xyz returned by SS8LocalSystem, as described in §2.3.

The module returns

Kw The 24×24 stiffness matrix of the warped brick in local coordinates.

```
SS8TransformToGlobal[Ke0_,Td_]:=Module[{
  st1=st2=st3=Table[0,{24}],
  t11,t12,t13,t21,t22,t23,t31,t32,t33,
  tst11,tst12,tst13,tst21,tst22,tst23,
  tst31,tst32,tst33,i,j,ib,jb, Ke=Ke0},
  {{t11,t12,t13},{t21,t22,t23},{t31,t32,t33}}=Td;
  Print[{{t11,t12,t13},{t21,t22,t23},{t31,t32,t33}}];
  For[jb=1,jb<=8,jb++,j=3*(jb-1);
    For[i=1,i<=24,i++,
      st1[[i]]=Ke[[i,j+1]]*t11+Ke[[i,j+2]]*t21+Ke[[i,j+3]]*t31;
      st2[[i]]=Ke[[i,j+1]]*t12+Ke[[i,j+2]]*t22+Ke[[i,j+3]]*t32;
      st3[[i]]=Ke[[i,j+1]]*t13+Ke[[i,j+2]]*t23+Ke[[i,j+3]]*t33;
      For[ib=1,ib<=jb,ib++,i=3*(ib-1);
        tst11=t11*st1[[i+1]]+t21*st1[[i+2]]+t31*st1[[i+3]];
        tst21=t12*st1[[i+1]]+t22*st1[[i+2]]+t32*st1[[i+3]];
        tst31=t13*st1[[i+1]]+t23*st1[[i+2]]+t33*st1[[i+3]];
        tst12=t11*st2[[i+1]]+t21*st2[[i+2]]+t31*st2[[i+3]];
        tst22=t12*st2[[i+1]]+t22*st2[[i+2]]+t32*st2[[i+3]];
        tst32=t13*st2[[i+1]]+t23*st2[[i+2]]+t33*st2[[i+3]];
        tst13=t11*st3[[i+1]]+t21*st3[[i+2]]+t31*st3[[i+3]];
        tst23=t12*st3[[i+1]]+t22*st3[[i+2]]+t32*st3[[i+3]];
        tst33=t13*st3[[i+1]]+t23*st3[[i+2]]+t33*st3[[i+3]];
        Ke[[i+1,j+1]]=tst11; Ke[[j+1,i+1]]=tst11;
        Ke[[i+1,j+2]]=tst12; Ke[[j+2,i+1]]=tst12;
        Ke[[i+1,j+3]]=tst13; Ke[[j+3,i+1]]=tst13;
        Ke[[i+2,j+1]]=tst21; Ke[[j+1,i+2]]=tst21;
        Ke[[i+2,j+2]]=tst22; Ke[[j+2,i+2]]=tst22;
        Ke[[i+2,j+3]]=tst23; Ke[[j+3,i+2]]=tst23;
        Ke[[i+3,j+1]]=tst31; Ke[[j+1,i+3]]=tst31;
        Ke[[i+3,j+2]]=tst32; Ke[[j+2,i+3]]=tst32;
        Ke[[i+3,j+3]]=tst33; Ke[[j+3,i+3]]=tst33;
      ];
    ];
  Return[Ke];
```

Figure 12. Module to transform the warped brick stiffness to global coordinates.

Transformation to Global Geometry. The stiffness matrix $\mathbf{K}_w^{(e)}$ produced by (32.32) is still in local coordinates. The final congruential transformation refers it to global $\{X, Y, Z\}$ coordinates:

$$\mathbf{K}^{(e)} = \mathbf{T}_d^T \mathbf{K}_w^{(e)} \mathbf{T}_d, \quad (32.34)$$

where $\mathbf{T}_d^{(e)}$ is a 24×24 block diagonal matrix built from the 3×3 matrix of direction cosines as discussed in §3.2.

This operation is implemented in module SS8TransformToGlobal, which is listed in Figure 12. The implementation takes advantage of the block-diagonal nature of matrix $\mathbf{T}_d^{(e)}$ to reduce the volume of operations.

The module is referenced as

$$\mathbf{K} = \text{SS8TransformToGlobal}[\mathbf{K}_w, \mathbf{T}_d] \quad (32.35)$$

where the arguments are

- K_w The warped brick stiffness matrix returned by `SS8TransformToWarped`.
 T_d The 3×3 direction cosine matrix returned by `SS8LocalSystem`.

The module returns

- K_e The 24×24 stiffness matrix of the warped brick in global coordinates.

§32.5. Constitutive Properties

This section describes the method by which a layered wall configuration is treated by through-the-thickness integration. Readers not interested in these details may skip directly to Section 6.

§32.5.1. Laminate Fabrication

The SS8 element is permitted to have a laminate composite fabrication in the thickness (z or ζ) direction. If this is the case, the development of thickness-integrated constitutive equations is tricky. The complication is due to the presence of the *thickness strain*, which is an effect not present in ordinary plate and shell elements.

There are only two node layers to capture across-the-thickness deformations. It follows that only an average thickness strain $\bar{e}_{zz}$ can be sensed. But if $\bar{e}_{zz}$ is inserted in the constitutive equations of a laminated composite, the thickness stress σ_{zz} will jump between layers. This jump would grossly violate traction continuity conditions at the layer interfaces. The contradiction leads to an effect called *thickness locking*, with consequent wrong stress predictions.⁶

Note that thickness locking is not present in conventional thin plate and shell elements because for these it is assumed from the start that $\sigma_{zz} \approx 0$, which allows the use of plane stress constitutive relations in each layer.

The cure for thickness locking is in principle easy: assume a uniform thickness stress $\sigma_{zz} = \bar{\sigma}_{zz}$ instead of a uniform strain $e_{zz} = \bar{e}_{zz}$. This is in fact automatic when one derives the element by hybrid stress methods [22]. But for such elements one must assume all stress components, and one faces the opposite dilemma: the inplane stress components must be allowed to jump between layers. This is exceedingly difficult to achieve. One comes to the conclusion that the best solution is a mixture of strain and stress assumptions:

- Assume inplane strains $\{e_{xx}, e_{yy}, e_{xy}\}$ varying linearly across the thickness
- Assume a normal stress $\bar{\sigma}_{zz}$ uniform across the thickness

The two assumptions can be combined through a careful derivation of integrated stress-strain relations, as carried out below. Note that transverse shear strain effects do not come in the foregoing assumption because of the decoupling assumption made in §4.1.

§32.5.2. Wall Fabrication Assumptions

The SS8 element may have an arbitrary number of layers in the thickness direction. The properties of each layer in the inplane directions are taken to be uniform.

Layers are separated by constant- ζ lines, as illustrated in Figure 13. This assumption greatly simplifies the integration of the constitutive relations across the thickness. For variable thickness the separation assumption is not very realistic. Fortunately plates and shells fabricated as laminate composites frequently have uniform thickness because of fabrication constraints.

⁶ The effect has been called also Poisson's locking by some authors.

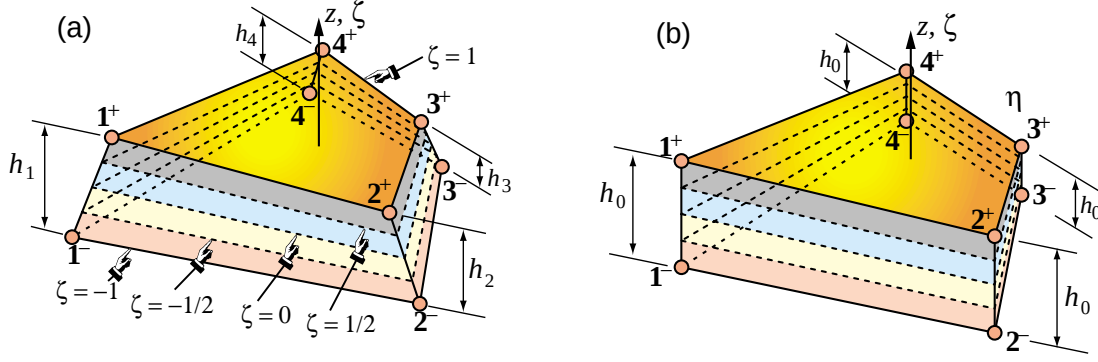


Figure 13. Wall fabrication assumption: layers are assumed to be separated by surfaces of constant ζ . Both figures illustrates the case of four equal thickness layers, separated at $\zeta = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. If the element has variable thickness as illustrated in (a), these are not surfaces of constant z . In the case of constant thickness illustrated in (b), the separation also occurs at constant z .

§32.5.3. MBT Constitutive Equations

The laminated composite has n_L layers identified by index $k = 1, \dots, n_L$. Layers are numbered from the bottom surface $\zeta = -1$ up; thus $k = 1$ is the bottom layer.

The point of departure are the transverse-shear-uncoupled stress-strain relations (32.29) for each k . From these we extract the relation for the inplane and thickness stresses and strains:

$$\begin{bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \sigma_{zz}^k \\ \sigma_{xy}^k \end{bmatrix} = \begin{bmatrix} E_{11}^k & E_{12}^k & E_{13}^k & E_{14}^k \\ E_{12}^k & E_{22}^k & E_{23}^k & E_{24}^k \\ E_{13}^k & E_{23}^k & E_{33}^k & E_{34}^k \\ E_{14}^k & E_{24}^k & E_{34}^k & E_{44}^k \end{bmatrix} \begin{bmatrix} e_{xx}^k \\ e_{yy}^k \\ e_{zz}^k \\ 2e_{xy}^k \end{bmatrix} \quad (32.36)$$

which for an isotropic material of elastic modulus E^k and Poisson's ratio ν^k reduce to

$$\begin{bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \sigma_{zz}^k \\ \sigma_{xy}^k \end{bmatrix} = \frac{E^k}{(1 - 2\nu^k)(1 + \nu^k)} \begin{bmatrix} 1 - \nu^k & \nu^k & \nu^k & 0 \\ \nu^k & 1 - \nu^k & \nu^k & 0 \\ \nu^k & \nu^k & 1 - \nu^k & 0 \\ 0 & 0 & 0 & \frac{1}{2} - \nu^k \end{bmatrix} \begin{bmatrix} e_{xx}^k \\ e_{yy}^k \\ e_{zz}^k \\ 2e_{xy}^k \end{bmatrix} \quad (32.37)$$

For subsequent manipulations this relation is rearranged and partitioned as follows

$$\begin{bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \sigma_{zz}^k \\ \sigma_{xy}^k \end{bmatrix} = \begin{bmatrix} E_{11}^k & E_{12}^k & E_{14}^k & \vdots & E_{13}^k \\ E_{12}^k & E_{22}^k & E_{24}^k & \vdots & E_{23}^k \\ E_{14}^k & E_{24}^k & E_{44}^k & \vdots & E_{34}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{13}^k & E_{23}^k & E_{34}^k & \vdots & E_{33}^k \end{bmatrix} \begin{bmatrix} e_{xx}^k \\ e_{yy}^k \\ 2e_{xy}^k \\ \vdots \\ e_{zz}^k \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{\equiv}^k & \mathbf{E}_{\times}^k \\ (\mathbf{E}_{\times}^k)^T & E_{zz}^k \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\equiv}^k \\ e_{zz}^k \end{bmatrix}. \quad (32.38)$$

where subscripts $\equiv, zz$ and $\times$ denote in-plane, transverse and coupled in-plane-transverse directions, respectively. The inverse relation is

$$\begin{bmatrix} \mathbf{e}_{\equiv}^k \\ e_{zz}^k \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{\equiv}^k & \mathbf{E}_{\times}^k \\ (\mathbf{E}_{\times}^k)^T & E_{zz}^k \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{\equiv}^k \\ \sigma_{zz}^k \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\equiv}^k & \mathbf{C}_{\times}^k \\ (\mathbf{C}_{\times}^k)^T & C_{zz}^k \end{bmatrix} \begin{bmatrix} \sigma_{\equiv}^k \\ \sigma_{zz}^k \end{bmatrix}. \quad (32.39)$$

This relation is exhibited only to establish notation since the inversion need not be done explicitly as shown later. Partial inversion yields

$$\begin{bmatrix} \sigma_{\equiv}^k \\ e_{zz}^k \end{bmatrix} = \begin{bmatrix} (\mathbf{C}_{\equiv}^k)^{-1} & -(\mathbf{C}_{\equiv}^k)^{-1} \mathbf{C}_{\times}^k \\ (\mathbf{C}_{\times}^k)^T (\mathbf{C}_{\equiv}^k)^{-1} & \mathbf{C}_{zz}^k - (\mathbf{C}_{\times}^k)^T (\mathbf{C}_{\equiv}^k)^{-1} \mathbf{C}_{\times}^k \end{bmatrix} \begin{bmatrix} e_{\equiv}^k \\ \sigma_{zz}^k \end{bmatrix} = \begin{bmatrix} \mathbf{S}^k & \mathbf{D}^k \\ -(\mathbf{D}^k)^T & \mathbf{R}^k \end{bmatrix} \begin{bmatrix} e_{\equiv}^k \\ \sigma_{zz}^k \end{bmatrix}. \quad (32.40)$$

§32.5.4. MBT Thickness Integration

Because $\mathbf{e}_{\equiv} = \mathbf{e}_m + \zeta \mathbf{e}_b$ varies linearly in ζ and $\sigma_{zz} = \sigma_{zz}$ is constant along ζ , it is feasible to integrate (32.40) along ζ while taking $\mathbf{e}_m$, $\mathbf{e}_b$ and $\bar{\sigma}_{zz}$ out of the integral. Integration of $\sigma_{\equiv}$ and $\zeta \sigma_{\equiv}$ provided by the first matrix equation gives effective membrane and bending stresses:

$$\begin{aligned} \begin{bmatrix} \bar{\sigma}_m \\ \bar{\sigma}_b \end{bmatrix} &= \frac{1}{2} \int_{-1}^1 \begin{bmatrix} \sigma_m \\ \zeta \sigma_b \end{bmatrix} d\zeta = \frac{1}{2} \int_{-1}^1 \begin{bmatrix} \mathbf{S}^k & \mathbf{D}^k \\ \zeta \mathbf{S}^k & \zeta \mathbf{D}^k \end{bmatrix} \begin{bmatrix} \mathbf{e}_m + \zeta \mathbf{e}_b \\ \sigma_{zz} \end{bmatrix} d\zeta \\ &= \frac{1}{2} \int_{-1}^1 \begin{bmatrix} \mathbf{S}^k & \zeta \mathbf{S}^k & \mathbf{D}^k \\ \zeta \mathbf{S}^k & \zeta^2 \mathbf{S}^k & \zeta \mathbf{D}^k \end{bmatrix} d\zeta \begin{bmatrix} \mathbf{e}_m \\ \mathbf{e}_b \\ \sigma_{zz} \end{bmatrix}. \end{aligned} \quad (32.41)$$

Note that although $\bar{\sigma}_m$ and $\bar{\sigma}_b$ are not the stress resultants of conventional plates, they have a similar meaning. Integrating the second matrix equation one obtains the average thickness :

$$\begin{aligned} \bar{e}_z &= \frac{1}{2} \int_{-1}^1 e_{zz} d\zeta = \frac{1}{2} \int_{-1}^1 [-(\mathbf{D}^k)^T \quad \mathbf{R}^k] \begin{bmatrix} \mathbf{e}_p \\ \sigma_{zz} \end{bmatrix} d\zeta = \frac{1}{2} \int_{-1}^1 [-(\mathbf{D}^k)^T \quad \mathbf{R}^k] \begin{bmatrix} \mathbf{e}_m + \zeta \mathbf{e}_b \\ \sigma_{zz} \end{bmatrix} d\zeta \\ &= \frac{1}{2} \int_{-1}^1 [-(\mathbf{D}^k)^T \quad -\zeta (\mathbf{D}^k)^T \quad \mathbf{R}^k] d\zeta \begin{bmatrix} \mathbf{e}_m \\ \mathbf{e}_b \\ \sigma_{zz} \end{bmatrix}. \end{aligned} \quad (32.42)$$

Introduce here the integrated constitutive matrices

$$\begin{bmatrix} \mathbf{S}_0 & \mathbf{S}_1 & \mathbf{D}_0 \\ \mathbf{S}_1 & \mathbf{S}_2 & \mathbf{D}_1 \\ -\mathbf{D}_0^T & -(\mathbf{D}_1)^T & \mathbf{R}_0 \end{bmatrix} = \frac{1}{2} \int_{-1}^1 \begin{bmatrix} \mathbf{S}^k & \zeta \mathbf{S}^k & \mathbf{D}^k \\ \zeta \mathbf{S}^k & \zeta^2 \mathbf{S}^k & \mathbf{D}^k \\ -(\mathbf{D}^k)^T & -\zeta (\mathbf{D}^k)^T & \mathbf{R}^k \end{bmatrix} d\zeta. \quad (32.43)$$

Then collecting (32.41) and (32.42) into one matrix relation and passing $\bar{e}_{zz}$ back to the right hand side gives

$$\begin{bmatrix} \bar{\sigma}_m \\ \bar{\sigma}_b \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_0 + \mathbf{D}_0 \mathbf{D}_0^T / \mathbf{R}_0 & \mathbf{S}_1 + \mathbf{D}_0 \mathbf{D}_1^T / \mathbf{R}_0 & \mathbf{D}_0 / \mathbf{R}_0 \\ \mathbf{S}_1 + \mathbf{D}_1 \mathbf{D}_0^T / \mathbf{R}_0 & \mathbf{S}_2 + \mathbf{D}_1 \mathbf{D}_1^T / \mathbf{R}_0 & \mathbf{D}_1 / \mathbf{R}_0 \\ \mathbf{D}_0^T / \mathbf{R}_0 & \mathbf{D}_1 / \mathbf{R}_0 & 1 / \mathbf{R}_0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_m \\ \mathbf{e}_b \\ \bar{e}_{zz} \end{bmatrix} = \bar{\mathbf{E}}_{mbt} \begin{bmatrix} \mathbf{e}_m \\ \mathbf{e}_b \\ \bar{e}_{zz} \end{bmatrix}, \quad (32.44)$$

The 4×4 constitutive matrix $\bar{\mathbf{E}}_{mbt}$ accounts for integrated membrane, bending and thickness effects, and can be used directly in that component of the stiffness matrix for an arbitrary layer configuration that meets the bounding assumptions of §5.2.

§32.5.5. Implementation of MBT Integration

For the computer implementation of the calculation of $\mathbf{E}_{mbt}$ the following simplification of (32.40) should be noted:

$$\begin{aligned} \mathbf{S}^k &= \begin{bmatrix} E_{11}^k - E_{13}^k E_{13}^k / E_{33}^k & E_{12}^k - E_{13}^k E_{23}^k / E_{33}^k & E_{14}^k - E_{13}^k E_{34}^k / E_{33}^k \\ E_{12}^k - E_{13}^k E_{23}^k / E_{33}^k & E_{22}^k - E_{23}^k E_{23}^k / E_{33}^k & E_{24}^k - E_{23}^k E_{34}^k / E_{33}^k \\ E_{14}^k - E_{13}^k E_{34}^k / E_{33}^k & E_{24}^k - E_{23}^k E_{34}^k / E_{33}^k & E_{44}^k - E_{34}^k E_{34}^k / E_{33}^k \end{bmatrix} \\ \mathbf{D}^k &= [E_{13}^k \quad E_{23}^k \quad E_{34}^k] / E_{33}^k, \quad \mathbf{R}^k = 1 / E_{33}^k \end{aligned} \quad (32.45)$$

Using these simplifications the results for a one layer with

$$\mathbf{E}^1 = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & E_{34} \\ E_{14} & E_{24} & E_{34} & E_{44} \end{bmatrix} \quad (32.46)$$

is easily worked out to be

$$\bar{\mathbf{E}}_{bmt} = \begin{bmatrix} E_{11} & E_{12} & E_{14} & 0 & 0 & 0 & E_{13} \\ E_{12} & E_{22} & E_{24} & 0 & 0 & 0 & E_{23} \\ E_{14} & E_{24} & E_{44} & 0 & 0 & 0 & E_{34} \\ 0 & 0 & 0 & \frac{1}{3}(E_{11} - \frac{E_{13}E_{13}}{E_{33}}) & \frac{1}{3}(E_{12} - \frac{E_{13}E_{23}}{E_{33}}) & \frac{1}{3}(E_{14} - \frac{E_{13}E_{34}}{E_{33}}) & 0 \\ 0 & 0 & 0 & \frac{1}{3}(E_{12} - \frac{E_{13}E_{23}}{E_{33}}) & \frac{1}{3}(E_{22} - \frac{E_{23}E_{23}}{E_{33}}) & \frac{1}{3}(E_{24} - \frac{E_{23}E_{34}}{E_{33}}) & 0 \\ 0 & 0 & 0 & \frac{1}{3}(E_{14} - \frac{E_{13}E_{34}}{E_{33}}) & \frac{1}{3}(E_{24} - \frac{E_{23}E_{34}}{E_{33}}) & \frac{1}{3}(E_{44} - \frac{E_{34}E_{34}}{E_{33}}) & 0 \\ E_{13} & E_{23} & E_{34} & 0 & 0 & 0 & E_{33} \end{bmatrix} \quad (32.47)$$

If the one-layer wall is isotropic with modulus E and Poisson's ratio ν , (32.47) reduces to

$$\bar{\mathbf{E}}_{bmt} = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 & 0 & 0 & \nu \\ \nu & 1-\nu & 0 & 0 & 0 & 0 & \nu \\ 0 & 0 & \frac{1}{2}-\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)(1+\nu)}{3(1-\nu^2)} & \frac{\nu(1-2\nu)(1+\nu)}{3(1-\nu^2)} & \frac{\nu(1-2\nu)(1+\nu)}{3(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{\nu(1-2\nu)(1+\nu)}{3(1-\nu^2)} & \frac{(1-2\nu)(1+\nu)}{3(1-\nu^2)} & \frac{\nu(1-2\nu)(1+\nu)}{3(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{\nu(1-2\nu)(1+\nu)}{3(1-\nu^2)} & \frac{\nu(1-2\nu)(1+\nu)}{3(1-\nu^2)} & \frac{(1-2\nu)(1+\nu)}{1-\nu^2} & 0 \\ \nu & \nu & 0 & 0 & 0 & 0 & 1-\nu \end{bmatrix} \quad (32.48)$$

A *Mathematica* implementation that incorporates this simplification is shown in Figure 14. The module is referenced as

$$\text{Embt} = \text{SS8WallIntegMBT}[\text{ElayerMBT}, \text{layers}] \quad (32.49)$$

ElayerMBT A list of 4×4 layer constitutive matrices (32.36), ordered from bottom to top.

layers is a list of scaled layer thicknesses adding up to one.

The module returns the integrated constitutive matrix $\bar{\mathbf{E}}_{bmt}$ stored in **Embt**.

To give an example, suppose that the total wall thickness is h fabricated with three layers whose thicknesses are $1/2$, $1/3$ and $1/6$ of the total thickness, respectively. The constitutive matrices are taken to be

$$\mathbf{E}^k = \begin{bmatrix} 2k & 1 & 1 & 1 \\ 1 & 2k & 1 & 1 \\ 1 & 1 & 2k & 1 \\ 1 & 1 & 1 & k \end{bmatrix}, \quad k = 1, 2, 3. \quad (32.50)$$

[This is admittedly a contrived example, only used as benchmark.] The integrated matrix (32.44) is produced by the *Mathematica* statements

```
ElayerMBT1={{2,1,1,1},{1,2,1,1},{1,1,2,1},{1,1,1,1}};
ElayerMBT2={{4,1,1,1},{1,4,1,1},{1,1,4,1},{1,1,1,2}};
```

```

SS8ThickIntegMBT[ElayerMBT_, layers_] :=
Module[{nlayers=Length[ElayerMBT], E11, E12, E13, E14, E22, E23, E24,
E33, E34, E44, Ek, Sk, Dk, Rk, S0, S1, S2, D0, D1, R0, R, k, hlay,
gbot, gtop, dg, gavg, Embt=Table[0, {7}, {7}] },
S0=S1=S2=Table[0, {3}, {3}]; D0=D1=Table[0, {3}]; R=0;
gbot=gtop=-1;
For [k=1, k<=nlayers, k++, hlay=layers[[k]];
gtop=gbot+2*hlay; dg=(gtop-gbot)/2; gavg=(gtop+gbot)/2;
Ek=ElayerMBT[[k]];
{{E11, E12, E13, E14}, {E12, E22, E23, E24},
{E13, E23, E33, E34}, {E14, E24, E34, E44}}=Ek;
Sk={{E11-E13*E13/E33, E12-E13*E23/E33, E14-E13*E34/E33},
{E12-E13*E23/E33, E22-E23*E23/E33, E24-E23*E34/E33},
{E14-E13*E34/E33, E24-E23*E34/E33, E44-E34*E34/E33}};
Sk=Simplify[Sk];
Dk={E13, E23, E34}/E33; Rk=1/E33;
S0=S0+dg*Sk; S1=S1+dg*gavg*Sk;
S2=S2+(gtop^2+gtop*gbot+gbot^2)*dg*Sk/3;
D0=D0+dg*Dk; D1=D1+dg*gavg*Dk; R=R+dg*Rk;
gbot=gtop
]; {S0, S1, D0, D1, R}=Simplify[ {S0, S1, D0, D1, R}];
S0=S0+(Transpose[{D0}].{D0})/R;
S1=S1+(Transpose[{D0}].{D1})/R;
S2=S2+(Transpose[{D1}].{D1})/R;
D0=D0/R; D1=D1/R; R0=1/R;
Embt=
{{S0[[1,1]], S0[[1,2]], S0[[1,3]], S1[[1,1]], S1[[1,2]], S1[[1,3]], D0[[1]]},
{S0[[2,1]], S0[[2,2]], S0[[2,3]], S1[[2,1]], S1[[2,2]], S1[[2,3]], D0[[2]]},
{S0[[3,1]], S0[[3,2]], S0[[3,3]], S1[[3,1]], S1[[3,2]], S1[[3,3]], D0[[3]]},
{S1[[1,1]], S1[[2,1]], S1[[3,1]], S2[[1,1]], S2[[1,2]], S2[[1,3]], D1[[1]]},
{S1[[1,2]], S1[[2,2]], S1[[3,2]], S2[[2,1]], S2[[2,2]], S2[[2,3]], D1[[2]]},
{S1[[1,3]], S1[[2,3]], S1[[3,3]], S2[[3,1]], S2[[3,2]], S2[[3,3]], D1[[3]]},
{D0[[1]], D0[[2]], D0[[3]], D1[[1]], D1[[2]], D1[[3]], R0 }};
Return[Simplify[Embt]];

```

Figure 14. *Mathematica* implementation of through-the-thickness integration process for the MBT constitutive equations.

```

ElayerMBT3={{6,1,1,1},{1,6,1,1},{1,1,6,1},{1,1,1,3}};
Embt=SS8ThickIntegMBT[{ElayerMBT1,ElayerMBT2,ElayerMBT3},{1/2,1/3,1/6}];
Print["Embt=",Embt//MatrixForm];

```

The result is

$$\bar{\mathbf{E}}_{mbt} = \frac{1}{6318} \begin{bmatrix} 21060 & 6318 & 6318 & 4914 & 0 & 0 & 6318 \\ 6318 & 21060 & 6318 & 0 & 4914 & 0 & 6318 \\ 6318 & 6318 & 10530 & 0 & 0 & 2457 & 6318 \\ 4914 & 0 & 0 & 7168 & 1474 & 1474 & -1296 \\ 0 & 4914 & 0 & 1474 & 7168 & 1474 & -1296 \\ 0 & 0 & 2457 & 1474 & 1474 & 3268 & -1296 \\ 6318 & 6318 & 6318 & -1296 & -1296 & -1296 & 17496 \end{bmatrix}. \quad (32.51)$$

Note that for unsymmetric wall fabrication, as in this example, the integrated matrix is generally full. Two restrictions on the constitutive equations of a layer should be noted:

- If the material of the k^{th} layer is exactly incompressible, $R^k = 0$ and division by zero will occur. In the isotropic case, this is obvious from a glance at (32.37) since the denominator vanishes if $\nu = 1/2$. Hence no layer can be incompressible.
- A “void layer”, as in a sandwich construction, cannot be accommodated by setting $\mathbf{E}^k = \mathbf{0}$ since division by $E_{33}^k = 0$ will occur. To represent that case scale a nonzero $\mathbf{E}^k$ by a small number.

```

SS8ThickIntegS[ElayerS_, layers_]:=
Module[{nlayers=Length[ElayerS], Ek, k, zbot, ztop, dz,
Es=Table[0, {2}, {2}] },
zbot=ztop=-1;
For[k=1, k<=nlayers, k++,
ztop=zbot+2*layers[[k]]; dz=(ztop-zbot)/2;
Es+=dz*ElayerS[[k]]; zbot=ztop;
];
Return[Simplify[Es]];

```

Figure 15. *Mathematica* implementation of through-the-thickness integration process for transverse shear constitutive equations.

These two restrictions should be contrasted with conventional plates and shells, in which both incompressible and zero-stiffness layers are acceptable on account of *a priori* enforcement of plane stress conditions. The difference is that consideration of the thickness strain introduces three-dimensional effects that cannot be circumvented if two node layers are kept.

§32.5.6. Transverse Shear Constitutive Equations

The corresponding process for the transverse shear energy portion is much simpler. The transverse strains e_{xz} and e_{yz} are assumed to be uniform through the thickness, and the constitutive cross-coupling with the in-plane and thickness effects is ignored, as stated by (32.29).

Without going into the derivation details, the integrated constitutive relation to be used in the transverse shear energy is simply

$$\bar{\mathbf{E}}_s = \int_{-1}^1 \begin{bmatrix} E_{55}^k & E_{56}^k \\ E_{66}^k & E_{66}^k \end{bmatrix} d\zeta. \quad (32.52)$$

An implementation of the integration process in *Mathematica* is shown in Figure 15.

The module is referenced as

$$\mathbf{E}_s = \text{SS8WallIntegS}[\text{ElayerS}, \text{layers}] \quad (32.53)$$

The arguments are

ElayerS	A list of 2×2 layer constitutive matrices relating transverse shear stresses to transverse shear strains, as per the lower 2×2 block of (32.29).
layers	Same as in SS8WallIntegMBT.

The module returns the integrated constitutive matrix $\bar{\mathbf{E}}_s$ stored in $\mathbf{E}_s$ as function result.

§32.6. Membrane, Bending and Thickness (MBT) Stiffness

In this and the next section we cover the element stiffness computations that form the lower level schematized in Figure 16.

In solid shell elements, the contributions of inplane effects: membrane and bending, are coupled to the thickness effects (deformations in the z direction) through the constitutive equations. They are collectively abbreviated to MBT effects. They are decoupled from transverse shear (TS) effects by *a priori* assumptions on the constitutive equations as discussed in §4.1. Consequently they can be separately considered and their contributions to the stiffness matrix added.

The contribution of MBT effects to the element stiffness is handled by the assumed strain method. Initial attempts to use the assumed natural strain method developed by other authors [2,8,24,25,26] led

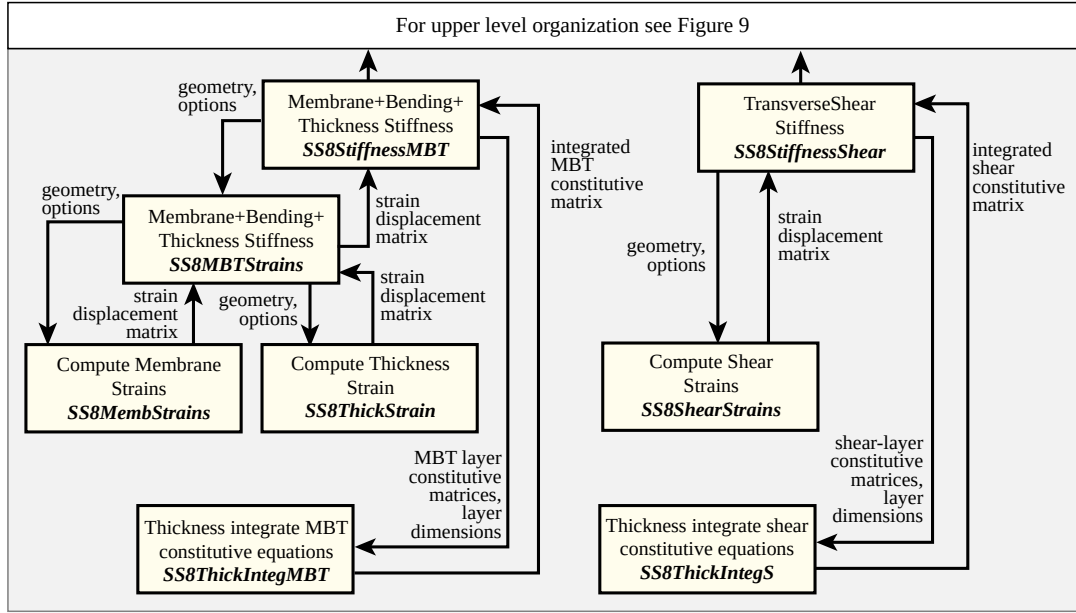


Figure 16. The lower level of the SS8 stiffness calculations.

to element with high sensitivity to inplane FE distortion. A related but different method [12] was then developed over four months of experimentation with *Mathematica*. The performance of the element under distortion improved with this new field.

For flat plate elements of constant thickness, the new field can be shown to be optimal in a variational sense; that is, it cannot be improved without violating certain orthogonality and patch test satisfaction conditions. The construction of the strain field is first described with reference to a flat 4-node quadrilateral, and later transported to the SS8 element.

§32.6.1. Four-Noded Quadrilateral Geometry

We consider the quadrilateral obtained by cutting the SS8 flattened-brick geometry with the surface $\zeta = 0$, as shown in Figure 17(a). This quadrilateral is flat. Consequently it will be defined by the coordinates of the four corners: $\{x_i^Q, y_i^Q\}$, $i = 1, 2, 3, 4$. These are functions of ζ .

We shall employ the abbreviations $x_{ij}^Q = x_i^Q - x_j^Q$ and $y_{ij}^Q = y_i^Q - y_j^Q$. The coordinates of the quadrilateral center $\{\xi = 0, \eta = 0\}$, denoted by 0 in Figure x.1(a), are

$$x_0^Q = \frac{1}{4}(x_1^Q + x_2^Q + x_3^Q + x_4^Q), \quad y_0^Q = \frac{1}{4}(y_1^Q + y_2^Q + y_3^Q + y_4^Q). \quad (32.54)$$

This point lies at the intersection of the medians. Denote by A_{ijk}^Q the signed area of the triangle spanned by the corners $\{i, j, k\}$, and A is the quadrilateral area. We have

$$\begin{aligned} A_{123}^Q &= \frac{1}{2}(x_{13}^Q y_{21}^Q - x_{21}^Q y_{13}^Q), & A_{234}^Q &= \frac{1}{2}(x_{24}^Q y_{32}^Q - x_{32}^Q y_{24}^Q), & A_{341}^Q &= \frac{1}{2}(x_{31}^Q y_{43}^Q - x_{43}^Q y_{31}^Q), \\ A_{412}^Q &= \frac{1}{2}(x_{42}^Q y_{14}^Q - x_{14}^Q y_{42}^Q), & A^Q &= \frac{1}{2}(x_{31}^Q y_{42}^Q - x_{42}^Q y_{31}^Q) = A_{123}^Q + A_{341}^Q = A_{234}^Q + A_{412}^Q. \end{aligned} \quad (32.55)$$

The following identities are readily verified algebraically:

$$\begin{bmatrix} x_{14}^Q & x_{42}^Q & x_{34}^Q & 0 \\ x_{13}^Q & x_{32}^Q & 0 & x_{34}^Q \\ x_{12}^Q & 0 & x_{32}^Q & x_{24}^Q \\ 0 & x_{12}^Q & x_{31}^Q & x_{14}^Q \end{bmatrix} \begin{bmatrix} A_{234}^Q \\ A_{341}^Q \\ A_{412}^Q \\ A_{123}^Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y_{14}^Q & y_{42}^Q & y_{34}^Q & 0 \\ y_{13}^Q & y_{32}^Q & 0 & y_{34}^Q \\ y_{12}^Q & 0 & y_{32}^Q & y_{24}^Q \\ 0 & y_{12}^Q & y_{31}^Q & y_{14}^Q \end{bmatrix} \begin{bmatrix} A_{234}^Q \\ A_{341}^Q \\ A_{412}^Q \\ A_{123}^Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (32.56)$$

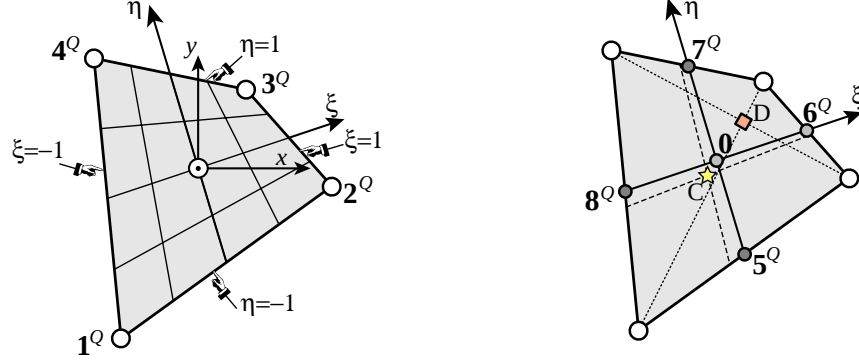


Figure 17. (a) four-node quadrilateral geometry and natural coordinates $\{\xi, \eta\}$. (b) Some special points: 5,6,7,8,0 designate midpoints and center, respectively; C is the centroid and D the intersection of diagonals.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1^Q & x_2^Q & x_3^Q & x_4^Q \\ y_1^Q & y_2^Q & y_3^Q & y_4^Q \end{bmatrix} \begin{bmatrix} -A_{234}^Q \\ A_{341}^Q \\ -A_{412}^Q \\ A_{123}^Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (32.57)$$

The Jacobian determinant of the transformation between $\{x, y\}$ and $\{\xi, \eta\}$ is

$$J^Q = \left| \frac{\partial \{x, y\}}{\partial \{\xi, \eta\}} \right| = \frac{1}{8} (J_0^Q + J_1^Q \xi + J_2^Q \eta). \quad (32.58)$$

where

$$J_0^Q = 2A^Q, \quad J_1^Q = x_{34}^Q y_{12}^Q - x_{12}^Q y_{34}^Q = 2(A_{234}^Q - A_{341}^Q), \quad J_2^Q = x_{23}^Q y_{14}^Q - x_{14}^Q y_{23}^Q = 2(A_{341}^Q - A_{412}^Q). \quad (32.59)$$

We note that

$$A_{123}^Q = \frac{1}{4}(J_0^Q + J_1^Q - J_2^Q), \quad A_{234}^Q = \frac{1}{4}(J_0^Q + J_1^Q + J_2^Q), \quad A_{412}^Q = \frac{1}{4}(J_0^Q - J_1^Q - J_2^Q), \quad A_{341}^Q = \frac{1}{4}(J_0^Q - J_1^Q + J_2^Q). \quad (32.60)$$

The identities corresponding to (32.57) is $(x_{12}^Q + x_{34}^Q)J_0^Q + (x_{13}^Q + x_{24}^Q)J_1^Q + (x_{12}^Q + x_{43}^Q)J_2^Q = (x_{14}^Q + x_{32}^Q)J_0^Q + (x_{14}^Q + x_{23}^Q)J_1^Q + (x_{13}^Q + x_{42}^Q)J_2^Q = 0$ and two similar ones with y 's. Let $\{\xi_C, \eta_C\}$ denote the isoparametric coordinates of the quadrilateral centroid C. (The intersection of diagonals D, shown in Figure x.1(b), is not necessary in this development.) It can be verified that

$$\xi_C = \frac{J_1^Q}{3J_0^Q}, \quad \eta_C = \frac{J_2^Q}{3J_0^Q} \quad (32.61)$$

For a rectangular or parallelogram geometry $J_1^Q = J_2^Q = 0$, and $\xi_C = \eta_C = 0$. For any other geometry C does not lie at the intersection of the medians.

§32.6.2. Quadrilateral Invariant Relations

Quadratic combinations of J_0^Q , J_1^Q and J_2^Q may be express rationally in terms of the squared lengths of sides and diagonals. The following formulas were found by *Mathematica* (of which only the first one

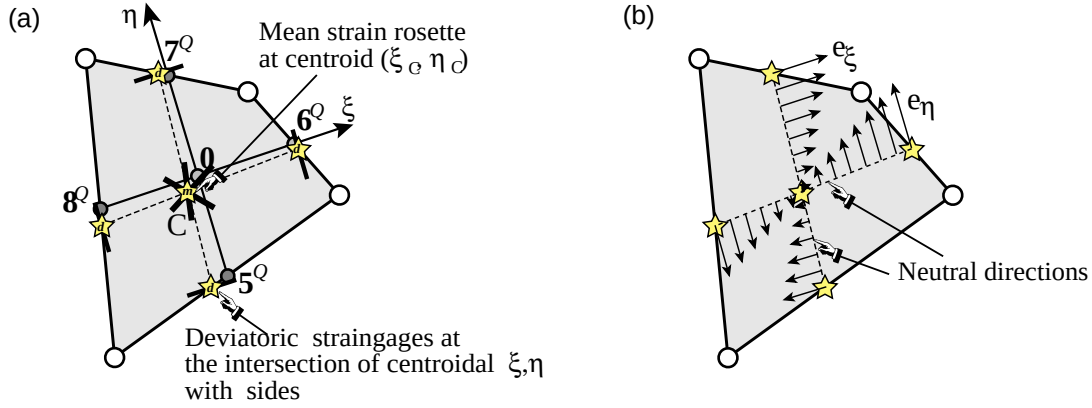


Figure 18. The strain field over the quadrilateral: (a) gages; (b) deviatoric bending strains about the centroidal medians (shown in dotted lines).

is well known):

$$\begin{aligned}
 (J_0^Q)^2 &= L_{31}^2 L_{42}^2 - P^2, \\
 4(J_1^Q)^2 &= 4L_{21}^2 L_{43}^2 - (L_{14}^2 - L_{31}^2 + L_{32}^2 - L_{42}^2)^2, \quad 4(J_2^Q)^2 = 4L_{14}^2 L_{32}^2 - (L_{21}^2 - L_{31}^2 - L_{42}^2 + L_{43}^2)^2, \\
 4J_0^Q J_1^Q &= 2(L_{32}^2 - L_{14}^2)P + L_{31}^2 Q + L_{42}^2 R, \quad 4J_0^Q J_2^Q = 2(L_{21}^2 - L_{43}^2)P - L_{31}^2 Q + L_{42}^2 R, \\
 4J_1^Q J_2^Q &= L_{14}^2 L_{21}^2 - L_{21}^2 L_{32}^2 + L_{32}^2 L_{43}^2 - L_{43}^2 L_{14}^2 + (L_{42}^2 - L_{31}^2)(S - L_{31}^2 - L_{42}^2) \\
 &= \frac{1}{4}(R^2 - Q^2)(S - L_{31}^2 - L_{42}^2),
 \end{aligned} \tag{32.62}$$

in which

$$\begin{aligned}
 P &= \frac{1}{2}(L_{21}^2 - L_{32}^2 + L_{43}^2 - L_{14}^2), \quad Q = L_{21}^2 + L_{32}^2 - L_{43}^2 - L_{14}^2, \\
 R &= -L_{21}^2 + L_{32}^2 + L_{43}^2 - L_{14}^2, \quad S = L_{21}^2 + L_{32}^2 + L_{43}^2 + L_{14}^2.
 \end{aligned} \tag{32.63}$$

The geometry of an arbitrary quadrilateral is defined by five independent quantities. It is convenient to select these to be invariants. For example, the four side lengths and one diagonal length may be chosen, but the specification is not symmetric respect to the choice of diagonal. A more elegant specification consists of taking the three invariants J_0^Q , J_1^Q , J_2^Q plus the semisum and semidifference of the squared diagonal lengths: $K_1 = \frac{1}{2}(L_{31}^2 + L_{42}^2)$ and $K_2 = \frac{1}{2}(L_{31}^2 - L_{42}^2)$. We get

$$\begin{aligned}
 P &= -\sqrt{L_{31}^2 L_{42}^2 - (J_0^Q)^2} = -\sqrt{K_1^2 - K_2^2 - (J_0^Q)^2}, \\
 Q &= 2 \frac{(J_1^Q(K_1 - K_2 - P) - J_2^Q(K_1 - K_2 + P))}{J_0^Q}, \quad R = 2 \frac{(J_1^Q(K_1 + K_2 - P) - J_2^Q(K_1 + K_2 + P))}{J_0^Q}, \\
 S &= 2 \frac{(J_0^Q)^2(K_1 K_2 - J_1^Q J_2^Q) + (J_2^Q K_2 + J_1^Q K_1 - J_1^Q P)(J_1^Q K_2 + J_2^Q K_1 + J_2^Q P)}{(J_0^Q)^2 K_2},
 \end{aligned} \tag{32.64}$$

The side lengths are then recovered from $L_{21}^2 = \frac{1}{4}(2P + Q - R + S)$, $L_{32}^2 = \frac{1}{4}(-2P + Q + R + S)$, $L_{43}^2 = \frac{1}{4}(2P - Q + R + S)$ and $L_{14}^2 = \frac{1}{4}(-2P - Q - R + S)$. The squared median lengths can also be expressed in terms of invariants, as discovered by *Mathematica*:

$$\begin{aligned}
 L_{57}^2 &= \frac{1}{4}(L_{42}^2 + L_{31}^2 - L_{21}^2 + L_{32}^2 - L_{43}^2 + L_{14}^2) = \frac{1}{2}(K_1 - P) \\
 L_{68}^2 &= \frac{1}{4}(L_{42}^2 + L_{31}^2 + L_{21}^2 - L_{32}^2 + L_{43}^2 - L_{14}^2) = \frac{1}{2}(K_1 + P)
 \end{aligned}$$

§32.6.3. The Assumed Strains

The assumed strain field is obtained from the gage locations shown in Figure 18(a). The strains $\mathbf{e} = \bar{\mathbf{e}} + \mathbf{e}_d$ are decomposed into a mean part $\bar{\mathbf{e}} = \mathbf{e}_C$, which is the value at the quadrilateral centroid $\{\xi_C = J_1/(3J_0), \eta_C = J_2/(3J_2)\}$, and an energy-orthogonal deviatoric part that varies linearly in $\{\xi, \eta\}$.

The mean strain $\bar{\mathbf{e}}$ is obtained by putting a strain rosette at the centroid as depicted in Figure 18(a).

The deviatoric strains are assumed to consist of the response of the quadrilateral to bending about the two dotted lines sketched in Figure 18(b), which pass through the centroid. These will be called the *neutral lines*. Two appropriate choices for the neutral lines are:

1. Lines parallel to the medians 57 and 68
2. Lines of coordinates $\xi = \xi_C, \eta = \eta_C$, called the *centroidal medians*

The two possibilities can be subsumed in a single form by introducing a free parameter α , such that if $\alpha = 0$ or $\alpha = 1$ the neutral lines are parallel to the medians or centroidal medians, respectively.

Both patterns consist of one-dimensional strains e_ξ and e_η where e_ξ is oriented along the axis ξ and e_η along the axis η . The latter varies linearly in $\xi - \xi_C$ and the former linearly in $\eta - \eta_C$. Four “deviatoric gages” are placed as shown to measure the amplitudes of these patterns. Note that these locations are not generally the side midpoints. Because of the assumed linear variations two of these gages are actually redundant and could be removed. For example we could keep 5 and 6 and get rid of 7 and 8. However they are retained in the configurations depicted in Figure 18 for visual symmetry.

A one dimensional strain state e_m along a direction m forming direction cosines $\{c_m, s_m\}$ with respect to $\{x, y\}$ generates the Cartesian components $e_{xx} = c_m^2 e_m$, $e_{yy} = s_m^2 e_m$ and $2e_{xy} = 2s_m c_m e_m$. Hence the foregoing strain field may be written in terms of five strain amplitudes χ_1 through χ_5 as

$$\begin{aligned} e_{xx} &= \chi_1 + c_{m\xi}^2 (\xi - \xi_C) \chi_4 + c_{m\eta}^2 (\eta - \eta_C) \chi_5, \\ e_{yy} &= \chi_2 + s_{m\xi}^2 (\xi - \xi_C) \chi_4 + s_{m\eta}^2 (\eta - \eta_C) \chi_5, \\ 2e_{xy} &= \chi_3 + 2s_{m\xi} c_{m\xi} (\xi - \xi_C) \chi_4 + 2s_{m\eta} c_{m\eta} (\eta - \eta_C) \chi_5, \end{aligned} \quad (32.65)$$

where $\{c_{m\xi}, s_{m\xi}\}$ and $\{c_{m\eta}, s_{m\eta}\}$ are the direction cosines of the neutral directions. In matrix form

$$\mathbf{e}^Q = \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & c_{m\xi}^2 (\xi - \xi_C) & c_{m\eta}^2 (\eta - \eta_C) \\ 0 & 1 & 0 & s_{m\xi}^2 (\xi - \xi_C) & s_{m\eta}^2 (\eta - \eta_C) \\ 0 & 0 & 1 & 2s_{m\xi} c_{m\xi} (\xi - \xi_C) & 2s_{m\eta} c_{m\eta} (\eta - \eta_C) \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{bmatrix} = \mathbf{B}^Q \boldsymbol{\chi}^Q. \quad (32.66)$$

What is the rationale behind (32.65)-(32.66)? Partly experience: for parallelograms and rectangles this distribution is known to lead to optimal elements. Partly simplicity: a linear variation in $\{\xi, \eta\}$ is the simplest one. But rooting this distribution away from the center point 0 has been missed by many authors. For the displacement derived strains we take the isoparametric interpolation. Then differentiation gives

$$\begin{aligned} \mathbf{B}_{iso}^Q &= \frac{J_0}{8J} \mathbf{B}_c + \frac{\xi}{8J} \begin{bmatrix} y_{43}^Q & 0 & y_{34}^Q & 0 & y_{12}^Q & 0 & y_{21}^Q & 0 \\ 0 & x_{34}^Q & 0 & x_{43}^Q & 0 & x_{21}^Q & 0 & x_{12}^Q \\ x_{34}^Q & y_{43}^Q & x_{43}^Q & y_{34}^Q & x_{21}^Q & y_{12}^Q & x_{12}^Q & y_{21}^Q \end{bmatrix} \\ &+ \frac{\eta}{8J} \begin{bmatrix} y_{41}^Q & 0 & y_{23}^Q & 0 & y_{41}^Q & 0 & y_{23}^Q & 0 \\ 0 & x_{14}^Q & 0 & x_{32}^Q & 0 & x_{14}^Q & 0 & x_{32}^Q \\ x_{14}^Q & y_{41}^Q & x_{32}^Q & y_{23}^Q & x_{14}^Q & y_{41}^Q & x_{32}^Q & y_{23}^Q \end{bmatrix} \end{aligned} \quad (32.67)$$

Evaluation of this matrix at the center pivot location P at $\xi_C^\beta = \beta J_1^Q / (3J_0^Q)$, $\eta_C^\beta = \beta J_2^Q / (3J_0^Q)$ yields

$$\mathbf{B}_C^Q = \frac{1}{3(J_0^Q)^2 + \beta((J_1^Q)^2 + (J_2^Q)^2)} \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 & B_{17} & 0 \\ 0 & B_{22} & 0 & B_{24} & 0 & B_{26} & 0 & B_{28} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} & B_{37} & B_{38} \end{bmatrix} \quad (32.68)$$

in which

$$\begin{aligned} B_{11} &= 3J_0^Q y_{24}^Q + \beta(J_1^Q y_{43}^Q + J_2^Q y_{32}^Q), & B_{13} &= 3J_0^Q y_{31}^Q + \beta(J_1^Q y_{34}^Q + J_2^Q y_{14}^Q), \\ B_{15} &= 3J_0^Q y_{42}^Q + \beta(J_1^Q y_{12}^Q + J_2^Q y_{41}^Q), & B_{17} &= 3J_0^Q y_{13}^Q + \beta(J_1^Q y_{21}^Q + J_2^Q y_{23}^Q), \\ B_{22} &= 3J_0^Q x_{42}^Q + \beta(J_1^Q x_{34}^Q + J_2^Q x_{23}^Q), & B_{24} &= 3J_0^Q x_{13}^Q + \beta(J_1^Q x_{43}^Q + J_2^Q x_{41}^Q), \\ B_{26} &= 3J_0^Q x_{24}^Q + \beta(J_1^Q x_{21}^Q + J_2^Q x_{14}^Q), & B_{28} &= 3J_0^Q x_{31}^Q + \beta(J_1^Q x_{12}^Q + J_2^Q x_{32}^Q), \\ B_{31} &= 3J_0^Q x_{42}^Q + \beta(J_1^Q x_{34}^Q + J_2^Q x_{23}^Q), & B_{32} &= 3J_0^Q y_{24}^Q + \beta(J_1^Q y_{43}^Q + J_2^Q y_{32}^Q), \\ B_{33} &= 3J_0^Q x_{13}^Q + \beta(J_1^Q x_{43}^Q + J_2^Q x_{41}^Q), & B_{34} &= 3J_0^Q y_{31}^Q + \beta(J_1^Q y_{34}^Q + J_2^Q y_{14}^Q), \\ B_{35} &= 3J_0^Q x_{24}^Q + \beta(J_1^Q x_{21}^Q + J_2^Q x_{14}^Q), & B_{36} &= 3J_0^Q y_{42}^Q + \beta(J_1^Q y_{12}^Q + J_2^Q y_{41}^Q), \\ B_{37} &= 3J_0^Q x_{31}^Q + \beta(J_1^Q x_{12}^Q + J_2^Q x_{32}^Q), & B_{38} &= 3J_0^Q y_{13}^Q + \beta(J_1^Q y_{21}^Q + J_2^Q y_{23}^Q), \end{aligned} \quad (32.69)$$

§32.6.4. The Fitted Strain Field

The strain fitting (SF) problem consists of constructing a matrix $\mathbf{T}^Q$ such that $\chi = \mathbf{T}^Q \mathbf{u}^Q$ minimizes a dislocation energy functional. Details of the fitting procedure are explained in another document in preparation [12]. Only the final result is shown here:

$$\mathbf{T}^Q = \begin{bmatrix} \mathbf{B}_C^Q \\ \mathbf{W}\mathbf{H}_h \end{bmatrix} \quad (32.70)$$

where matrices $\mathbf{W}$ and $\mathbf{H}_h$ are given by

$$\mathbf{H}_h = \begin{bmatrix} H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4 & 0 \\ 0 & H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4 \end{bmatrix} \quad (32.71)$$

$$H_1 = \frac{J_0^Q + J_1^Q + J_2^Q}{J_0^Q}, \quad H_2 = \frac{-J_0^Q + J_1^Q - J_2^Q}{J_0^Q}, \quad H_3 = \frac{J_0^Q - J_1^Q + J_2^Q}{J_0^Q}, \quad H_4 = \frac{-J_0^Q - J_1^Q - J_2^Q}{J_0^Q}, \quad (32.72)$$

$$\begin{aligned} \mathbf{W} &= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \\ W_{11} &= \frac{(3(J_0^Q)^2 + (J_2^Q)^2 \alpha)(3J_0^Q(x_{21}^Q + x_{34}^Q) + J_2^Q \alpha(x_{12}^Q + x_{34}^Q))}{18(J_0^Q)^3 L_{68C}^2}, \\ W_{12} &= \frac{(3(J_0^Q)^2 + (J_2^Q)^2 \alpha)(3J_0^Q(y_{21}^Q + y_{34}^Q) + J_2^Q \alpha(y_{12}^Q + y_{34}^Q))}{18(J_0^Q)^3 L_{68C}^2}, \\ W_{21} &= \frac{(3(J_0^Q)^2 + (J_1^Q)^2 \alpha)(3J_0^Q(x_{32}^Q + x_{41}^Q) + J_1^Q \alpha(x_{12}^Q + x_{34}^Q))}{18(J_0^Q)^3 L_{57C}^2}, \\ W_{22} &= \frac{(3(J_0^Q)^2 + (J_1^Q)^2 \alpha)(3J_0^Q(y_{32}^Q + y_{41}^Q) + J_1^Q \alpha(y_{12}^Q + y_{34}^Q))}{18(J_0^Q)^3 L_{57C}^2} \end{aligned} \quad (32.73)$$

If $\alpha = 0$, which means the neutral directions are parallel to the quadrilateral medians

$$W_{11} = \frac{x_{21}^Q + x_{34}^Q}{2L_{68}^2}, \quad W_{12} = \frac{y_{21}^Q + y_{34}^Q}{2L_{68}^2}, \quad W_{21} = \frac{x_{32}^Q + x_{41}^Q}{2L_{57}^2}, \quad W_{22} = \frac{y_{32}^Q + y_{41}^Q}{2L_{57}^2}. \quad (32.74)$$

The inplane Cartesian strains are then

$$\mathbf{e}^Q = \mathbf{B}_\chi^Q \mathbf{T}^Q \mathbf{u}^Q = \mathbf{B}^Q \mathbf{u}^Q. \quad (32.75)$$

It is not difficult to check that this strain field exactly represents rigid body motions and constant inplane strain states for any quadrilateral geometry. This should be expected since the derivation (not presented here) takes care of such conditions. What is surprising is that the field preserves those states also for a solid shell element of fairly general geometry, as discussed next.

§32.6.5. Extending the Quadrilateral Strains to SS8

The strain field derived in the preceding subsections for the quadrilateral will be now extended to the SS8. The two basic assumptions are:

1. The geometry is that of the flattened brick element. That is, the midsurface $\zeta = 0$ is flat, but the element may have thickness taper and variable thickness. See Figure 3.
2. The field $\mathbf{e}^Q$ evaluated for the midsurface quadrilateral is extended to any $-1 \leq \zeta \leq 1$ by taking a section of the flattened brick at a constant ζ .

Notice that the second assumption is relevant if the element is thickness tapered, because if so the constant- ζ section furnish different quadrilaterals.

The geometric constraints are less restrictive than those imposed for the transverse shear strains in Section 7. Why? Because it will be shown that the field $\mathbf{e}^Q$ exactly satisfies consistency conditions for the geometry (32.3) for the membrane component, and no further developments are required.⁷

§32.6.6. Membrane Strain Implementation

A *Mathematica* implementation of the foregoing assumptions for computing the matrix components $\mathbf{T}^Q$ and $\mathbf{B}_\chi^Q$ is listed in Figure 19.

The module is referenced as

$$\{\text{TQ}, \text{BQ}\} = \text{SS8MemStrains}[\text{xyzcoor}, \{\xi, \eta, \zeta\}, \text{Jlist}, \{\alpha, \beta\}, \text{options}] \quad (32.76)$$

where

xyzcoor The local coordinates of the flattened brick, arranged as
 $\{\{x_1, x_2, \dots, x_8\}, \{y_1, y_2, \dots, y_8\}, \{z_1, z_2, \dots, z_8\}\}$
 Here $z_1 = -h_1/2$, $z_2 = -h_2/2$, etc.

$\{\xi, \eta, \zeta\}$ ξ and η are quadrilateral coordinates, while ζ specifies the cutting surface that produces the quadrilateral. The ability to evaluate strains at the bottom surface $\zeta = -1$ and top surface $\zeta = 1$ will be found important in the treatment of the bending response.

⁷ Exactness for a warped element would demand additional work, however, which presently seems beyond the power of computer algebra systems.

```

SS8MembStrains[xyzcoor_, {ξ_, η_, ξ_}, Jlist_, {α_, β_}, options_] :=
Module[{x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8,
z1, z2, z3, z4, z5, z6, z7, z8, cb, ct,
x10, x20, x30, x40, y10, y20, y30, y40, h1, h2, h3, h4,
x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43,
y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43,
Δx1, Δx2, Δx3, Δx4, Δy1, Δy2, Δy3, Δy4, numer, Jgiven,
JQ0, JQ1, JQ2, ξC, ηC, ξ0β, η0β, dξ, dη, x5C, y5C, x6C, y6C, x7C, y7C,
x8C, y8C, x57C, y57C, x68C, y68C, x75C, y75C, x86C, y86C, y68C,
LL57C, LL68C, H1, H2, H3, H4, W, WH, JC, Tm, Bm},
{{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}, {x4, y4, z4},
{x5, y5, z5}, {x6, y6, z6}, {x7, y7, z7}, {x8, y8, z8}}=xyzcoor;
cb=(1-ξ)/2; ct=(1+ξ)/2;
{x10, x20, x30, x40}={x5, x6, x7, x8}*ct+{x1, x2, x3, x4}*cb;
{y10, y20, y30, y40}={y5, y6, y7, y8}*ct+{y1, y2, y3, y4}*cb;
{Δx1, Δx2, Δx3, Δx4}={x5, x6, x7, x8}*ct-{x1, x2, x3, x4}*cb;
{Δy1, Δy2, Δy3, Δy4}={y5, y6, y7, y8}*ct-{y1, y2, y3, y4}*cb;
{x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43}=
{x10-x20, x10-x30, x10-x40, x20-x10, x20-x30, x20-x40,
x30-x10, x30-x20, x30-x40, x40-x10, x40-x20, x40-x30};
{y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43}=
{y10-y20, y10-y30, y10-y40, y20-y10, y20-y30, y20-y40,
y30-y10, y30-y20, y30-y40, y40-y10, y40-y20, y40-y30};
{numer, Jgiven}=options;
If [Jgiven, {JQ0, JQ1, JQ2}=Jlist, JQ0=x31*y42-x42*y31;
JQ1=x34*y12-x12*y34; JQ2=x23*y14-x14*y23];
{ξ0, η0}={JQ1, JQ2}/(3*JQ0); {ξC, ηC}=α*{ξ0, η0}; ξ0β=β*ξ0; η0β=β*η0;
x5C=(x1*(1-ξC)+x2*(1+ξC))/2; x6C=(x2*(1-ηC)+x3*(1+ηC))/2;
x7C=(x3*(1+ξC)+x4*(1-ξC))/2; x8C=(x4*(1+ηC)+x1*(1-ηC))/2;
y5C=(y1*(1-ξC)+y2*(1+ξC))/2; y6C=(y2*(1-ηC)+y3*(1+ηC))/2;
y7C=(y3*(1+ξC)+y4*(1-ξC))/2; y8C=(y4*(1+ηC)+y1*(1-ηC))/2;
x57C=x5C-x7C; y57C=y5C-y7C; x75C=-x57C; y75C=-y57C;
x68C=x6C-x8C; y68C=y6C-y8C; x86C=-x68C; y86C=-y68C;
LL57C=x57C^2+y57C^2; LL68C=x68C^2+y68C^2;
H1=(JQ0+JQ1+JQ2)/(2*JQ0); H2=(-JQ0+JQ1-JQ2)/(2*JQ0);
H3=(JQ0-JQ1-JQ2)/(2*JQ0); H4=(-JQ0-JQ1+JQ2)/(2*JQ0);
H=Simplify[{{H1, 0, H2, 0, H3, 0, H4, 0}, {0, H1, 0, H2, 0, H3, 0, H4}}];
W={{(3*JQ0^2+JQ2^2*α)*(3*JQ0*(x21+x34)+JQ2*α*(x12+x34)),
(3*JQ0^2+JQ2^2*α)*(3*JQ0*(y21+y34)+JQ2*α*(y12+y34))}/
(18*JQ0^3*LL68C),
{(3*JQ0^2+JQ1^2*α)*(3*JQ0*(x32+x41)+JQ1*α*(x12+x34)),
(3*JQ0^2+JQ1^2*α)*(3*JQ0*(y32+y41)+JQ1*α*(y12+y34))}/
(18*JQ0^3*LL57C)};
WH=Simplify[W.H]; JC=3*JQ0^2+β*(JQ1^2+JQ2^2);
Tm={{3*JQ0*y24+β*(JQ1*y43+JQ2*y32), 0,
3*JQ0*y31+β*(JQ1*y34+JQ2*y14), 0,
3*JQ0*y42+β*(JQ1*y12+JQ2*y41), 0,
3*JQ0*y13+β*(JQ1*y21+JQ2*y23), 0}/JC,
{0, 3*JQ0*x42+β*(JQ1*x34+JQ2*x23),
0, 3*JQ0*x13+β*(JQ1*x43+JQ2*x41),
0, 3*JQ0*x24+β*(JQ1*x21+JQ2*x14),
0, 3*JQ0*x31+β*(JQ1*x12+JQ2*x32)}/JC,
{3*JQ0*x42+β*(JQ1*x34+JQ2*x23), 3*JQ0*y24+β*(JQ1*y43+JQ2*y32),
3*JQ0*x13+β*(JQ1*x43+JQ2*x41), 3*JQ0*y31+β*(JQ1*y34+JQ2*y14),
3*JQ0*x24+β*(JQ1*x21+JQ2*x14), 3*JQ0*y42+β*(JQ1*y12+JQ2*y41),
3*JQ0*x31+β*(JQ1*x12+JQ2*x32), 3*JQ0*y13+β*(JQ1*y21+JQ2*y23)}/JC,
WH[[1]], WH[[2]]];
dη=η-η0β; dξ=ξ-ξ0β;
Bm={{1, 0, 0, (x68C^2/LL68C)*dη, (x75C^2/LL57C)*dξ},
{0, 1, 0, (y68C^2/LL68C)*dη, (y75C^2/LL57C)*dξ},
{0, 0, 1, 2*(x68C*y68C/LL68C)*dη, 2*(x75C*y75C/LL57C)*dξ}};
If [numer, Return[N[{Bm, Tm}]]]; Return[{Bm, Tm}];

```

Figure 19. *Mathematica* implementation of the fitted strain field over a quadrilateral

Jlist

If logical flag Jgiven in options is True, a list {J0, J1, J2} of Jacobians to be used instead of internally calculated from the corner coordinates. Primarily used in element development and testing.

$\{\alpha, \beta\}$ Free parameters α and β . The second one is far more important as regard element performance. In the present SS8 implementation both parameters are set to zero by the higher levels, because optimal values are not precisely known yet. They are left as arguments for future developments.

options Logical flags {numer, Jgiven}

The module returns the matrix list $\{\mathbf{B}_m, \mathbf{T}_m\}$. The first one is the 3×5 matrix $\mathbf{B}^Q$ that transforms $\chi^Q = \mathbf{T}^Q \mathbf{u}^{(e)}$ to Cartesian components as per (32.66). The second one is the 5×8 strain-amplitude-displacement matrix $\mathbf{T}^Q$.

§32.6.7. Thickness Interpolation

To find a strain interpolation along ζ , relation (32.75) is manipulated as follows. First, $\mathbf{u}_Q$ at a fixed ζ is expanded to include all degrees of freedom of the SS8 element:

$$\begin{aligned} \mathbf{u}^Q(\zeta) &= [u_{x1}^Q(\zeta) \ u_{y1}^Q(\zeta) \ u_{x2}^Q(\zeta) \ \dots \ u_{x4}^Q(\zeta)]^T = \frac{1}{2}(1-\zeta)\mathbf{u}^{Q-} + \frac{1}{2}(1+\zeta)\mathbf{u}^{Q+} \\ \mathbf{u}^{Q-} &= [u_{x1}^- \ u_{y1}^- \ u_{x2}^- \ u_{y2}^- \ u_{x3}^- \ u_{y3}^- \ u_{x4}^- \ u_{y4}^-]^T \\ \mathbf{u}^{Q+} &= [u_{x1}^+ \ u_{y1}^+ \ u_{x2}^+ \ u_{y2}^+ \ u_{x3}^+ \ u_{y3}^+ \ u_{x4}^+ \ u_{y4}^+]^T \end{aligned} \quad (32.77)$$

In turn $\mathbf{u}^{Q-} = \mathbf{L}^- \mathbf{u}^{(e)}$, $\mathbf{u}^{Q+} = \mathbf{L}^+ \mathbf{u}^{(e)}$ where $\mathbf{L}^-$ and $\mathbf{L}^+$ are 8×24 localization matrices that extract the freedoms in $\mathbf{u}^{Q-}$ and $\mathbf{u}^{Q+}$ from the 24 freedoms of the full node displacement vector:

$$\begin{aligned} \mathbf{L}^- &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathbf{L}^+ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (32.78)$$

Substitution into the first of (32.77) yields

$$\mathbf{u}^Q = \frac{1}{2}(\mathbf{L}^+ + \mathbf{L}^-)\mathbf{u}^{(e)} + \frac{1}{2}(\mathbf{L}^+ - \mathbf{L}^-)\zeta\mathbf{u}^{(e)} \quad (32.79)$$

As can be seen the strain $\mathbf{e}^Q = \mathbf{T}^Q \mathbf{u}^Q$ splits naturally into a membrane and a bending part, each containing 3 strain components:

$$\mathbf{e}^Q = \mathbf{e}_m + \mathbf{e}_b = \mathbf{T}^Q \frac{1}{2}(\mathbf{L}^+ + \mathbf{L}^-)\mathbf{u}^{(e)} + \mathbf{T}^Q \frac{1}{2}(\mathbf{L}^+ - \mathbf{L}^-)\zeta\mathbf{u}^{(e)} = \mathbf{B}_m \mathbf{u}^{(e)} + \mathbf{B}_b \mathbf{u}^{(e)} \quad (32.80)$$

The thickness strain $e_{zz} = \mathbf{B}_t \mathbf{u}^{(e)}$ is adjoined to the components of $\mathbf{e}_m$ and $\mathbf{e}_b$ to complete a 7-vector:

$$\mathbf{e}_{mbt} = \begin{bmatrix} \mathbf{e}_m \\ \mathbf{e}_b \\ \mathbf{e}_t \end{bmatrix} = \begin{bmatrix} \mathbf{B}_m \\ \mathbf{B}_b \\ \mathbf{B}_t \end{bmatrix} \mathbf{u}^{(e)} \quad (32.81)$$

The expression of $\mathbf{B}_t$ is developed in §6.9.

§32.6.8. Strain Field Consistency Checks

Consider the 12×24 basic-states matrix of the Free Formulation [4,5,6]:

$$\mathbf{G}_{rc}^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots & 0 & 0 & 1 \\ y_1 & -x_1 & 0 & y_2 & -x_2 & 0 & y_3 & -x_3 & 0 & y_4 & -x_4 & 0 & \dots & y_8 & -x_8 & 0 \\ 0 & z_1 & -y_1 & 0 & z_2 & -y_2 & 0 & z_3 & -y_3 & 0 & z_4 & -y_4 & \dots & 0 & z_8 & -y_8 \\ -z_1 & 0 & x_1 & -z_2 & 0 & x_2 & -z_3 & 0 & x_3 & -z_4 & 0 & x_4 & \dots & -z_8 & 0 & x_8 \\ x_1 & 0 & 0 & x_2 & 0 & 0 & x_3 & 0 & 0 & x_4 & 0 & 0 & \dots & x_8 & 0 & 0 \\ 0 & y_1 & 0 & 0 & y_2 & 0 & 0 & y_3 & 0 & 0 & y_4 & 0 & \dots & 0 & y_8 & 0 \\ 0 & 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 & 0 & z_4 & \dots & 0 & 0 & z_8 \\ y_1 & x_1 & 0 & y_2 & x_2 & 0 & y_3 & x_3 & 0 & y_4 & x_4 & 0 & \dots & y_8 & x_8 & 0 \\ 0 & z_1 & y_1 & 0 & z_2 & y_2 & 0 & z_3 & y_3 & 0 & z_4 & y_4 & \dots & 0 & z_8 & y_8 \\ z_1 & 0 & x_1 & z_2 & 0 & x_2 & z_3 & 0 & x_3 & z_4 & 0 & x_4 & \dots & z_8 & 0 & x_8 \end{bmatrix}. \quad (32.82)$$

The first six columns of $\mathbf{G}_{rc}$ specify the six rigid body modes of three-dimensional space. The next six columns specify uniform strain states $e_{xx} = 1$, $e_{yy} = 1$, $e_{zz} = 1$, $2e_{xy} = 2$, $2e_{yz} = 2$ and $2e_{zx} = 2$, in that order.

Insert in (32.82) the coordinates (32.4) of an arbitrary flattened brick in symbolic form. Perform the multiplication $\mathbf{e}^Q(\zeta) = \mathbf{T}^Q[(\mathbf{L}^+ + \mathbf{L}^-) + (\mathbf{L}^+ - \mathbf{L}^-)]\mathbf{G}_{rc}$. The result should be

$$\begin{bmatrix} e_{xx}^Q \\ e_{yy}^Q \\ 2e_{xy}^Q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \quad (32.83)$$

And indeed a symbolic computation verifies (32.83) exactly for any combination of coordinates in (32.5) and arbitrary ζ . Note that this verification takes care of the membrane strains at any ζ . A similar check for the bending strains $\mathbf{e}_b$ gave the following result: uniform bending strain states are exactly reproduced if the element has constant thickness $h_1 = h_2 = h_4 = h_4 = h_0$, even if it is thickness tapered. If the thickness varies, uniform bending states are not exactly preserved.⁸

§32.6.9. The Thickness Strain

The thickness strain is the average extensional strain $\bar{e}_{zz}$ in the z direction normal to the midplane. Note that this is only an average value because an element with only two node layers in the z direction cannot resolve more. If this average strain is taken as the actual strain e_{zz} stress-jump contradictions arise in laminate wall constructions; these are resolved as explained in Section 5.

⁸ It is not presently known if the strain field definition can be adjusted to extend bending strain exactness to variable thickness configurations. An attempt in that direction failed because the symbolic computations “exploded” in intermediate stages and expressions could not be simplified over a reasonable amount of time.

To get $\bar{e}_{zz}$ from the nodal displacements the most effective method is the use of the conventional isoparametric interpolation, differentiated and evaluate it at $\zeta = 0$. The derivation offers no surprises and only the final result is given here.

$$\bar{e}_{zz} = \bar{e}_{zz1}N_1^Q(\xi, \eta) + \bar{e}_{zz2}N_2^Q(\xi, \eta) + \bar{e}_{zz3}N_3^Q(\xi, \eta) + \bar{e}_{zz4}N_4^Q(\xi, \eta) = \mathbf{B}_{zz}\mathbf{u}^{(e)}. \quad (32.84)$$

whwre N_i^Q are the bilinear quadrilateral shape functions, and $\bar{e}_{zzi}$ are the average thickness strains at the corners $i = 1, 2, 3, 4$. These are given by the relations

$$\bar{e}_{zz1} = \mathbf{B}_{zz1}\mathbf{u}^{(e)}, \quad \bar{e}_{zz2} = \mathbf{B}_{zz2}\mathbf{u}^{(e)}, \quad e_{zz3} = \mathbf{B}_{zz3}\mathbf{u}^{(e)}, \quad e_{zz4} = \mathbf{B}_{zz4}\mathbf{u}^{(e)}. \quad (32.85)$$

The $\mathbf{B}_{zzi}$ are the 1×24 strain-displacement matrices

$$\begin{aligned} \mathbf{B}_{t1} &= \begin{bmatrix} 0 & 0 & B_{z11} - \frac{1}{h_1} & 0 & 0 & B_{z12} & 0 & 0 & 0 & 0 & 0 & B_{z14} \\ 0 & 0 & B_{z11} + \frac{1}{h_1} & 0 & 0 & B_{z12} & 0 & 0 & 0 & 0 & 0 & B_{z14} \end{bmatrix} \\ \mathbf{B}_{t2} &= \begin{bmatrix} 0 & 0 & B_{z21} & 0 & 0 & B_{z22} - \frac{1}{h_2} & 0 & 0 & B_{z23} & 0 & 0 & 0 \\ 0 & 0 & B_{z21} & 0 & 0 & B_{z22} + \frac{1}{h_2} & 0 & 0 & B_{z23} & 0 & 0 & 0 \end{bmatrix} \\ \mathbf{B}_{t3} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{z32} & 0 & 0 & B_{z33} - \frac{1}{h_3} & 0 & 0 & B_{z34} \\ 0 & 0 & 0 & 0 & 0 & B_{z32} & 0 & 0 & B_{z33} + \frac{1}{h_3} & 0 & 0 & B_{z34} \end{bmatrix} \\ \mathbf{B}_{t4} &= \begin{bmatrix} 0 & 0 & B_{z41} & 0 & 0 & 0 & 0 & 0 & B_{z43} & 0 & 0 & B_{z44} - \frac{1}{h_4} \\ 0 & 0 & B_{z41} & 0 & 0 & 0 & 0 & 0 & B_{z43} & 0 & 0 & B_{z44} + \frac{1}{h_4} \end{bmatrix} \end{aligned} \quad (32.86)$$

where

$$\begin{aligned} B_{z11} &= (\Delta y_1 \bar{x}_{24} + \Delta x_1 \bar{y}_{42}) / (8J_{c1}), & B_{z12} &= (\Delta y_1 \bar{x}_{41} + \Delta x_1 \bar{y}_{14}) / (8J_{c1}) \\ B_{z14} &= (\Delta y_1 \bar{x}_{12} + \Delta x_1 \bar{y}_{21}) / (8J_{c1}), & B_{z21} &= (\Delta y_2 \bar{x}_{23} + \Delta x_2 \bar{y}_{32}) / (8J_{c2}) \\ B_{z22} &= (\Delta y_2 \bar{x}_{31} + \Delta x_2 \bar{y}_{13}) / (8J_{c2}), & B_{z23} &= (\Delta y_2 \bar{x}_{12} + \Delta x_2 \bar{y}_{21}) / (8J_{c2}) \\ B_{z32} &= (\Delta y_3 \bar{x}_{34} + \Delta x_3 \bar{y}_{43}) / (8J_{c3}), & B_{z33} &= (\Delta y_3 \bar{x}_{42} + \Delta x_3 \bar{y}_{24}) / (8J_{c3}) \\ B_{z34} &= (\Delta y_3 \bar{x}_{23} + \Delta x_3 \bar{y}_{32}) / (8J_{c3}), & B_{z41} &= (\Delta y_4 \bar{x}_{34} + \Delta x_4 \bar{y}_{43}) / (8J_{c4}) \\ B_{z43} &= (\Delta y_4 \bar{x}_{41} + \Delta x_4 \bar{y}_{14}) / (8J_{c4}), & B_{z44} &= (\Delta y_4 \bar{x}_{13} + \Delta x_4 \bar{y}_{31}) / (8J_{c4}), \\ J_{c1} &= (J_0^Q - J_1^Q - J_2^Q)h_1, & J_{c1} &= (J_0^Q + J_1^Q - J_2^Q)h_2, \\ J_{c3} &= (J_0^Q + J_1^Q + J_2^Q)h_3, & J_{c4} &= (J_0^Q - J_1^Q + J_2^Q)h_4. \end{aligned} \quad (32.87)$$

in which J_0^Q , J_1^Q and J_2^Q are the Jacobian coefficients of the quadrilateral at $\zeta = 0$. Obviously

$$\mathbf{B}_{zz} = \mathbf{B}_{zz1}N_1^Q + \mathbf{B}_{zz2}N_2^Q + \mathbf{B}_{zz3}N_3^Q + \mathbf{B}_{zz4}N_4^Q. \quad (32.88)$$

This 1×4 matrix is a function of $\{\xi, \eta\}$ only through the N_i^Q 's. Postmultiplying $\mathbf{B}_{zz}$ by $\mathbf{G}_{rc}$ gives

$$\mathbf{B}_{zz}\mathbf{G}_{rc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (32.89)$$

for any combination of coordinates. Hence the strain-displacement relation satisfies conservation of basic states.

It should be noted that if one attempts to use the tensorial thickness strains to get $\bar{e}_{zz}$ as done by several authors, the resulting strains are generally incorrect unless the element is thickness prismatic: $\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = \Delta y_1 = \Delta y_2 = \Delta y_3 = \Delta y_4 = 0$.


```

SS8ThickStrain[xyzcoor_, {ξ_, η_}, options_] := Module[
{
x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8,
z1, z2, z3, z4, z5, z6, z7, z8,
x10, x20, x30, x40, y10, y20, y30, y40, h1, h2, h3, h4,
x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43,
y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43,
Δx1, Δx2, Δx3, Δx4, Δy1, Δy2, Δy3, Δy4, JQ0, JQ1, JQ2, numer,
NQ1, NQ2, NQ3, NQ4, Jc1, Jc2, Jc3, Jc4, Bz11, Bz12, Bz14, Bz21, Bz22,
Bz23, Bz32, Bz33, Bz34, Bz41, Bz42, Bz43, Bz44, Bz},
{
{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}, {x4, y4, z4},
{x5, y5, z5}, {x6, y6, z6}, {x7, y7, z7}, {x8, y8, z8} = xyzcoor;
{x10, x20, x30, x40} = {x5+x1, x6+x2, x7+x3, x8+x4}/2;
{y10, y20, y30, y40} = {y5+y1, y6+y2, y7+y3, y8+y4}/2;
{Δx1, Δx2, Δx3, Δx4} = {x5-x1, x6-x2, x7-x3, x8-x4}/2;
{Δy1, Δy2, Δy3, Δy4} = {y5-y1, y6-y2, y7-y3, y8-y4}/2;
{h1, h2, h3, h4} = {z5-z1, z6-z2, z7-z3, z8-z4};
{x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43} =
{x10-x20, x10-x30, x10-x40, x20-x10, x20-x30, x20-x40,
x30-x10, x30-x20, x30-x40, x40-x10, x40-x20, x40-x30};
{y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43} =
{y10-y20, y10-y30, y10-y40, y20-y10, y20-y30, y20-y40,
y30-y10, y30-y20, y30-y40, y40-y10, y40-y20, y40-y30};
JQ0=x31*y42-x42*y31; JQ1=x34*y12-x12*y34; JQ2=x23*y14-x14*y23;
Jc1=(JQ0-JQ1-JQ2)*h1/2; Jc2=(JQ0+JQ1-JQ2)*h2/2;
Jc3=(JQ0+JQ1+JQ2)*h3/2; Jc4=(JQ0-JQ1+JQ2)*h4/2;
Bz11=(Δy1*x24+Δx1*y42)/Jc1; Bz12=(Δy1*x41+Δx1*y14)/Jc1;
Bz14=(Δy1*x12+Δx1*y21)/Jc1; Bz21=(Δy2*x23+Δx2*y32)/Jc2;
Bz22=(Δy2*x31+Δx2*y13)/Jc2; Bz23=(Δy2*x12+Δx2*y21)/Jc2;
Bz32=(Δy3*x34+Δx3*y43)/Jc3; Bz33=(Δy3*x42+Δx3*y24)/Jc3;
Bz34=(Δy3*x23+Δx3*y32)/Jc3; Bz41=(Δy4*x34+Δx4*y43)/Jc4;
Bz43=(Δy4*x41+Δx4*y14)/Jc4; Bz44=(Δy4*x13+Δx4*y31)/Jc4;
NQ1=(1-ξ)*(1-η)/4; NQ2=(1+ξ)*(1-η)/4; NQ3=(1+ξ)*(1+η)/4;
NQ4=(1-ξ)*(1+η)/4; numer=options[[1]];
Bz= {0,0,Bz11-1/h1, 0,0,Bz12, 0,0,0, 0,0,Bz14,
0,0,Bz11+1/h1, 0,0,Bz12, 0,0,0, 0,0,Bz14}*NQ1+
{0,0,Bz21, 0,0,Bz22-1/h2, 0,0,Bz23, 0,0,0,
0,0,Bz21, 0,0,Bz22+1/h2, 0,0,Bz23, 0,0,0}*NQ2+
{0,0,0, 0,0,Bz32, 0,0,Bz33-1/h3, 0,0,Bz34,
0,0,0, 0,0,Bz32, 0,0,Bz33+1/h3, 0,0,Bz34}*NQ3+
{0,0,Bz41, 0,0,0, 0,0,Bz43, 0,0,Bz44-1/h4,
0,0,Bz41, 0,0,0, 0,0,Bz43, 0,0,Bz44+1/h4}*NQ4;
If [numer, Return[N[Bz]]]; Return[Bz] ];

```

Figure 20. *Mathematica* implementation of thickness strain-displacement matrix.

§32.6.10. Thickness Strain Implementation

The *Mathematica* module SS8ThickStrain listed in Figure 20 implements the computation of $\mathbf{B}_{zz}$. The module is referenced as

$$\mathbf{B}_z = \text{SS8ThickStrain}[\text{xyzcoor}, \{\xi, \eta\}, \text{options}] \quad (32.90)$$

where

xyzcoor The local coordinates of the flattened brick, arranged as
 $\{\{x1, x2, \dots, x8\}, \{y1, y2, \dots, y8\}, \{z1, z2, \dots, z8\}\}$
 Here $z1 = -h1/2$, $z2 = -h2/2$, etc.

$\{\xi, \eta\}$ Quadrilateral coordinates of point at which $\mathbf{B}_{zz}$ will be evaluated.

options Logical flag {numer}. If True, compute $\mathbf{B}_{zz}$ using floating point arithmetic.

The module returns Bz, which is the 1×24 matrix $\mathbf{B}_{zz}$ defined in (32.88).

$x_1^- = -\frac{3}{8}$	$x_2^- = \frac{89}{35}$	$x_3^- = \frac{29}{18}$	$x_4^- = \frac{23}{70}$	$x_1^+ = -\frac{7}{24}$	$x_2^+ = \frac{79}{35}$	$x_3^+ = \frac{31}{18}$	$x_4^+ = \frac{33}{70}$
$y_1^- = \frac{4}{15}$	$y_2^- = -\frac{47}{126}$	$y_3^- = \frac{157}{66}$	$y_4^- = \frac{59}{30}$	$y_1^+ = \frac{2}{15}$	$y_2^+ = -\frac{61}{126}$	$y_3^+ = \frac{151}{66}$	$y_4^+ = \frac{49}{30}$
$z_1^- = -\frac{3}{5}$	$z_2^- = -\frac{4}{7}$	$z_3^- = -\frac{2}{5}$	$z_4^- = -\frac{3}{10}$	$z_1^+ = \frac{3}{5}$	$z_2^+ = \frac{4}{7}$	$z_3^+ = \frac{2}{5}$	$z_4^+ = \frac{3}{10}$

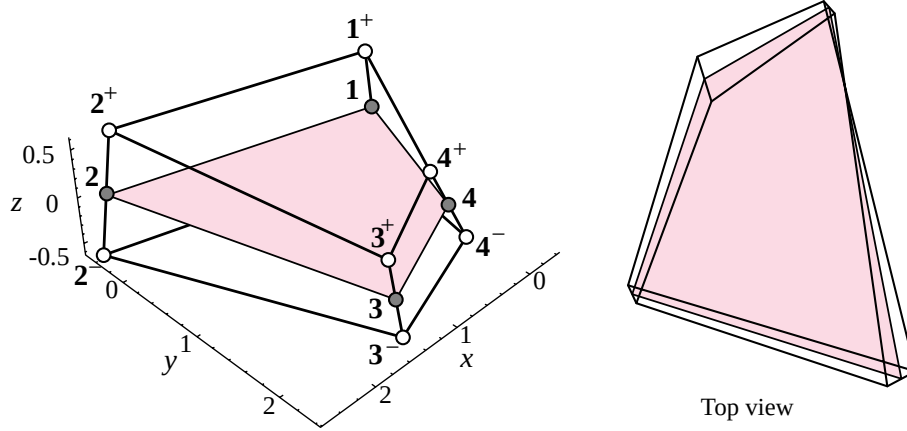


Figure 21. Example of thickness strain calculations for a thickness tapered element of variable thickness.

As an example, consider the element with the node coordinates shown in Figure 21. From this

$$\begin{aligned}
 x_1^0 &= -1/3, & x_2^0 &= 12/5, & x_3^0 &= 5/3, & x_4^0 &= 2/5, \\
 y_1^0 &= 1/5, & y_2^0 &= -3/7, & y_3^0 &= 7/3, & y_4^0 &= 9/5, \\
 \Delta x_1 &= 1/12, & \Delta x_2 &= -2/7, & \Delta x_3 &= 1/9, & \Delta x_4 &= 1/7, \\
 \Delta y_1 &= -2/15, & \Delta y_2 &= -1/9, & \Delta y_3 &= -1/11, & \Delta y_4 &= -1/3, \\
 h_1 &= 6/5, & h_2 &= 8/7, & h_3 &= 4/5, & h_4 &= 3/5,
 \end{aligned} \tag{32.91}$$

Running the *Mathematica* code for this geometry and evaluating at the corners gives

$$\begin{aligned}
 \mathbf{B}_{zz1} &= \begin{bmatrix} 0 & 0 & -\frac{51185}{60912} & 0 & 0 & -\frac{455}{22842} & 0 & 0 & 0 & 0 & \frac{4915}{182736} & 0 & 0 & \frac{50335}{60912} & 0 & 0 & -\frac{455}{22842} & 0 & 0 & 0 & 0 & \frac{4915}{182736} \end{bmatrix} \\
 &= [0 \ 0 \ -0.8403 \ 0 \ 0 \ -0.0199 \ 0 \ 0 \ 0 \ 0 \ 0.0269 \ 0 \ 0 \ 0.8264 \ 0 \ 0 \ -0.01992 \ 0 \ 0 \ 0 \ 0 \ 0.0269] \\
 \mathbf{B}_{zz2} &= \begin{bmatrix} 0 & 0 & -\frac{28795}{535872} & 0 & 0 & -\frac{76013}{89312} & 0 & 0 & \frac{15985}{535872} & 0 & 0 & 0 & 0 & -\frac{28795}{535872} & 0 & 0 & \frac{80283}{89312} & 0 & 0 & \frac{15985}{535872} & 0 & 0 \end{bmatrix} \\
 &= [0 \ 0 \ -0.05373 \ 0 \ 0 \ -0.8511 \ 0 \ 0 \ 0.02983 \ 0 \ 0 \ 0 \ 0 \ -0.05373 \ 0 \ 0 \ 0.8989 \ 0 \ 0 \ 0.02983 \ 0 \ 0 \ 0] \\
 \mathbf{B}_{zz3} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{45325}{1617264} & 0 & 0 & -\frac{28315}{22462} & 0 & 0 & \frac{5675}{147024} & 0 & 0 & 0 & 0 & -\frac{45325}{1617264} & 0 & 0 & \frac{13920}{11231} & 0 & 0 \end{bmatrix} \\
 &= [0 \ 0 \ 0 \ 0 \ 0 \ -0.02803 \ 0 \ 0 \ -1.261 \ 0 \ 0 \ 0.03860 \ 0 \ 0 \ 0 \ 0 \ 0 \ -0.02803 \ 0 \ 0 \ 1.239 \ 0 \ 0 \ 0.0386] \\
 \mathbf{B}_{zz4} &= \begin{bmatrix} 0 & 0 & -\frac{3925}{15456} & 0 & 0 & 0 & 0 & -\frac{3725}{15456} & 0 & 0 & -\frac{9055}{7728} & 0 & 0 & -\frac{3925}{15456} & 0 & 0 & 0 & 0 & -\frac{3725}{15456} & 0 & 0 & \frac{16705}{7728} \end{bmatrix} \\
 &= [0 \ 0 \ -0.2539 \ 0 \ 0 \ 0 \ 0 \ 0 \ -0.2410 \ 0 \ 0 \ -1.172 \ 0 \ 0 \ -0.2539 \ 0 \ 0 \ 0 \ 0 \ 0 \ -0.2410 \ 0 \ 0 \ 2.162]
 \end{aligned} \tag{32.92}$$

§32.6.11. MBT Strain Implementation

The *Mathematica* module listed in Figure 22 merges the strain-displacement matrices produced by SS8MembStrains and SS8MBTStrains to form the 7×24 matrix $\mathbf{B}_{mbt}$ defined by (32.81).

```

SS8MBTStrains[xyzcoor_, {ξ_, η_}, {α_, β_}, {numer_}] := Module[
{Bm, Tm, Bz, B, NQ1, NQ2, NQ3, NQ4, i, j, n, c0, c1, c2, Bmbt, Grc, Gh},
{Bm1, Tm1} = SS8MembStrains[xyzcoor, {ξ, η, -1}, {}, {α, β}, {numer, False}];
{Bm2, Tm2} = SS8MembStrains[xyzcoor, {ξ, η, 1}, {}, {α, β}, {numer, False}];
{Bm0, Tm0} = SS8MembStrains[xyzcoor, {ξ, η, 0}, {}, {α, β}, {numer, False}];
Bz = SS8ThickStrain[xyzcoor, {ξ, η}, {numer}];
B1 = Simplify[Bm1.Tm1]; B2 = Simplify[Bm2.Tm2];
B0 = Simplify[Bm0.Tm0];
{B1, B2, B0} = Simplify[{B1, B2, B0}];
Bmbt = Table[0, {7}, {24}];
For [n=1, n<=8, n++, j={1, 2, 4, 5, 7, 8, 10, 11}[[n]];
  For [i=1, i<=3, i++,
    c1=B1[[i, n]]/2; c2=B2[[i, n]]/2; c0=B0[[i, n]]/2;
    Bmbt[[i, j]] = c0; Bmbt[[i, j+12]] = c0;
    Bmbt[[i+3, j]] = -c1; Bmbt[[i+3, j+12]] = c2;
    Bmbt[[7, 3*n]] = Bz[[3*n]];
  If [numer, Return[N[Bmbt]]]; Return[Bmbt] ];

```

Figure 22. *Mathematica* module for evaluating $\mathbf{B}_{mbt}$.

The module is referenced as

$$\mathbf{B}_{mbt} = \text{SS8MBTStrains}[\text{xyzcoor}, \{\xi, \eta\}, \{\alpha, \beta\}, \text{options}] \quad (32.93)$$

where

- xyzcoor The local coordinates of the flattened brick, arranged as
 $\{\{x_1, x_2, \dots, x_8\}, \{y_1, y_2, \dots, y_8\}, \{z_1, z_2, \dots, z_8\}\}$
 Here $z_1 = -h_1/2$, $z_2 = -h_2/2$, etc.
- $\{\xi, \eta\}$ Quadrilateral coordinates of midsurface point at which $\mathbf{B}_{mbt}$ will be evaluated.
- $\{\alpha, \beta\}$ Parameters passed to SS8MembStrains.
- options Logical flag {numer}. If True, compute $\mathbf{B}_{mbt}$ using floating point arithmetic.

The module returns Bmbt, which is the 7×24 matrix $\mathbf{B}_{mbt}$ defined in (32.82).

§32.6.12. The MBT Stiffness Matrix

The hard work has been to obtain the strain-displacement matrix $\mathbf{B}_{mbt}$. The rest is easier. The MBT stiffness matrix is

$$\mathbf{K}_{mbt} = \int_{V^{(e)}} \mathbf{B}_{mbt}^T \bar{\mathbf{E}}_{mbt} \mathbf{B}_{mbt} dV \quad (32.94)$$

where $V^{(e)}$ is the volume of the flattened brick and $\bar{\mathbf{E}}_{mbt}$ the thickness-integrated MBT constitutive relation derived in §5.4.

The stiffness may be evaluated by a 2×2 two-dimensional Gauss rule since the volume Jacobian splits as $J^Q h$, in which $h(\xi, \eta) = h_1 N_1^Q + h_2 N_2 + h_3 N_3^Q + h_4 N_4^Q$ is the interpolated thickness:

$$\mathbf{K}_{mbt} = \sum_{i=1}^2 \sum_{j=1}^2 \mathbf{B}_{mbt}^T(\xi_i, \eta_j) \bar{\mathbf{E}}_{mbt} \mathbf{B}_{mbt}(\xi_i, \eta_j) J^Q(\xi_i, \eta_j) h(\xi_i, \eta_j) w_{ij}. \quad (32.95)$$

Here the weights $w_{ij} = 1$ for the 2×2 rule can be omitted.

A *Mathematica* implementation is listed in Figure 23. The module is referenced as

$$\mathbf{K}_{mbt} = \text{SS8StiffnessMBT}[\text{xyzcoor}, \text{ElayerMBT}, \text{layers}, \{\alpha, \beta\}, \text{options}] \quad (32.96)$$

where

```

SS8StiffnessMBT[xyzcoor_, ElayerMBT_, layers_, {α_, β_}, options_] := Module[
  {x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8,
   z1, z2, z3, z4, z5, z6, z7, z8, h1, h2, h3, h4, h0, h,
   x10, x20, x30, x40, y10, y20, y30, y40,
   Δx1, Δx2, Δx3, Δx4, Δy1, Δy2, Δy3, Δy4, numer, Jconst,
   JQ0, JQ1, JQ2, J, ξ, η, g1=1/Sqrt[3], gp1, gp2, gp3, gp4,
   Jg1, Jg2, Jg3, Jg4, Bg1, Bg2, Bg3, Bg4, Bmbt, Kmbt},
  {{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}, {x4, y4, z4},
   {x5, y5, z5}, {x6, y6, z6}, {x7, y7, z7}, {x8, y8, z8}} = xyzcoor;
  {x10, x20, x30, x40} = {x5+x1, x6+x2, x7+x3, x8+x4}/2;
  {y10, y20, y30, y40} = {y5+y1, y6+y2, y7+y3, y8+y4}/2;
  {h1, h2, h3, h4} = {z5-z1, z6-z2, z7-z3, z8-z4};
  NQ1=(1-ξ)*(1-η)/4; NQ2=(1+ξ)*(1-η)/4;
  NQ3=(1+ξ)*(1+η)/4; NQ4=(1-ξ)*(1+η)/4;
  h=Simplify[h1*NQ1+h2*NQ2+h3*NQ3+h4*NQ4];
  JQ0=(x30-x10)*(y40-y20)-(x40-x20)*(y30-y10);
  JQ1=(x30-x40)*(y10-y20)-(x10-x20)*(y30-y40);
  JQ2=(x20-x30)*(y10-y40)-(x10-x40)*(y20-y30);
  {numer, Jconst}=options;
  h0=(h1+h2+h3+h4)/4; If [Jconst, JQ1=JQ2=0; h=h0];
  Embt=SS8ThickIntegMBT[ElayerMBT, layers];
  Bmbt=SS8MBTStrains[xyzcoor, {ξ, η}, {α, β}, {numer}];
  J=h*(JQ0+JQ1*ξ+JQ2*η)/8; J=Simplify[J];
  gp1={ξ->-g1, η->-g1}; gp2={ξ-> g1, η->-g1};
  gp3={ξ->-g1, η-> g1}; gp4={ξ->-g1, η-> g1};
  Jg1=J/.gp1; Jg2=J/.gp2; Jg3=J/.gp3; Jg4=J/.gp4;
  {Jg1, Jg2, Jg3, Jg4}=Simplify[{Jg1, Jg2, Jg3, Jg4}];
  Bg1=(Bmbt/.gp1); Bg2=(Bmbt/.gp2);
  Bg3=(Bmbt/.gp3); Bg4=(Bmbt/.gp4);
  Kmbt= Transpose[Bg1].Embt.Bg1*Jg1+
        Transpose[Bg2].Embt.Bg2*Jg2+
        Transpose[Bg3].Embt.Bg3*Jg3+
        Transpose[Bg4].Embt.Bg4*Jg4;
  Return[Kmbt]];

```

Figure 23. *Mathematica* module for evaluating $\mathbf{K}_{mbt}$.

xyzcoor The local coordinates of the flattened brick, arranged as
 $\{\{x1, x2, \dots, x8\}, \{y1, y2, \dots, y8\}, \{z1, z2, \dots, z8\}\}$
 Here $z1 = -h1/2$, $z2 = -h2/2$, etc.

ElayerMBT MBT constitutive inputs. See Section 5 for details.

layers Layer extent list; see Section 5 for details.

$\{\alpha, \beta\}$ Parameters passed to `SS8MembStrains`.

options Logical flag $\{\text{numer}\}$. If `True`, compute $\mathbf{B}_{mbt}$ using floating point arithmetic.

The module returns $\mathbf{K}_{mbt}$, which is the 24×24 matrix $\mathbf{K}_{mbt}$ defined in (32.85).

§32.7. Transverse Shear Stiffness

The contribution of the transverse shear to the element stiffness is handled by the assumed natural strain method. To make the construction of the strain field practical a number of simplifying assumptions will be made.

§32.7.1. Assumptions and Requirements

The construction of the assumed strains proceeds under the following constraints.

Geometric restriction. The flattened brick may be of variable thickness but it must be thickness prismatic. The Jacobian at a point $\{\xi, \eta\}$ is

$$J = \frac{1}{32}(x_{2134}y_{3241} - x_{3241}y_{2134})h = J^Q h. \quad (32.97)$$

where J^Q is the Jacobian determinant of the flat midsurface quadrilateral, with $x_{2134} = x_{21}(1 - \eta) + x_{34}(1 + \eta)$, $y_{2134} = y_{21}(1 - \eta) + y_{34}(1 + \eta)$, $x_{3241} = x_{32}(1 + \xi) + x_{41}(1 - \xi)$ and $y_{3241} = y_{32}(1 + \xi) + y_{41}(1 - \xi)$.

Natural strain interpolation. The natural transverse shear strains

$$\gamma_{\zeta\xi} = e_{\xi\zeta} + e_{\zeta\xi}, \quad \gamma_{\zeta\eta} = e_{\eta\zeta} + e_{\zeta\eta} \quad (32.98)$$

are uniform across the thickness; that is, they do not depend on ζ . (Note that generally $e_{\xi\zeta} \neq e_{\zeta\xi}$ and $e_{\eta\zeta} \neq e_{\zeta\eta}$, thus the sum in (32.98) cannot be replaced by duplication.) The dependence on ξ and η is defined by the midpoint interpolations

$$\begin{aligned} \gamma_{\zeta\xi} &= \gamma_{\xi\zeta}(\xi, \eta) = \frac{1}{2}\gamma_{\xi,\zeta|5}(1 - \eta) + \frac{1}{2}\gamma_{\xi,\zeta|7}(1 + \eta), \\ \gamma_{\zeta\eta} &= \gamma_{\eta\zeta}(\xi, \eta) = \frac{1}{2}\gamma_{\eta,\zeta|8}(1 - \xi) + \frac{1}{2}\gamma_{\eta,\zeta|6}(1 + \xi) \\ \gamma_{\xi\zeta|5} &= \gamma_{\zeta\xi}|_{\xi=0,\eta=-1,\zeta=0}, \gamma_{\xi\zeta|7} = \gamma_{\zeta\xi}|_{\xi=0,\eta=1,\zeta=0}, \\ \gamma_{\eta\zeta|8} &= \gamma_{\zeta\eta}|_{\xi=-1,\eta=0,\zeta=0}, \gamma_{\eta\zeta|6} = \gamma_{\zeta\eta}|_{\xi=1,\eta=0,\zeta=0}. \end{aligned} \quad (32.99)$$

Cartesian shear strain recovery. The Cartesian transverse shear strains are recovered by the following transformation

$$\gamma_z = \begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} 2e_{yz} \\ 2e_{zx} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} \gamma_{\zeta\eta} \\ \gamma_{\zeta\xi} \end{bmatrix} = \mathbf{Y}\gamma_\zeta. \quad (32.100)$$

Matrix $\mathbf{Y}$ must preserve exactness of the two constant transverse shear states

$$\begin{aligned} u_y = Cz, \quad u_z = Cy, & \rightarrow \text{evaluate at nodes} \rightarrow \gamma_{yz} = 2C, \\ u_x = Cz, \quad u_z = Cx, & \rightarrow \text{evaluate at nodes} \rightarrow \gamma_{zx} = 2C, \end{aligned} \quad (32.101)$$

for arbitrary quadrilateral planforms and corner thicknesses, but within the constraint of no thickness taper.

§32.7.2. Comment on the Foregoing Assumptions

The geometric restriction is a concession to the limited time available to develop this component of the puzzle. If the element is allowed to be thickness tapered, symbolic computations with *Mathematica* to meet (32.101) became far too complicated to be finalized within reasonable time. Fulfillment of that condition in fact requires consideration of additional natural strain components at the midpoints, not just two. It is possible that such condition be eventually fulfilled by the simplification noted in §10.3.

The natural strain interpolation (32.99) was discovered to be optimal for C^0 flat plate elements by researchers in the mid 1980s [2] so it may be taken as a given. True, optimality has not been proven for solid shell configurations, and several benchmarks discussed in §9 show this to be far from optimal for thickness tapered elements. Nonetheless the particular choice of interpolation in $\{\xi, \eta\}$ turns out to be unimportant. Any deviations from it are compensated by the far stricter condition (32.101).

A new advance is the exact recovery of constant transverse shear states for varying thickness. While (32.99) has been well studied, the backtransformation process (32.100) available in the literature only works correctly for constant thickness $h_1 = h_2 = h_3 = h_4$. Several weeks of computations with *Mathematica* eventually produced, after harrowing simplifications, a relatively manageable yet exact form for the matrix Y_{ij} . This brings hope that an equally simple form could be eventually be discovered for thickness-tapered geometries.

§32.7.3. Strain Computation

Getting to (32.99) is the easy part. The natural strains are obtained as follows. Recall the definition of the 8-node isoparametric brick

$$\begin{aligned} x &= \mathbf{N}_{iso} \mathbf{x}, & y &= \mathbf{N}_{iso} \mathbf{y}, & z &= \mathbf{N}_{iso} \mathbf{z}, \\ u_x &= \mathbf{N}_{iso} \mathbf{u}_x, & u_y &= \mathbf{N}_{iso} \mathbf{u}_y, & u_z &= \mathbf{N}_{iso} \mathbf{u}_z, \end{aligned} \quad (32.102)$$

with

$$\begin{aligned} \mathbf{x} &= [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8] \\ \mathbf{y} &= [y_1 \ y_2 \ y_3 \ y_4 \ y_5 \ y_6 \ y_7 \ y_8] \\ \mathbf{z} &= [z_1 \ z_2 \ z_3 \ z_4 \ z_5 \ z_6 \ z_7 \ z_8] \\ \mathbf{N}_{iso} &= [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6 \ N_7 \ N_8] \\ N_1 &= \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta), & N_2 &= \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta), \\ N_3 &= \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta), & N_4 &= \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta), \\ N_5 &= \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta), & N_6 &= \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta), \\ N_7 &= \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta), & N_8 &= \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta). \end{aligned} \quad (32.103)$$

Let $\mathbf{X} = [x \ y \ z]^T$ and $\mathbf{U} = [u_x \ u_y \ u_z]^T$. The natural transverse shear strains are defined as

$$\gamma_{\xi\eta} = \frac{\partial \mathbf{X}^T}{\partial \zeta} \frac{\partial \mathbf{U}}{\partial \xi} + \frac{\partial \mathbf{X}^T}{\partial \xi} \frac{\partial \mathbf{U}}{\partial \zeta}, \quad \gamma_{\xi\eta} = \frac{\partial \mathbf{X}^T}{\partial \zeta} \frac{\partial \mathbf{U}}{\partial \eta} + \frac{\partial \mathbf{X}^T}{\partial \eta} \frac{\partial \mathbf{U}}{\partial \zeta}, \quad (32.104)$$

According to geometric restriction stated in §7.1, the geometry of the brick is specialized to

$$\begin{aligned} \mathbf{x} &= [x_1^0 \ x_2^0 \ x_3^0 \ x_4^0 \ x_1^0 \ x_2^0 \ x_3^0 \ x_4^0] \\ \mathbf{y} &= [y_1^0 \ y_2^0 \ y_3^0 \ y_4^0 \ y_1^0 \ y_2^0 \ y_3^0 \ y_4^0] \\ \mathbf{z} &= [-\frac{1}{2}h_1 \ -\frac{1}{2}h_2 \ -\frac{1}{2}h_3 \ -\frac{1}{2}h_4 \ \frac{1}{2}h_1 \ \frac{1}{2}h_2 \ \frac{1}{2}h_3 \ \frac{1}{2}h_4] \end{aligned} \quad (32.105)$$

Insertion of (32.102), (32.103) and (32.105) into (32.104) yields a natural strain field that contains up to cubic terms such as $\xi^2\eta$. This is filtered by evaluating at the midpoints 5, 6, 7 and 8 to yield

$$\begin{bmatrix} \gamma_{\xi\eta} \\ \gamma_{\xi\xi} \end{bmatrix} = \mathbf{B}_{\gamma u} \mathbf{u}^{(e)}. \quad (32.106)$$

Here $\mathbf{u}^{(e)}$ are the solid shell node displacements arranged as per (32.23), and

$$\begin{aligned} \mathbf{B}_{\gamma u} &= \frac{1}{16} \begin{bmatrix} x_{14}(1-\xi) & y_{14}(1-\xi) & -\bar{h}_{41}(1-\xi) & x_{23}(1+\xi) & y_{23}(1+\xi) & -\bar{h}_{23}(1+\xi) \\ x_{12}(1-\eta) & y_{12}(1-\eta) & -\bar{h}_{12}(1-\eta) & x_{12}(1-\eta) & y_{12}(1-\eta) & \bar{h}_{12}(1-\eta) \\ x_{23}(1+\xi) & y_{23}(1+\xi) & \bar{h}_{23}(1+\xi) & x_{14}(1-\xi) & y_{14}(1-\xi) & -\bar{h}_{41}(1-\xi) \\ x_{43}(1+\eta) & y_{43}(1+\eta) & \bar{h}_{34}(1+\eta) & x_{43}(1+\eta) & y_{43}(1+\eta) & -\bar{h}_{34}(1+\eta) \\ x_{41}(1-\xi) & y_{41}(1-\xi) & -\bar{h}_{41}(1-\xi) & x_{32}(1+\xi) & y_{32}(1+\xi) & -\bar{h}_{23}(1+\xi) \\ x_{21}(1-\eta) & y_{21}(1-\eta) & -\bar{h}_{12}(1-\eta) & x_{21}(1-\eta) & y_{21}(1-\eta) & \bar{h}_{12}(1-\eta) \\ x_{32}(1+\xi) & y_{32}(1+\xi) & \bar{h}_{23}(1+\xi) & x_{41}(1-\xi) & y_{41}(1-\xi) & -\bar{h}_{41}(1-\xi) \\ x_{34}(1+\eta) & y_{34}(1+\eta) & \bar{h}_{34}(1+\eta) & x_{34}(1+\eta) & y_{34}(1+\eta) & -\bar{h}_{34}(1+\eta) \end{bmatrix} \end{aligned} \quad (32.107)$$

```

SS8ShearStrains[xyzcoord_, {ξ_, η_}, numer_] := Module[
  {x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8,
   z1, z2, z3, z4, z5, z6, z7, z8, x10, x20, x30, x40, y10, y20, y30, y40,
   h1, h2, h3, h4, h, h12m, h23m, h34m, h41m,
   x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43,
   y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43,
   ξm, ξp, ηm, ηp, hx1243, hx2314, hy1243, hy2314, Bγu, Y, Bs},
  {{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}, {x4, y4, z4},
   {x5, y5, z5}, {x6, y6, z6}, {x7, y7, z7}, {x8, y8, z8}} = xyzcoord;
  {x10, x20, x30, x40} = ({x5, x6, x7, x8} + {x1, x2, x3, x4})/2;
  {y10, y20, y30, y40} = ({y5, y6, y7, y8} + {y1, y2, y3, y4})/2;
  {h1, h2, h3, h4} = {z5 - z1, z6 - z2, z7 - z3, z8 - z4};
  {x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43} =
    {x10 - x20, x10 - x30, x10 - x40, x20 - x10, x20 - x30, x20 - x40,
     x30 - x10, x30 - x20, x30 - x40, x40 - x10, x40 - x20, x40 - x30};
  {y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43} =
    {y10 - y20, y10 - y30, y10 - y40, y20 - y10, y20 - y30, y20 - y40,
     y30 - y10, y30 - y20, y30 - y40, y40 - y10, y40 - y20, y40 - y30};
  h12m = (h1 + h2)/2; h23m = (h2 + h3)/2; h34m = (h3 + h4)/2;
  h41m = (h1 + h4)/2; ξm = 1 - ξ; ξp = 1 + ξ; ηm = 1 - η; ηp = 1 + η;
  Bγu = (1/16)*
  {{x14*ξm, y14*ξm, -h41m*ξm, x23*ξp, y23*ξp, -h23m*ξp,
    x23*ξp, y23*ξp, h23m*ξp, x14*ξm, y14*ξm, h41m*ξm,
    -x14*ξm, -y14*ξm, -h41m*ξm, x32*ξp, y32*ξp, -h23m*ξp,
    x32*ξp, y32*ξp, h23m*ξp, -x14*ξm, -y14*ξm, h41m*ξm},
   {x12*ηm, y12*ηm, -h12m*ηm, x12*ηm, y12*ηm, h12m*ηm,
    x43*ηp, y43*ηp, h34m*ηp, x43*ηp, y43*ηp, -h34m*ηp,
    -x12*ηm, -y12*ηm, -h12m*ηm, -x12*ηm, -y12*ηm, h12m*ηm,
    x34*ηp, y34*ηp, h34m*ηp, x34*ηp, y34*ηp, -h34m*ηp}};
  x2134 = x21*ηm + x34*ηp; y2134 = y21*ηm + y34*ηp;
  x3241 = x32*ξp + x41*ξm; y3241 = y32*ξp + y41*ξm;
  hx1243 = h12m*x12*ηm + h34m*x43*ηp;
  hx2314 = h23m*x23*ξp + h41m*x14*ξm;
  hy1243 = h12m*y12*ηm + h34m*y43*ηp;
  hy2314 = h23m*y23*ξp + h41m*y14*ξm;
  Y = 8*{{hx1243, -hx2314}, {-hy1243, hy2314}}/
    (hx2314*hy1243 - hx1243*hy2314);
  If [numer, Bγu = N[Bγu]; Y = N[Y]; Bs = Simplify[Y.Bγu];
  Return[Bs]];

```

Figure 24. *Mathematica* module for evaluating the shear-strain-displacement matrix $\mathbf{B}_s$.

in which $\bar{h}_{12} = \frac{1}{2}(h_1 + h_2)$, $\bar{h}_{23} = \frac{1}{2}(h_2 + h_3)$, $\bar{h}_{34} = \frac{1}{2}(h_3 + h_4)$ and $\bar{h}_{41} = \frac{1}{2}(h_1 + h_4)$.

The next step is more difficult and only possible with the help of a computer algebra package. Consider the basic-states matrix $\mathbf{G}_{rc}$ of the Free Formulation, given by (32.82). The first 6 columns pertain to rigid body modes, and the last 6 to constant strain states ordered e_{xx} , e_{yy} , e_{zz} , $2e_{xy}$, $2e_{yz}$ and $2e_{zx}$. Replace the coordinates (32.105) into (32.107) and require that

$$\gamma_z = \mathbf{YB}_{\gamma u} \mathbf{G}_{rc} = \mathbf{B}_s \mathbf{G}_{rc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad (32.108)$$

identically for any $\{\xi, \eta\}$ and any combination of coordinates and thicknesses. The meaning of (32.108) is: rigid body motions and constant strain states should produce zero transverse shear strains, except for the two uniform strain states (32.101), which for $C = 1$ are represented by the last two columns of $\mathbf{G}_{rc}$. This requirement provides a unique $\mathbf{Y}$, which turns out to be

$$\mathbf{Y} = \frac{8}{x_{2314}^h y_{1243}^h - x_{1243}^h y_{2314}^h} \begin{bmatrix} x_{1243}^h & -x_{2314}^h \\ -y_{1243}^h & y_{2314}^h \end{bmatrix} \quad (32.109)$$

in which

$$\begin{aligned} x_{1243}^h &= h_{12}^m x_{12}(1 - \eta) + h_{34}^m x_{43}(1 + \eta), & x_{2314}^h &= h_{23}^m x_{23}(1 + \xi) + h_{41}^m x_{14}(1 - \xi), \\ y_{1243}^h &= h_{12}^m y_{12}(1 - \eta) + h_{34}^m y_{43}(1 + \eta), & y_{2314}^h &= h_{23}^m y_{23}(1 + \xi) + h_{41}^m y_{14}(1 - \xi). \end{aligned} \quad (32.110)$$

§32.7.4. Transverse Shear Strain Implementation

The *Mathematica* module listed in Figure 24 implements the computation of the shear-strain-displacement matrix. The module is referenced as

$$\mathbf{B}_s = \text{SS8ShearStrains}[\text{xyzcoor}, \{\xi, \eta\}, \text{numer}] \quad (32.111)$$

where

- `xyzcoor` The local coordinates of the flattened brick, arranged as $\{\{x_1, x_2, \dots, x_8\}, \{y_1, y_2, \dots, y_8\}, \{z_1, z_2, \dots, z_8\}\}$. Here $z_1 = -h_1/2$, $z_2 = -h_2/2$, etc.
- $\{\xi, \eta\}$ Quadrilateral coordinates of midsurface point at which $\mathbf{B}_s$ will be evaluated.
- `numer` Logical flag. If True, compute $\mathbf{B}_s$ using floating point arithmetic.

The module returns $\mathbf{B}_s$, which is the 2×24 matrix $\mathbf{B}_s = \mathbf{Y}\mathbf{B}_{\gamma u}$.

§32.7.5. Shear Stiffness Matrix Implementation

The contribution of the transverse shear to the SS8 element stiffness is given by

$$\mathbf{K}_s = \int_{V^{(e)}} \mathbf{B}_s^T \bar{\mathbf{E}}_s \mathbf{B}_s dV = \int_{-1}^1 \int_{-1}^1 \mathbf{B}_s^T \bar{\mathbf{E}}_s \mathbf{B}_s J^Q h d\xi d\eta. \quad (32.112)$$

where $\bar{\mathbf{E}}_s$ is the thickness-integrated shear constitutive matrix defined in §5.4. This evaluation is done by a 2×2 Gauss integration rule:

$$\mathbf{K}_s = \sum_{i=1}^2 \sum_{j=1}^2 \mathbf{B}_g(\xi_i, \eta_j) \bar{\mathbf{E}}_s \mathbf{B}_g(\xi_i, \eta_j) J^Q(\xi_i, \eta_j) h(\xi_i, \eta_j) w_{ij}. \quad (32.113)$$

in which $\{\xi_i, \eta_j\}$ are abscissas $\pm\sqrt{1/3}$ of the sample points of the 2×2 Gauss rule, which has unit weights $w_{ij} = 1$.

The *Mathematica* module `SS8ShearStiffness` listed in Figure 25 implements the calculation of $\mathbf{K}_s$. The module is called by

$$\mathbf{K}_s = \text{SS8ShearStiffness}[\text{xyhcoor}, \text{ElayerS}, \text{layers}, \text{options}] \quad (32.114)$$

The arguments are:

- `xyzcoor` The local coordinates of the flattened brick, arranged as $\{\{x_1, x_2, \dots, x_8\}, \{y_1, y_2, \dots, y_8\}, \{z_1, z_2, \dots, z_8\}\}$. Here $z_1 = -h_1/2$, $z_2 = -h_2/2$, etc.
- `ElayerS` List of transverse-shear layer constitutive matrices. See section 5.
- `layers` Extents of layer thicknesses. See Section 5.
- `options` A list of two logical flags: $\{\text{numer}, \text{Jconst}\}$
 - `numer` If True carry calculations in floating point arithmetic
 - `Jconst` If True use a constant Jacobian $J^Q = J_0^Q = J^Q|_{\xi=0, \eta=0}$ and average thickness $h_0 = (h_1 + h_2 + h_3 + h_4)/4$ in the Gauss integration rule (32.113).

As an example, suppose the element is unit square with unit thickness and fabricated of a single layer of isotropic material with $G = 48$. The *Mathematica* statements


```

SS8StiffnessShear[xyzcoor_, ElayersS_, layers_, options_] :=
Module[{x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4,
y5, y6, y7, y8, z1, z2, z3, z4, z5, z6, z7, z8,
x10, x20, x30, x40, y10, y20, y30, y40, h1, h2, h3, h4, h,
x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43,
y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43,
numer, Jconst, ξ, η, ξm, ξp, ηm, ηp, NQ1, NQ2, NQ3, NQ4,
x2134, y3241, x3241, y2134, R, WR, Rcorr,
Eshr, J, Jg1, Jg2, Jg3, Jg4, Bg1, Bg2, Bg3, Bg4,
g1=1/Sqrt[3], gp1, gp2, gp3, gp4, Ks}, Rcorr=False;
{{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}, {x4, y4, z4},
{x5, y5, z5}, {x6, y6, z6}, {x7, y7, z7}, {x8, y8, z8}}=xyzcoor;
{x10, x20, x30, x40}={x5, x6, x7, x8}+{x1, x2, x3, x4}/2;
{y10, y20, y30, y40}={y5, y6, y7, y8}+{y1, y2, y3, y4}/2;
{h1, h2, h3, h4}={z5-z1, z6-z2, z7-z3, z8-z4};
{x12, x13, x14, x21, x23, x24, x31, x32, x34, x41, x42, x43}=
{x10-x20, x10-x30, x10-x40, x20-x10, x20-x30, x20-x40,
x30-x10, x30-x20, x30-x40, x40-x10, x40-x20, x40-x30};
{y12, y13, y14, y21, y23, y24, y31, y32, y34, y41, y42, y43}=
{y10-y20, y10-y30, y10-y40, y20-y10, y20-y30, y20-y40,
y30-y10, y30-y20, y30-y40, y40-y10, y40-y20, y40-y30};
gp1={ξ->-g1, η->-g1}; gp2={ξ->-g1, η->-g1};
gp3={ξ->-g1, η->-g1}; gp4={ξ->-g1, η->-g1};
{numer, Jconst}=options;
Eshr=SS8ThickIntegS[ElayersS, layers];
Bg1=SS8ShearStrains[xyzcoor, {-g1, -g1}, numer];
Bg2=SS8ShearStrains[xyzcoor, {g1, -g1}, numer];
Bg3=SS8ShearStrains[xyzcoor, {g1, g1}, numer];
Bg4=SS8ShearStrains[xyzcoor, {-g1, g1}, numer];
If [Rcorr,
R=Transpose[
{{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0},
{0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0},
{0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0},
{y1, -x1, 0, y2, -x2, 0, y3, -x3, 0, y4, -x4, 0,
y5, -x5, 0, y6, -x6, 0, y7, -x7, 0, y8, -x8, 0},
{0, z1, -y1, 0, z2, -y2, 0, z3, -y3, 0, z4, -y4,
0, z5, -y5, 0, z6, -y6, 0, z7, -y7, 0, z8, -y8},
{-z1, 0, x1, -z2, 0, x2, -z3, 0, x3, -z4, 0, x4,
-z5, 0, x5, -z6, 0, x6, -z7, 0, x7, -z8, 0, x8}}];
If [numer, R=N[R]]; W=Simplify[Transpose[R].R];
WR=Inverse[W].Transpose[R];
Bg1=Bg1-(Bg1.R).WR; Bg2=Bg2-(Bg2.R).WR;
Bg3=Bg3-(Bg3.R).WR; Bg4=Bg4-(Bg4.R).WR];
ξm=1-ξ; ξp=1+ξ; ηm=1-η; ηp=1+η;
NQ1=ξm*ηm/4; NQ2=ξp*ηm/4; NQ3=ξp*ηp/4; NQ4=ξm*ηp/4;
h=Simplify[h1*NQ1+h2*NQ2+h3*NQ3+h4*NQ4];
x2134=x21*ηm+x34*ηp; y2134=y21*ηm+y34*ηp;
x3241=x32*ξp+x41*ξm; y3241=y32*ξp+y41*ξm;
J=Simplify[h*(x2134*y3241-x3241*y2134)/16];
h0=(h1+h2+h3+h4)/4;
Jg1=J/.gp1; Jg2=J/.gp2; Jg3=J/.gp3; Jg4=J/.gp4;
If [Jconst, Jg1=Jg2=Jg3=Jg4=J/.{ξ->0, η->0}; h=h0];
{Jg1, Jg2, Jg3, Jg4}=Simplify[{Jg1, Jg2, Jg3, Jg4}];
Ks= Transpose[Bg1].Eshr.Bg1*Jg1+Transpose[Bg2].Eshr.Bg2*Jg2+
Transpose[Bg3].Eshr.Bg3*Jg3+Transpose[Bg4].Eshr.Bg4*Jg4;
If [numer, Return[Ks], Return[Simplify[Ks]] ] ];

```

Figure 25. Mathematica implementation of transverse shear stiffness computations.

```

a=b=1; {x1,x2,x3,x4}={-a,a,a,-a}/2; {y1,y2,y3,y4}={-b,-b,b,b}/2;
h0=1; {h1,h2,h3,h4}={h0,h0,h0,h0};
xyhcoor={{x1,y1,h1},{x2,y2,h2},{x3,y3,h3},{x4,y4,h4}}; Eshr={{48,0},{0,48}};
Ks=SS8ShearStiffness[xyhcoor,{Eshr},{1},{False,True}];
Print["Ks=",Ks//InputForm];

```

are run producing

$$\mathbf{K}_s = 2 \begin{bmatrix} 2 & 0 & 2 & 2 & 0 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & -2 & 0 & 2 & -2 & 0 & -2 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 2 & -2 & 0 & -2 & 2 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & -2 & -2 \\ 2 & 2 & 4 & 2 & 1 & -1 & 1 & 1 & -2 & 1 & 2 & -1 & -2 & -2 & 4 & -2 & -1 & -1 & -1 & -2 & -1 & -2 & -1 \\ 2 & 0 & 2 & 2 & 0 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & -2 & 0 & 2 & -2 & 0 & -2 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 & 2 & -2 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & -2 & 2 & 0 & -2 & -2 & 0 & -1 & -1 \\ -2 & 1 & -1 & -2 & 2 & 4 & -1 & 2 & -1 & -1 & 1 & -2 & 2 & -1 & -1 & 2 & -2 & 4 & 1 & -2 & -1 & 1 & -1 & -2 \\ 1 & 0 & 1 & 1 & 0 & -1 & 2 & 0 & -2 & 2 & 0 & 2 & -1 & 0 & 1 & -1 & 0 & -1 & -2 & 0 & -2 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 & 2 & -2 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & -2 & 2 & 0 & -2 & -2 & 0 & -1 & -1 \\ -1 & -1 & -2 & -1 & -2 & -1 & -2 & -2 & 4 & -2 & -1 & -1 & 1 & 1 & -2 & 1 & 2 & -1 & 2 & 2 & 4 & 2 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 & -1 & 2 & 0 & -2 & 2 & 0 & 2 & -1 & 0 & 1 & -1 & 0 & -1 & -2 & 0 & -2 & -2 & 0 & 2 \\ 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 2 & -2 & 0 & -2 & 2 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & -2 & -2 \\ 1 & -2 & -1 & 1 & -1 & -2 & 2 & -1 & -1 & 2 & -2 & 4 & -1 & 2 & -1 & -1 & 1 & -2 & -2 & 1 & -1 & -2 & 2 & 4 \\ -2 & 0 & -2 & -2 & 0 & 2 & -1 & 0 & 1 & -1 & 0 & -1 & 2 & 0 & -2 & 2 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -2 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & -2 & 2 & 0 & 2 & -2 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 2 & 2 \\ 2 & 2 & 4 & 2 & 1 & -1 & 1 & 1 & -2 & 1 & 2 & -1 & -2 & -2 & 4 & -2 & -1 & -1 & -1 & -1 & -2 & -1 & -2 & -1 \\ -2 & 0 & -2 & -2 & 0 & 2 & -1 & 0 & 1 & -1 & 0 & -1 & 2 & 0 & -2 & 2 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -2 & -2 & 0 & -2 & 2 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 2 & -2 & 0 & 2 & 2 & 0 & 1 & 1 \\ -2 & 1 & -1 & -2 & 2 & 4 & -1 & 2 & -1 & -1 & 1 & -2 & 2 & -1 & -1 & 2 & -2 & 4 & 1 & -2 & -1 & 1 & -1 & -2 \\ -1 & 0 & -1 & -1 & 0 & 1 & -2 & 0 & 2 & -2 & 0 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 2 & 0 & 2 & 2 & 0 & -2 \\ 0 & -1 & -1 & 0 & -2 & -2 & 0 & -2 & 2 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 2 & -2 & 0 & 2 & 2 & 0 & 1 & 1 \\ -1 & -1 & -2 & -1 & -2 & -1 & -2 & -2 & 4 & -2 & -1 & -1 & 1 & 1 & -2 & 1 & 2 & -1 & 2 & 2 & 4 & 2 & 1 & -1 \\ -1 & 0 & -1 & -1 & 0 & 1 & -2 & 0 & 2 & -2 & 0 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 2 & 0 & 2 & 2 & 0 & -2 \\ 0 & -2 & -2 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & -2 & 2 & 0 & 2 & -2 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 2 & 2 \\ 1 & -2 & -1 & 1 & -1 & -2 & 2 & -1 & -1 & 2 & -2 & 4 & -1 & 2 & -1 & -1 & 1 & -2 & -2 & 1 & -1 & -2 & 2 & 4 \end{bmatrix} \quad (32.115)$$

This matrix has rank four, with nonzero eigenvalues 12, 24, 48, 48.

§32.7.6. RBM Cleanup

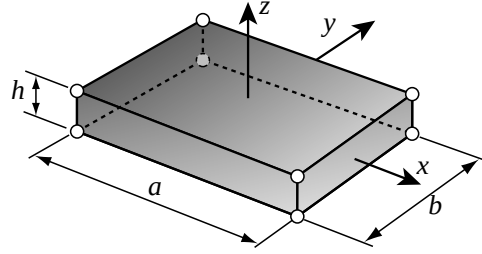
If the foregoing shear stiffness matrix is used for a thickness tapered element, two defects are detected upon transforming to global coordinates:

1. $\mathbf{K}_s$ becomes “polluted” with respect to rigid body modes (RBMs). Mathematically, let $\mathbf{R}$ denote the 24×6 RBM matrix for the original 8-node hexahedron geometry. $\mathbf{R}$ is constructed by taking the first 6 columns of (32.82) and replacing the global node coordinates. It is found that $\mathbf{K}_s \mathbf{R} \neq \mathbf{0}$. Physically: a RBM motion produces nonzero shear strains and hence nonzero node forces.
2. The shear and inplane strains interlock to stiffen the element in inextensional bending modes.

The first defect is cured by the projection method described in [10]. Let

$$\mathbf{P} = \mathbf{I} - \mathbf{R}(\mathbf{R}\mathbf{R})^{-1}\mathbf{R}^T = \mathbf{P}^T, \quad (32.116)$$

be the projector that filters out rigid body motions, in which $\mathbf{I}$ is the identity matrix of order 24. Then $\mathbf{P}\mathbf{K}_s\mathbf{P}$ is RBM clean. In the implementation of Figure 25 the projector is not applied to $\mathbf{K}_s$ but to $\mathbf{B}_s$: $\mathbf{B}_s\mathbf{P}$ is used in the numerical integration process that produces $\mathbf{K}_s$. The cure for the second defect: inextensional bending locking, is discussed in §10.3.

Figure 26. Rectangular parallelepiped element dimensioned $a \times b \times h$.

§32.8. A Special Geometry

§32.8.1. Stiffness of an Isotropic Rectangular-Prismatic Element

Consider the element of rectangular-prismatic geometry of inplane dimensions $a \times b$, constant thickness h and one layer of isotropic material with elastic modulus E and Poisson's ratio ν , as shown in Figure 26. The element stiffness matrix in local coordinates was computed in closed symbolic form by *Mathematica*.

This closed form is useful for spot checks of new implementations, as well as for investigations such as the torsional stiffness analysis reported in the next subsection. The stiffness entries were generated and compared symbolically for duplication; then *Mathematica* generated the necessary $\text{\TeX}$ expressions listed below.

$$\begin{aligned}
 K_{1,1}^{(e)} &= K_{4,4}^{(e)} = K_{7,7}^{(e)} = K_{10,10}^{(e)} = K_{13,13}^{(e)} = K_{16,16}^{(e)} = K_{19,19}^{(e)} = K_{22,22}^{(e)} = \\
 &\quad 2E(3a^2(b^2 + h^2)(1 - 3\nu + 2\nu^2) + 2b^2h^2(4 - 8\nu + 3\nu^2))/K_{fac}, \\
 K_{1,2}^{(e)} &= K_{4,11}^{(e)} = K_{5,10}^{(e)} = K_{7,8}^{(e)} = K_{13,14}^{(e)} = K_{16,23}^{(e)} = K_{17,22}^{(e)} = K_{19,20}^{(e)} = \\
 &\quad -3aEbh^2(-2 + 2\nu + \nu^2)/K_{fac}, \\
 K_{1,3}^{(e)} &= K_{4,15}^{(e)} = K_{6,13}^{(e)} = K_{7,24}^{(e)} = K_{9,22}^{(e)} = K_{10,12}^{(e)} = K_{16,18}^{(e)} = K_{19,21}^{(e)} = \\
 &\quad -6aEb^2h(-1 + \nu)/K_{fac}, \\
 K_{1,4}^{(e)} &= K_{7,10}^{(e)} = K_{13,16}^{(e)} = K_{19,22}^{(e)} = \\
 &\quad 2E(3a^2(b^2 + h^2)(1 - 3\nu + 2\nu^2) - 2b^2h^2(4 - 8\nu + 3\nu^2))/K_{fac}, \\
 K_{1,5}^{(e)} &= K_{2,10}^{(e)} = K_{4,8}^{(e)} = K_{7,11}^{(e)} = K_{13,17}^{(e)} = K_{14,22}^{(e)} = K_{16,20}^{(e)} = K_{19,23}^{(e)} = \\
 &\quad -3aEbh^2(2 - 10\nu + 9\nu^2)/K_{fac}, \\
 K_{1,6}^{(e)} &= K_{3,13}^{(e)} = K_{4,18}^{(e)} = K_{7,21}^{(e)} = K_{9,10}^{(e)} = K_{12,22}^{(e)} = K_{15,16}^{(e)} = K_{19,24}^{(e)} = \\
 &\quad -6aEb^2h(1 - \nu)(1 - 4\nu)/K_{fac}, \\
 K_{1,7}^{(e)} &= K_{4,10}^{(e)} = K_{13,19}^{(e)} = K_{16,22}^{(e)} = \\
 &\quad E(3a^2(b^2 - 2h^2)(1 - 3\nu + 2\nu^2) - 2b^2h^2(4 - 8\nu + 3\nu^2))/K_{fac}, \\
 K_{1,8}^{(e)} &= K_{2,7}^{(e)} = K_{4,5}^{(e)} = K_{10,11}^{(e)} = K_{13,20}^{(e)} = K_{14,19}^{(e)} = K_{16,17}^{(e)} = K_{22,23}^{(e)} = \\
 &\quad 3aEbh^2(-2 + 2\nu + \nu^2)/K_{fac}, \\
 K_{1,9}^{(e)} &= K_{3,22}^{(e)} = K_{4,21}^{(e)} = K_{6,10}^{(e)} = K_{7,18}^{(e)} = K_{12,13}^{(e)} = K_{15,19}^{(e)} = K_{16,24}^{(e)} = \\
 &\quad -3aEb^2h(1 - \nu)(1 - 4\nu)/K_{fac}, \\
 K_{1,10}^{(e)} &= K_{4,7}^{(e)} = K_{13,22}^{(e)} = K_{16,19}^{(e)} = \\
 &\quad E(3a^2(b^2 - 2h^2)(1 - 3\nu + 2\nu^2) + 2b^2h^2(4 - 8\nu + 3\nu^2))/K_{fac},
 \end{aligned} \tag{32.117}$$

$$\begin{aligned}
K_{1,11}^{(e)} &= K_{2,4}^{(e)} = K_{5,7}^{(e)} = K_{8,10}^{(e)} = K_{13,23}^{(e)} = K_{14,16}^{(e)} = K_{17,19}^{(e)} = K_{20,22}^{(e)} = \\
&\quad 3aEbh^2(2 - 10\nu + 9\nu^2)/K_{fac}, \\
K_{1,12}^{(e)} &= K_{3,10}^{(e)} = K_{4,24}^{(e)} = K_{6,22}^{(e)} = K_{7,15}^{(e)} = K_{9,13}^{(e)} = K_{16,21}^{(e)} = K_{18,19}^{(e)} = \\
&\quad -3aEb^2h(-1 + \nu)/K_{fac}, \\
K_{1,13}^{(e)} &= K_{4,16}^{(e)} = K_{7,19}^{(e)} = K_{10,22}^{(e)} = \\
&\quad E(-3a^2(2b^2 - h^2)(1 - 3\nu + 2\nu^2) + 4b^2h^2(2 - 4\nu + 3\nu^2))/K_{fac}, \\
K_{1,14}^{(e)} &= K_{2,13}^{(e)} = K_{4,23}^{(e)} = K_{5,22}^{(e)} = K_{7,20}^{(e)} = K_{8,19}^{(e)} = K_{10,17}^{(e)} = K_{11,16}^{(e)} = \\
&\quad 3aEbh^2(1 - \nu + \nu^2)/K_{fac}, \\
K_{1,15}^{(e)} &= K_{3,4}^{(e)} = K_{6,16}^{(e)} = K_{7,12}^{(e)} = K_{9,19}^{(e)} = K_{10,24}^{(e)} = K_{13,18}^{(e)} = K_{21,22}^{(e)} = \\
&\quad 6aEb^2h(1 - \nu)(1 - 4\nu)/K_{fac}, \\
K_{1,16}^{(e)} &= K_{4,13}^{(e)} = K_{7,22}^{(e)} = K_{10,19}^{(e)} = \\
&\quad -(E(3a^2(2b^2 - h^2)(1 - 3\nu + 2\nu^2) + 4b^2h^2(2 - 4\nu + 3\nu^2)))/K_{fac}, \\
K_{1,17}^{(e)} &= K_{2,22}^{(e)} = K_{4,20}^{(e)} = K_{5,13}^{(e)} = K_{7,23}^{(e)} = K_{8,16}^{(e)} = K_{10,14}^{(e)} = K_{11,19}^{(e)} = \\
&\quad -3aEbh^2(1 - 5\nu + 3\nu^2)/K_{fac}, \\
K_{1,18}^{(e)} &= K_{3,16}^{(e)} = K_{4,6}^{(e)} = K_{7,9}^{(e)} = K_{10,21}^{(e)} = K_{12,19}^{(e)} = K_{13,15}^{(e)} = K_{22,24}^{(e)} = \\
&\quad 6aEb^2h(-1 + \nu)/K_{fac}, \\
K_{1,19}^{(e)} &= K_{4,22}^{(e)} = K_{7,13}^{(e)} = K_{10,16}^{(e)} = \\
&\quad -(E(3a^2(b^2 + h^2)(1 - 3\nu + 2\nu^2) + 2b^2h^2(2 - 4\nu + 3\nu^2)))/K_{fac}, \\
K_{1,20}^{(e)} &= K_{2,19}^{(e)} = K_{4,17}^{(e)} = K_{5,16}^{(e)} = K_{7,14}^{(e)} = K_{8,13}^{(e)} = K_{10,23}^{(e)} = K_{11,22}^{(e)} = \\
&\quad -3aEbh^2(1 - \nu + \nu^2)/K_{fac}, \\
K_{1,21}^{(e)} &= K_{3,19}^{(e)} = K_{4,9}^{(e)} = K_{6,7}^{(e)} = K_{10,18}^{(e)} = K_{12,16}^{(e)} = K_{13,24}^{(e)} = K_{15,22}^{(e)} = \\
&\quad 3aEb^2h(-1 + \nu)/K_{fac}, \\
K_{1,22}^{(e)} &= K_{4,19}^{(e)} = K_{7,16}^{(e)} = K_{10,13}^{(e)} = \\
&\quad E(-3a^2(b^2 + h^2)(1 - 3\nu + 2\nu^2) + 2b^2h^2(2 - 4\nu + 3\nu^2))/K_{fac}, \\
K_{1,23}^{(e)} &= K_{2,16}^{(e)} = K_{4,14}^{(e)} = K_{5,19}^{(e)} = K_{7,17}^{(e)} = K_{8,22}^{(e)} = K_{10,20}^{(e)} = K_{11,13}^{(e)} = \\
&\quad 3aEbh^2(1 - 5\nu + 3\nu^2)/K_{fac}, \\
K_{1,24}^{(e)} &= K_{3,7}^{(e)} = K_{4,12}^{(e)} = K_{6,19}^{(e)} = K_{9,16}^{(e)} = K_{10,15}^{(e)} = K_{13,21}^{(e)} = K_{18,22}^{(e)} = \\
&\quad 3aEb^2h(1 - \nu)(1 - 4\nu)/K_{fac}, \\
K_{2,2}^{(e)} &= K_{5,5}^{(e)} = K_{8,8}^{(e)} = K_{11,11}^{(e)} = K_{14,14}^{(e)} = K_{17,17}^{(e)} = K_{20,20}^{(e)} = K_{23,23}^{(e)} = \\
&\quad 2E(3b^2h^2(1 - 3\nu + 2\nu^2) + a^2(2h^2(4 - 8\nu + 3\nu^2) + b^2(3 - 9\nu + 6\nu^2)))/K_{fac}, \\
K_{2,3}^{(e)} &= K_{5,6}^{(e)} = K_{8,18}^{(e)} = K_{9,17}^{(e)} = K_{11,15}^{(e)} = K_{12,14}^{(e)} = K_{20,21}^{(e)} = K_{23,24}^{(e)} = \\
&\quad -6a^2Ebh(-1 + \nu)/K_{fac}, \\
K_{2,5}^{(e)} &= K_{8,11}^{(e)} = K_{14,17}^{(e)} = K_{20,23}^{(e)} = \\
&\quad E(-6b^2h^2(1 - 3\nu + 2\nu^2) + a^2(2h^2(4 - 8\nu + 3\nu^2) + b^2(3 - 9\nu + 6\nu^2)))/K_{fac}, \\
K_{2,6}^{(e)} &= K_{3,5}^{(e)} = K_{8,15}^{(e)} = K_{9,14}^{(e)} = K_{11,18}^{(e)} = K_{12,17}^{(e)} = K_{20,24}^{(e)} = K_{21,23}^{(e)} = \\
&\quad -3a^2Ebh(-1 + \nu)/K_{fac}, \\
K_{2,8}^{(e)} &= K_{5,11}^{(e)} = K_{14,20}^{(e)} = K_{17,23}^{(e)} = \\
&\quad E(-6b^2h^2(1 - 3\nu + 2\nu^2) + a^2(-2h^2(4 - 8\nu + 3\nu^2) + b^2(3 - 9\nu + 6\nu^2)))/K_{fac}, \\
K_{2,9}^{(e)} &= K_{3,17}^{(e)} = K_{5,12}^{(e)} = K_{6,14}^{(e)} = K_{8,24}^{(e)} = K_{11,21}^{(e)} = K_{15,20}^{(e)} = K_{18,23}^{(e)} = \\
&\quad -3a^2Ebh(1 - \nu)(1 - 4\nu)/K_{fac},
\end{aligned} \tag{32.118}$$

$$\begin{aligned}
K_{2,11}^{(e)} &= K_{5,8}^{(e)} = K_{14,23}^{(e)} = K_{17,20}^{(e)} = \\
&\quad 2E(3b^2h^2(1-3\nu+2\nu^2) + a^2(-2h^2(4-8\nu+3\nu^2) + b^2(3-9\nu+6\nu^2)))/K_{fac}, \\
K_{2,12}^{(e)} &= K_{3,14}^{(e)} = K_{5,9}^{(e)} = K_{6,17}^{(e)} = K_{8,21}^{(e)} = K_{11,24}^{(e)} = K_{15,23}^{(e)} = K_{18,20}^{(e)} = \\
&\quad -6a^2Ebh(1-\nu)(1-4\nu)/K_{fac}, \\
K_{2,14}^{(e)} &= K_{5,17}^{(e)} = K_{8,20}^{(e)} = K_{11,23}^{(e)} = \\
&\quad E(3b^2h^2(1-3\nu+2\nu^2) - 2a^2(-2h^2(2-4\nu+3\nu^2) + b^2(3-9\nu+6\nu^2)))/K_{fac}, \\
K_{2,15}^{(e)} &= K_{3,11}^{(e)} = K_{5,18}^{(e)} = K_{6,8}^{(e)} = K_{9,20}^{(e)} = K_{12,23}^{(e)} = K_{14,24}^{(e)} = K_{17,21}^{(e)} = \\
&\quad 6a^2Ebh(1-\nu)(1-4\nu)/K_{fac}, \\
K_{2,17}^{(e)} &= K_{5,14}^{(e)} = K_{8,23}^{(e)} = K_{11,20}^{(e)} = \\
&\quad E(-3b^2h^2(1-3\nu+2\nu^2) + a^2(b^2(-3+9\nu-6\nu^2) + 2h^2(2-4\nu+3\nu^2)))/K_{fac}, \\
K_{2,18}^{(e)} &= K_{3,8}^{(e)} = K_{5,15}^{(e)} = K_{6,11}^{(e)} = K_{9,23}^{(e)} = K_{12,20}^{(e)} = K_{14,21}^{(e)} = K_{17,24}^{(e)} = \\
&\quad 3a^2Ebh(1-\nu)(1-4\nu)/K_{fac}, \\
K_{2,20}^{(e)} &= K_{5,23}^{(e)} = K_{8,14}^{(e)} = K_{11,17}^{(e)} = \\
&\quad -(E(3b^2h^2(1-3\nu+2\nu^2) + a^2(2h^2(2-4\nu+3\nu^2) + b^2(3-9\nu+6\nu^2))))/K_{fac}, \\
K_{2,21}^{(e)} &= K_{3,20}^{(e)} = K_{5,24}^{(e)} = K_{6,23}^{(e)} = K_{8,12}^{(e)} = K_{9,11}^{(e)} = K_{14,18}^{(e)} = K_{15,17}^{(e)} = \\
&\quad 3a^2Ebh(-1+\nu)/K_{fac}, \\
K_{2,23}^{(e)} &= K_{5,20}^{(e)} = K_{8,17}^{(e)} = K_{11,14}^{(e)} = \\
&\quad E(3b^2h^2(1-3\nu+2\nu^2) - 2a^2(2h^2(2-4\nu+3\nu^2) + b^2(3-9\nu+6\nu^2)))/K_{fac}, \\
K_{2,24}^{(e)} &= K_{3,23}^{(e)} = K_{5,21}^{(e)} = K_{6,20}^{(e)} = K_{8,9}^{(e)} = K_{11,12}^{(e)} = K_{14,15}^{(e)} = K_{17,18}^{(e)} = \\
&\quad 6a^2Ebh(-1+\nu)/K_{fac}, \\
K_{3,3}^{(e)} &= K_{6,6}^{(e)} = K_{9,9}^{(e)} = K_{12,12}^{(e)} = K_{15,15}^{(e)} = K_{18,18}^{(e)} = K_{21,21}^{(e)} = K_{24,24}^{(e)} = \\
&\quad -2E(1-\nu)(3b^2h^2(-1+2\nu) + a^2(8b^2(-1+\nu) + 3h^2(-1+2\nu)))/K_{fac}, \\
K_{3,6}^{(e)} &= K_{9,12}^{(e)} = K_{15,18}^{(e)} = K_{21,24}^{(e)} = \\
&\quad -E(1-\nu)(6b^2h^2(1-2\nu) + a^2(8b^2(-1+\nu) + 3h^2(-1+2\nu)))/K_{fac}, \\
K_{3,9}^{(e)} &= K_{6,12}^{(e)} = K_{15,21}^{(e)} = K_{18,24}^{(e)} = \\
&\quad E(a^2(3h^2(1-2\nu) + 4b^2(-1+\nu)) + 3b^2h^2(1-2\nu))(-1+\nu)/K_{fac}, \\
K_{3,12}^{(e)} &= K_{6,9}^{(e)} = K_{15,24}^{(e)} = K_{18,21}^{(e)} = \\
&\quad -E(1-\nu)(2a^2(3h^2(1-2\nu) + 4b^2(-1+\nu)) + 3b^2h^2(-1+2\nu))/K_{fac}, \\
K_{3,15}^{(e)} &= K_{6,18}^{(e)} = K_{9,21}^{(e)} = K_{12,24}^{(e)} = \\
&\quad -2E(1-\nu)(3b^2h^2(-1+2\nu) + a^2(-8b^2(-1+\nu) + 3h^2(-1+2\nu)))/K_{fac}, \\
K_{3,18}^{(e)} &= K_{6,15}^{(e)} = K_{9,24}^{(e)} = K_{12,21}^{(e)} = \\
&\quad -E(1-\nu)(6b^2h^2(1-2\nu) + a^2(-8b^2(-1+\nu) + 3h^2(-1+2\nu)))/K_{fac}, \\
K_{3,21}^{(e)} &= K_{6,24}^{(e)} = K_{9,15}^{(e)} = K_{12,18}^{(e)} = \\
&\quad E(a^2(3h^2(1-2\nu) - 4b^2(-1+\nu)) + 3b^2h^2(1-2\nu))(-1+\nu)/K_{fac}, \\
K_{3,24}^{(e)} &= K_{6,21}^{(e)} = K_{9,18}^{(e)} = K_{12,15}^{(e)} = \\
&\quad -E(1-\nu)(a^2(6h^2(1-2\nu) - 8b^2(-1+\nu)) + 3b^2h^2(-1+2\nu))/K_{fac},
\end{aligned} \tag{32.119}$$

The common denominator is $K_{fac} = 144abh(1-\nu^2)(1-2\nu)$. Only 48 entries (of the $24 \times 25/2 = 300$ entries of the upper triangle) are different.

For this geometry the α and β parameters of the membrane strain field introduced in §6.3 do not have any influence on the stiffness.

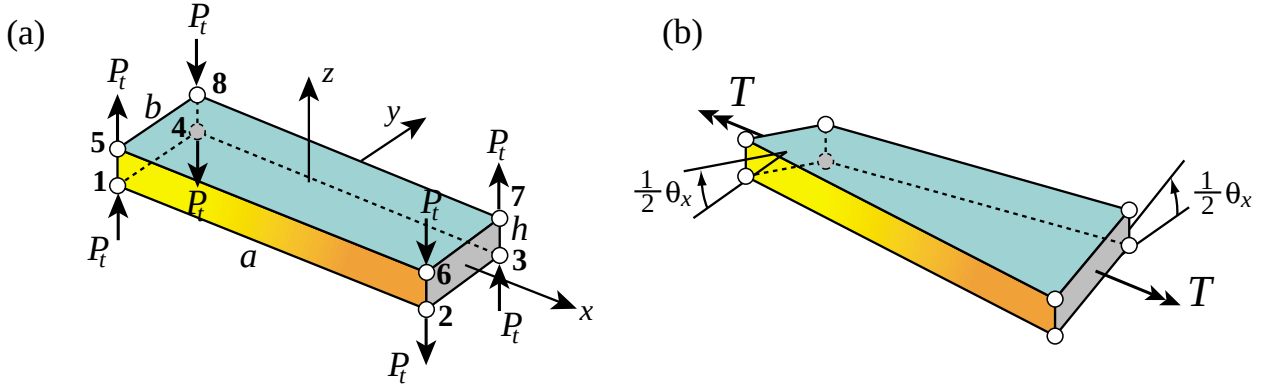


Figure 27. SS8 torsion response test.

Using the foregoing analytical form it is easily to verify symbolically that a single element of this (rectangular prismatic) geometry and homogeneous-isotropic wall construction is exact for any $\{a, b, h, E\}$ for the following actions:

- (i) Uniform in-plane stretching (any ν)
- (ii) Pure in-plane bending (if $\nu = 0$)
- (iii) Pure out-of-plane bending (if $\nu = 0$)
- (iv) Pure transverse shear (any ν)

Response exactness for (ii) and (iii) is not achievable with the standard 8-node brick element. This should not come as a surprise since the SS8 element was designed explicitly to be exact under those conditions, which are important for thin and thick shells. But the behavior under torsion was not explicitly considered in the element design. It remains to be checked *a posteriori*. This is done below.

§32.8.2. Torsion Response of Individual Element

To check torsion consider an element as in Figure 27, torqued about the local x axis. Apply the twist force system $\mathbf{f}_t$ shown in Figure 27(a). The eight P_z loads are equivalent to a torque $T = 4P_z(b/2) = 2P_z b$ about the x axis. Solve $\mathbf{K}^{(e)}\mathbf{u}_t = \mathbf{f}_t$ for $\mathbf{u}_t$ using the free-free flexibility to project out the rigid body motions [9,10]. The computed node displacements form the pattern

$$\mathbf{u}_t^T = [-u_x \ -u_y \ u_z \ -u_x \ u_y \ -u_z \ u_x \ u_y \ u_z \ u_x \ -u_y \ -u_z \ u_x \ u_y \ u_z \ u_x \ -u_y \ -u_z] \quad (32.120)$$

where

$$u_z = \frac{3}{2} \frac{a^2(b^2 + h^2) + b^2h^2}{Gabh^3} P_z, \quad u_x = \frac{a^2(b^2 - h^2) + b^2h^2}{a^3(b^2 + h^2) + ab^2h^2} hu_z, \quad u_y = \frac{a^2(b^2 + h^2) - b^2h^2}{a^2b(b^2 + h^2) + b^3h^2} hu_z. \quad (32.121)$$

The twist angle is $\theta_x = u_z/b$; whence the twist in terms of T and the effective torsional rigidity are given by

$$\theta_x = \frac{3(b^2h^2 + a^2(b^2 + h^2))T}{Gab^3h^3} \stackrel{\text{def}}{=} \frac{Ta}{GJ_t}, \quad \rightarrow \quad J_t = \frac{1}{3} \frac{a^2b^3h^3}{a^2(b^2 + h^2) + b^2h^2}. \quad (32.122)$$

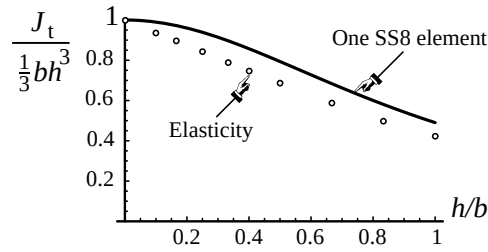


Figure 28. Effective J_t component of torsional rigidity $G J_t$ (normalized to $J_{thin} = \frac{1}{3}bh^3$) of individual SS8 element versus Saint-Venant's elasticity theory.

As $h \rightarrow 0$ while keeping a and b fixed, $J_t \rightarrow J_{thin} = \frac{1}{3}bh^3$, which is correct for a torqued thin rectangle. Consequently the SS8 element of this special geometry is torsion exact in the thin plate limit. For finite h/b the ratio J_t/J_{thin} is plotted in Figure 28, and compared to the J_t/J_{thin} given by the Saint-Venant's elasticity theory as tabulated in Timoshenko and Goodier [28].

The agreement is surprisingly good for two reasons: the element is very low order as regards shear stress representation, and no special tuning was done for the torsional response.⁹

⁹ Torsion tuning for this particular element is actually impossible because the transverse shear strain field, which dominates the torsional response, has no free parameters.

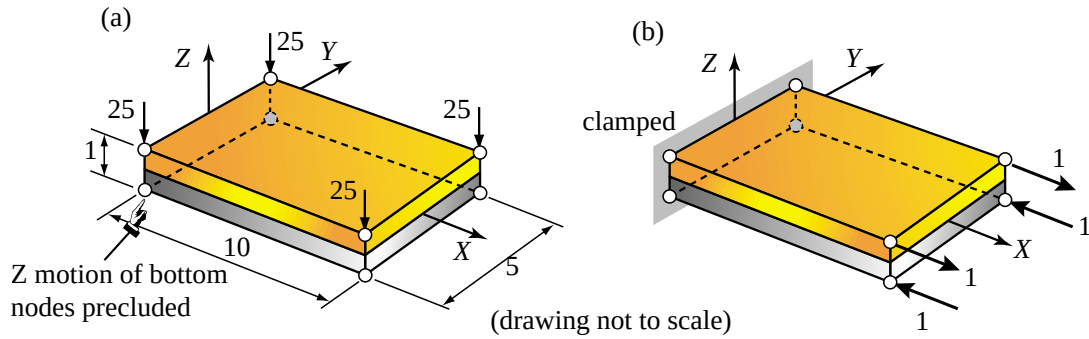


Figure 29. Two ply rectangular plate modeled by one element:
(a) transverse compression tests, (b) pure bending test.

§32.9. Numerical Tests

§32.9.1. Patch Tests

The SS8 element passes all rigid-body motion and constant strain mode patch tests exactly if it is thickness prismatic and of constant thickness. It passes rigid body motion tests for arbitrary geometries. It passes some higher order patch tests for rectangular planforms of constant thickness.

For bending patch tests in tapered thickness geometries see Sections 6 and 7. For the torsion test, which may be viewed as a special high order patch test, see §8.2.

§32.9.2. Invariance

The element is invariant with respect to global node numbering. This was a concern in the initial development, since for expediency the inplane stiffness component was treated with a selective-reduced integration scheme [18], which turned out to depend on node numbering.¹⁰ This problem was eliminated later when the development of the membrane field discussed in Section 7 permitted use of a standard 2×2 Gauss integration on the midsurface.

§32.9.3. Two-Ply Rectangular Plate

This test has been proposed by Sze, Yao and Cheung [27], and is used with some modifications, notably use of symbolic computation. The rectangular-prismatic 2-ply laminated plate shown in Figure 29, which is dimensioned $10 \times 5 \times 1$, is modeled with one SS8 element. Both layers are isotropic but have different properties: E_1 and ν_1 for the bottom layer, and E_2 and ν_2 for the top layer. Two types of tests are performed.

With the bottom surface restrained in the Z direction, but able to move inplane, the rectangular block is subjected to a pressure $p_z = 2$ on the top surface. This is lumped to four $f_{zi} = 25$ node forces as shown in Figure 29(a). The displacement results are exact for any combination $\{E_1, \nu_1, E_2, \nu_2\}$.

The $X = 0$ face is clamped and the plate is subjected to a pure bending load as shown in Figure 29(b). The mean tip deflection agrees closely with composite beam theory, and is exact if $\nu_1 = \nu_2 = 0$ because

¹⁰ Because the choice of local axes $\{x, y\}$ depends on which corner is numbered first. Selective integration relies on picking up a shear strain component to be 1-point integrated. This necessarily brings directional anisotropy, which can be highly disturbing for anisotropic laminate construction.

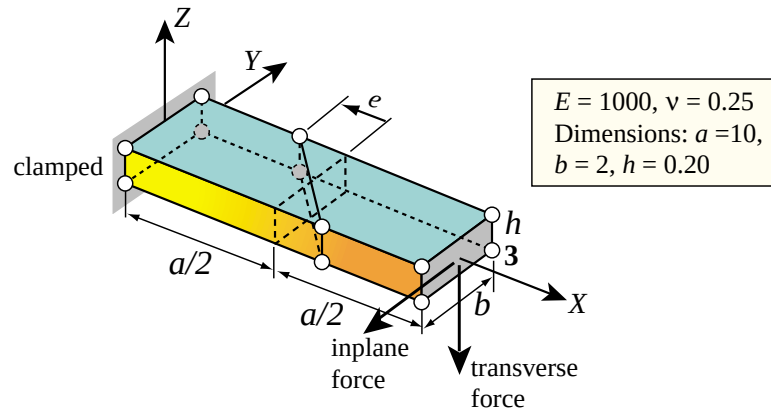


Figure 30. Two-element distortion sensitivity tests for inplane (membrane) and out of plane (bending) loading.

the clamped condition precludes lateral root expansion for nonzero Poisson's ratios.¹¹

§32.9.4. Inplane Distortion Sensitivity

Figure 30 shows a cantilever plate modeled by two elements whose inplane distortions are characterized by the eccentricity e . Inplane and out-of-plane end shear forces are applied. After normalized by the analytical solutions, the mean tip deflections for the two loading cases are reported in Table 2. They are compared against a solid-shell element developed by Sze, Yao and Cheung [27], labeled SYC therein.¹²

As can be observed the SS8 element has lower distortion sensitivity for in-plane bending. The distortion sensitivity for out-of-plane bending is similar for both elements.¹³

Table 2. Results for Distortion Sensitivity Test

Element Model	In-plane loading norm. deflection				Out-of-plane loading norm. deflection			
	$e = 0$	$e = 1$	$e = 2$	$e = 3$	$e = 0$	$e = 1$	$e = 2$	$e = 3$
SS8	0.8993	0.6029	0.5598	0.5878	0.9337	0.9319	0.9091	0.8696
SYC	0.8993	0.3480	0.2474	0.1812	0.9345	0.9285	0.9021	0.8526

¹¹ This test would be more convincing for arbitrary ν_1 and ν_2 if carried out on a free-free element, as done for the torsion test in §8.2, using free-free flexibility methods. It was not done that way since the symbolic computations became messy and could not be completed on time.

¹² Actually [27] tests more solid shell elements. The one compared to here is that reported to be their best.

¹³ These results were obtained with $\alpha = \beta = 0$ for the free parameters of the membrane strain field, as preset by SS8Stiff. If one sets $\beta = -1.5$, which is suspected (but not proven) of being close to the optimal value, the in-plane loading result for $e = 2$ rises to 0.7080, almost 3 times that of SYC.

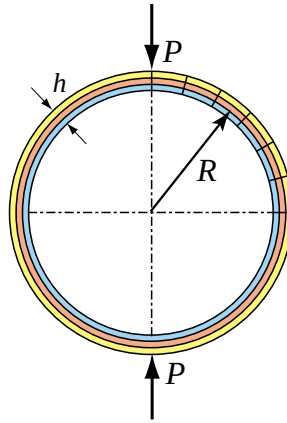


Figure 31. Pinched composite ring shell modeled with N_e elements over one quadrant; figure shows $N_e = 6$.

§32.9.5. Homogeneous and Laminated Pinched Ring

The problem is defined in Figure 31. The ring is pinched by two diametrically opposed loads P . The deflection response is strongly dominated by inextensional circumferential bending, particularly for high R/h ratios.

One quarter of the ring is modeled with N_e elements. Two isotropic materials with properties $E_1 = 10$, $\nu_1 = 0.30$ and $E_2 = 1$, $\nu_2 = 0.20$, respectively, are considered. The homogeneous ring is made up of material 1. Two and three ply laminations of equal thickness are also considered. The two ply lamination is unsymmetric and consists of materials 1 and 2 whereas the three-ply lamination is symmetric and consists of materials 1, 2 and 1. Two radius-to-thickness R/h ratios are examined. The vertical deflections under the line force are computed and normalized by curved Bernoulli-Euler beam solutions, which ignore the effects of membrane and shear.

Results for one and two plies are shown in Table 3. Results for the symmetric 3-ply laminate were similar to the one-ply case and are not shown. Once N_e exceeds 16 convergence is satisfactory. However, the response is too stiff for coarse meshes, especially for high R/h . This overstiffness was traced to transverse shear-membrane coupling that hinders inextensional bending response when the element is thickness tapered, as in the case here in the circumferential direction.

Table 3. Normalized Deflections for the Pinched Composite Ring

Wall:	One ply: $E = 10$, $\nu = .3$		Two plies: $E = 10$, 1; $\nu = .3$, .2	
N_e	$R/h = 20$	$R/h = 100$	$R/h = 20$	$R/h = 100$
2	0.0190	0.0008	0.0172	0.0009
4	0.5746	0.0062	0.5421	0.0072
6	0.8993	0.4322	0.8640	0.4158
8	0.9582	0.7813	0.9320	0.7502
16	0.9896	0.9659	0.9759	0.9480
32	0.9955	0.9753	0.9916	0.9691

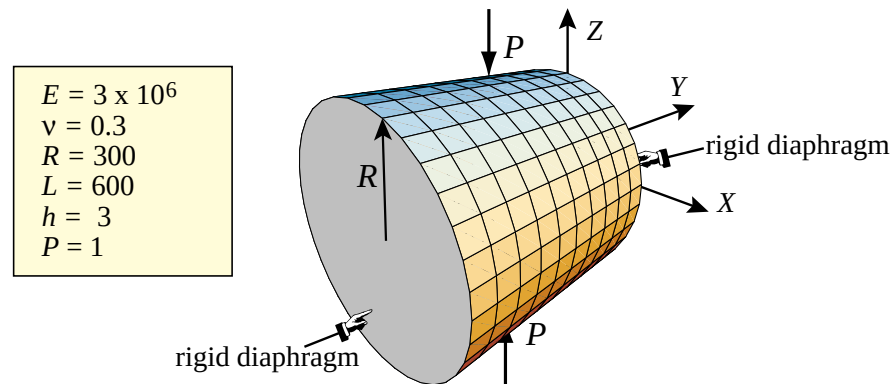


Figure 32. Pinched cylindrical shell benchmark.

§32.9.6. Pinched Cylindrical Shell

A test in the MacNeal-Harder benchmark suite [16,17]. The cylinder is pinched by two diametrically opposite point loads as shown in Figure 32. As in the previous case, the deflection under the load is dominated by inextensional bending response, which now is bidirectional.

Owing to symmetry, only one eighth of the shell needs to be considered. Over the rigid diaphragms the boundary conditions are $U_X^- = U_Y^- = U_Z^- = 0$. The deflection of the node under the point force is computed and normalized against the reference solution 0.18248×10^{-4} given by Belytschko, Wong and Stolarki [3]. The results are too stiff and the convergence unsatisfactory. As in the case of the pinched ring, this is caused by shear-membrane locking for thickness tapered elements. The effect is exacerbated here because the localization in the vicinity of the load is more pronounced.

Table 4. Normalized Deflections for the Pinched Cylinder Benchmark

Mesh	4×4	8×8	16×16	32×32
SS8	0.0762	0.2809	0.5366	0.8029

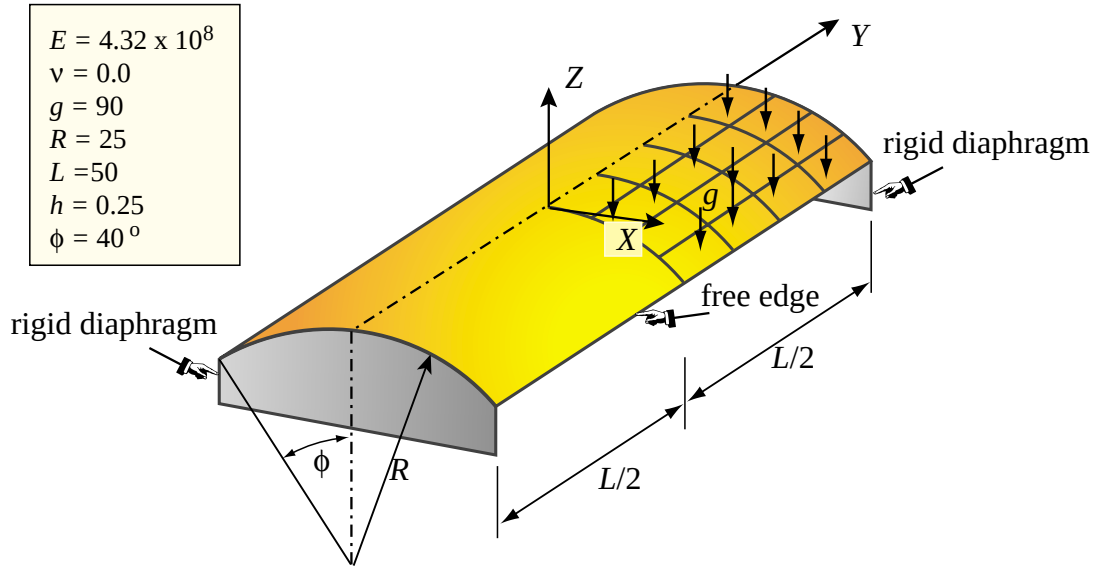


Figure 33. The Scordelis-Lo Roof Benchmark.

§32.9.7. Scordelis-Lo Roof

This widely used benchmark is defined in Figure 33. The roof is mounted on two rigid diaphragms and loaded under its own weight specified as g per unit of midsurface area. The boundary conditions at the diaphragms are $U_X^- = U_Y^- = U_Z^- = 0$. Only a quarter of the roof is analyzed due to symmetry. The vertical deflections at the midspan point of the free edge are normalized by the reference solution 0.3086 given by MacNeal and Harder[17].

The results for the SS8 element are shown in Table 5. The convergence is satisfactory and occurs from above. Good results are expected because in this problem the free edge deflection is dominated by longitudinal membrane-bending extensional coupling, and the elements are not tapered in the longitudinal (Y) direction.

It should be noted that the results appear to converge to the deep-shell analytical solution of 0.3012. This is about 3% lower than the MacNeal-Harder value of 0.3086 quoted above, which is based on a different shell theory.

Table 5. Normalized Deflections for the Scordelis-Lo Roof Benchmark

Mesh	2×2	4×4	8×8	16×16
SS8	1.2928	1.0069	0.9844	0.9772

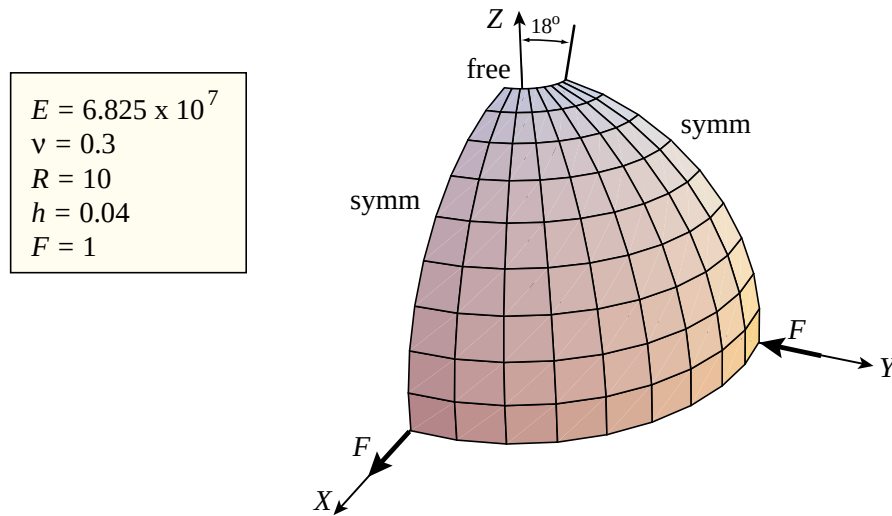


Figure 34. Pinched hemispherical shell with a 18° polar cutout. Only the FEM-modeled quadrant is shown.

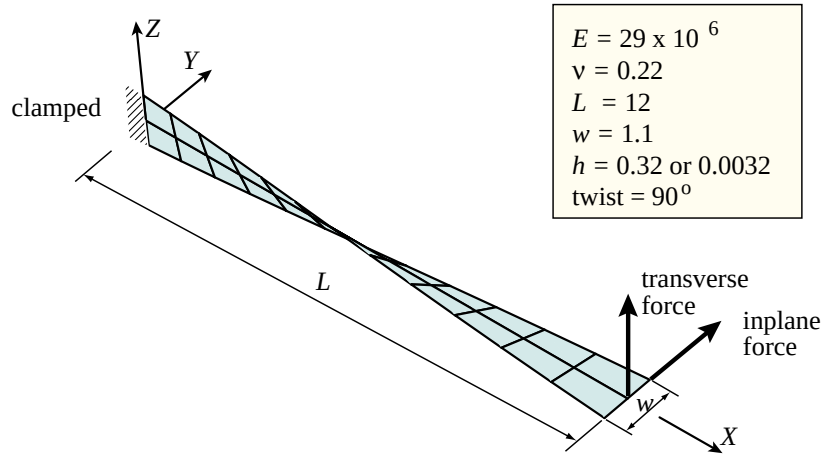
§32.9.8. Pinched Hemisphere

This is another widely used benchmark which is part of the test suite of MacNeal and Harder [17]. A quarter of a closed spherical shell is shown in Figure 34. The shell is acted upon by two pairs of orthogonal point loads. The reference solution is 0.0940 [17].

The normalized displacement of the nodes subjected to the point forces are listed in Table 6. The convergence is partly affected by inextensional bending over stiffness. The results of Key, Gunerud and Koteras [15], who use a stress-resultant-based, homogeneous solid shell element are listed for a 10×10 mesh and three different values of their hourglass control parameter s_{HG}

Table 6. Normalized Deflections for the Pinched Hemisphere

Mesh	2×2	4×4	8×8	10×10
SS8	0.6541	0.9219	0.9853	
Key et al, $s_{HG} = 0.000$				1.0191
Key et al, $s_{HG} = 0.015$				0.9979
Key et al, $s_{HG} = 0.030$				0.9766

Figure 35. Pretwisted beam modeled by a 2×12 mesh.

§32.9.9. Pretwisted Beam

In this problem, defined in Figure 35, the elements warp. For the beam thickness equal to 0.32 the forces acting are 1 unit. The deflections along the loading direction are 5.424×10^{-3} and 1.754×10^{-3} for the inplane and out of plane loadings, respectively. For the thickness of 0.0032, the applied force is 10^{-6} units. The deflections along the loading directions are 5.256×10^{-3} and 1.294×10^{-3} for the inplane and out of plane loadings, respectively [3,17].

The results for two coarse meshes run with $h = 0.32$ are listed in Table 7. Running $h = 0.0032$ caused the *Mathematica* equation solver to complain about a very high condition number (approximately 10^{14}) with consequent loss of solution accuracy.

Table 7. Normalized Deflections for the Pretwisted Beam Problem with $h = 0.32$

Mesh	In-plane loading	Out of plane loading
1×6	1.0257	0.9778
2×12	1.0041	0.9930

§32.10. Conclusions

Solid shells fill a modeling void in the grey area between thin shell and 3D solid elements. Such models are particularly useful for laminate wall constructions that do not need to be treated with high accuracy, and hence do not merit making each layer a separate element. One SS8 element can take arbitrary number of layers across the thickness while keeping the same node and freedom configuration.

Following is a summary of the strengths and weaknesses of this new element as assessed from benchmarks conducted as of this writing.

§32.10.1. General Strengths

Efficiency. SS8 does not need condensation of internal degrees of freedom, as required in most stress-hybrid and incompatible-mode elements. Consequently processing is highly efficient. Formation time in the local system is not much higher than that of the 8-node isoparametric brick element. This time is barely affected by the number of layers defined across the thickness. The transformations to the warped and global systems increase the local formation time by a mild factor.

Laminate Wall Construction. Through-the-thickness behavior of laminate wall construction is rendered physically correct by assuming an uniform normal transverse stress. This is corroborated by benchmarks.

§32.10.2. Special Strengths

Membrane Response. The assumed membrane strain field, obtained by a new energy fitting method discovered this fall [12] is believed to be the best to date as regards reduction of inplane distortion sensitivity. For thickness-prismatic elements the bending field is also insensitive to inplane distortion.

Consistency. The element has the correct rank of 18. It passes the patch test for flat thickness-prismatic configurations. The rectangular-prismatic element also verifies exactly some high order patch tests, as summarized in §8.1.

Torsion Response. The torsion behavior of an individual TP element of rectangular planform is surprisingly good, and becomes exact in the thin plate limit.

§32.10.3. Weaknesses and Question Marks

Inextensional Bending. Thickness-tapered elements do not capture inextensional bending well. This was evident from the pinched-ring and pinched-cylindrical-shell tests. In fact the element exhibits severe bending-shear lock if the taper exceeds a few degrees.

To cure this problem the transverse shear strain field will have to be corrected by coupling it to the membrane strain field as a function of taper, with the correction vanishing if the element is thickness prismatic. Initial attempts were made to find the correction by symbolic computations but expressions were forbiddingly complex for arbitrary geometries. Approximate corrections might be achieved by considering median transverse sections $\xi = 0$ and $\eta = 0$, but this scheme has not been tried.

Warping. The effect of severe midsurface warping on element performance is unknown. All benchmarks but one (the pretwisted beam) have dealt with flat-midsurface elements. The flat-to-warped displacement transformation geometry described in §3.3 remains a question mark, since it has not been proposed by any other author. Pending further validation the SS8 should be used only for slightly warped geometries bounded by $\Psi < 0.01$, where the warping measure Ψ is defined in §2.5.

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Chapter 4

Triangular Shell Elements.

An important advantage of the co-rotational formulation is the “reuse” of existing linear finite elements for large-rotation small-strain analysis. Chapters 4 and 5 develop high-performance shell elements that can efficiently provide the internal forces $\mathbf{f}_e$ and linear stiffness $\mathbf{K}_e$ used in the co-rotational nonlinear analysis presented in Chapters 2 and 3. The term “high performance” collectively identifies elements that can provide engineering accuracy with fairly coarse discretization.

Chapter 4 develops a triangular shell element whereas Chapter 5 develops a 4-node quadrilateral shell element. Both elements include drilling degrees of freedom as part of their membrane components.

In the exposition below, the special identifiers used in the previous chapters to distinguish linear and nonlinear components are dropped for clarity since the most of the development deals with the formulation of linear elements. Thus $\mathbf{K}_e$, for example, is written simply as $\mathbf{K}$.

4.1 Element stiffness by the ANDES formulation.

Let $\mathbf{K}$ denote the linear element stiffness matrix, $\mathbf{v}$ the visible element degrees of freedoms and $\mathbf{f}$ the corresponding element forces. The element stiffness equations for the elements developed below can be written as

$$\mathbf{K}\mathbf{v} = (\mathbf{K}_b + \mathbf{K}_h)\mathbf{v} = \mathbf{f} . \quad (4.1.1)$$

Here $\mathbf{K}_b$ and $\mathbf{K}_h$ are called *basic* and *higher-order* stiffness matrices respectively. This decomposition of the element stiffness equations also applies to the quadrilateral shell element constructed in the next chapter.

$\mathbf{K}_b$ is formulation independent in that it is entirely defined by an assumed constant stress together with an assumed boundary displacement field. This approach to forming the basic stiffness was first developed by Bergan and Hanssen [00] and later integrated in the more developed form of the Free Formulation (FF) by Bergan and Nygård [00, 00].

$\mathbf{K}_h$ can be formed using several different formulations, most notably the FF, the Extended Free Formulation (EFF) [00] and the Assumed Natural Deviatoric Strains (ANDES) formulation. The latter grew out of work done by Felippa to incorporate FF into a variational framework [00, 00], combined with further developments by Militello and Felippa [00, 00].

4.1.1 Basic stiffness construction.

The procedure for constructing the basic stiffness can be found in several sources. Militello has a very enlightening description in his Ph.D thesis [00]. This step by step outline of the basic stiffness construction is also described in Reference [00], and is outlined below here for easy reference.

- B1. Assume a constant stress state, $\bar{\boldsymbol{\sigma}}$ inside the element. This gives the associated boundary tractions $\bar{\boldsymbol{\sigma}}_n$:

$$\bar{\boldsymbol{\sigma}}_n = \bar{\boldsymbol{\sigma}} \cdot \mathbf{n} = \mathbf{T}_n \bar{\boldsymbol{\sigma}} , \quad (4.1.2)$$

where $\mathbf{n}$ is the outward unit normal vector on the element boundary and $\mathbf{T}_n$ is a transformation matrix that substitutes the tensor-product $\bar{\sigma}_{ni} = \bar{\sigma}_{ij} n_j$ with an equivalent matrix-multiply.

- B2. Connect a boundary displacement field, $\mathbf{d}$, to the visible degrees of freedom, $\mathbf{v}$ as

$$\mathbf{d} = \mathbf{N}_d \mathbf{v} . \quad (4.1.3)$$

Matrix $\mathbf{N}_d$ contains boundary displacement functions that must satisfy inter-element continuity, and exactly include rigid body and constant strains motion. Note, however, that the internal displacement field need not be defined here at this point; and in fact in the ANDES formulation such field is not explicitly constructed.

- B3. Construct the force-lumping matrix, $\mathbf{L}$, that consistently “lumps” the boundary tractions $\bar{\boldsymbol{\sigma}}_n$ to element node forces that are conjugate to the visible degrees of freedom $\mathbf{v}$ in the virtual work sense:

$$\int_S \delta \mathbf{d}^T \bar{\boldsymbol{\sigma}}_n dS = \int_S \delta \mathbf{v}^T \mathbf{N}_d^T \mathbf{T}_n \bar{\boldsymbol{\sigma}} dS = \delta \mathbf{v}^T \int_S \mathbf{N}_{dn}^T dS \bar{\boldsymbol{\sigma}} = \delta \mathbf{v}^T \mathbf{L} \bar{\boldsymbol{\sigma}} = \delta \mathbf{v}^T \bar{\mathbf{f}} . \quad (4.1.4)$$

This equation provides the lumping matrix $\mathbf{L}$ as

$$\mathbf{L} = \int_S \mathbf{N}_d^T \mathbf{T}_n dS = \int_S \mathbf{N}_{dn}^T dS . \quad (4.1.5)$$

- B4. The basic stiffness is constructed as

$$\mathbf{K}_b = \frac{1}{V} \mathbf{L} \mathbf{C} \mathbf{L}^T , \quad (4.1.6)$$

where $\mathbf{C}$ is the stress-strain constitutive matrix, and V is the volume of a three-dimensional element. (V is replaced by area and length measures for two-dimensional and one-dimensional elements, respectively.)

4.1.2 Higher order stiffness by the ANDES formulation.

Militello gives a thorough description of the ANDES formulation in his Ph.D thesis [00] . This includes a point by point description of the construction of the higher order stiffness. This is also described by Felippa and Militello in [00] . The essence of this development outlined below is the use of assumed strain distribution, rather than displacement modes, to characterize the higher-order behavior of the element.

- H1. Select locations in the element where “natural straingage” locations are to be chosen. For many ANDES elements these gages are placed on *reference lines* but this is not a general rule. By appropriate interpolation, express the element natural strains ϵ in terms of the “straingage readings” at those locations:

$$\epsilon = \mathbf{A}_\epsilon \mathbf{g} , \quad (4.1.7)$$

where ϵ is a strain field in natural coordinates that must include all constant strain states. (For structural elements the term “strain” is to be interpreted in a generalized sense, for example curvatures for beams or plate bending elements.)

- H2. Relate the Cartesian strains $\mathbf{e}$ to the natural strains:

$$\mathbf{e} = \mathbf{T}\epsilon = \mathbf{T}\mathbf{A}_\epsilon \mathbf{g} = \mathbf{A}\mathbf{g} \quad (4.1.8)$$

at each point in the element. (If $\mathbf{e} \equiv \epsilon$, or if it is possible to work throughout in natural coordinates, this step is skipped. This is often the case if $\mathbf{T}$ is constant over the element as for the triangular shell elements developed here.)

- H3. Relate the natural straingage readings $\mathbf{g}$ to the visible degrees of freedom

$$\mathbf{g} = \mathbf{Q}\mathbf{v} , \quad (4.1.9)$$

where $\mathbf{Q}$ is a straingage-to-node displacement transformation matrix. Techniques for doing this vary from element to element and it is difficult to state rules that apply to every situation. Often this step is amenable to breakdown into subproblems; for example

$$\mathbf{g} = \mathbf{Q}_1 \mathbf{v}_1 + \mathbf{Q}_2 \mathbf{v}_2 + \dots \quad (4.1.10)$$

where $\mathbf{v}_1, \mathbf{v}_2, \dots$ are conveniently selected subsets of $\mathbf{v}$. Some of these components may be derivable from displacements while others are not.

H4. Split the Cartesian strain field into mean (volume-averaged) and deviatoric strains:

$$\mathbf{e} = \bar{\mathbf{e}} + \mathbf{e}_d = (\bar{\mathbf{A}} + \mathbf{A}_d)\mathbf{g} , \quad (4.1.11)$$

where $\bar{\mathbf{A}} = \frac{1}{V} \int_V \mathbf{T} \mathbf{A}_\epsilon dV$, and $\mathbf{e}_d = \mathbf{A}_d \mathbf{g}$ has mean zero value over V . For elements with simple element geometry this decomposition can often be done in advance, and $\mathbf{e}_d$ constructed directly. Furthermore, this step may also be carried out on the natural strains if $\mathbf{T}$ is constant.

H5. The higher order stiffness matrix is given by

$$\mathbf{K}_h = \beta \mathbf{Q}^T \mathbf{K}_d \mathbf{Q} , \quad \text{with} \quad \mathbf{K}_d = \int_V \mathbf{A}_d^T \mathbf{C} \mathbf{A}_d dV , \quad (4.1.12)$$

where $\beta > 0$ is a scaling coefficient. It is often convenient to combine the product of $\mathbf{A}$ and $\mathbf{Q}$ into a single strain-displacement matrix called (as usual) $\mathbf{B}$, which splits into $\bar{\mathbf{B}}$ and $\mathbf{B}_d$:

$$\mathbf{e} = \mathbf{A} \mathbf{Q} = (\bar{\mathbf{A}} + \mathbf{A}_d) \mathbf{Q} \mathbf{v} = (\bar{\mathbf{B}} + \mathbf{B}_d) \mathbf{v} = \mathbf{B} \mathbf{v} , \quad (4.1.13)$$

in which case

$$\mathbf{K}_h = \beta \int_V \mathbf{B}_d^T \mathbf{C} \mathbf{B}_d dV . \quad (4.1.14)$$

We next apply these rules to the construction of a three-node triangular shell element. Because the element is flat, the membrane and bending can be developed separately. Both developments, however, share the geometric information presented in the following subsection.

4.2 Geometric definitions for a triangular element.

The geometry of a three-node triangular element is graphically defined in Figure 4.1.

By defining l_i to be length of side edge opposite to node i and h_i as height from node i to side i according to Figure 4.1 one obtains

$$l_i = \sqrt{x_{jk}^2 + y_{jk}^2} \quad \text{and} \quad h_i = \frac{2A}{l_i} , \quad (4.2.1)$$

where A is the triangle area, which may be calculated as

$$2A = x_{21}y_{31} - x_{31}y_{21} = x_{31}y_{12} - x_{12}y_{32} = x_{13}y_{23} - x_{23}y_{13} . \quad (4.2.2)$$

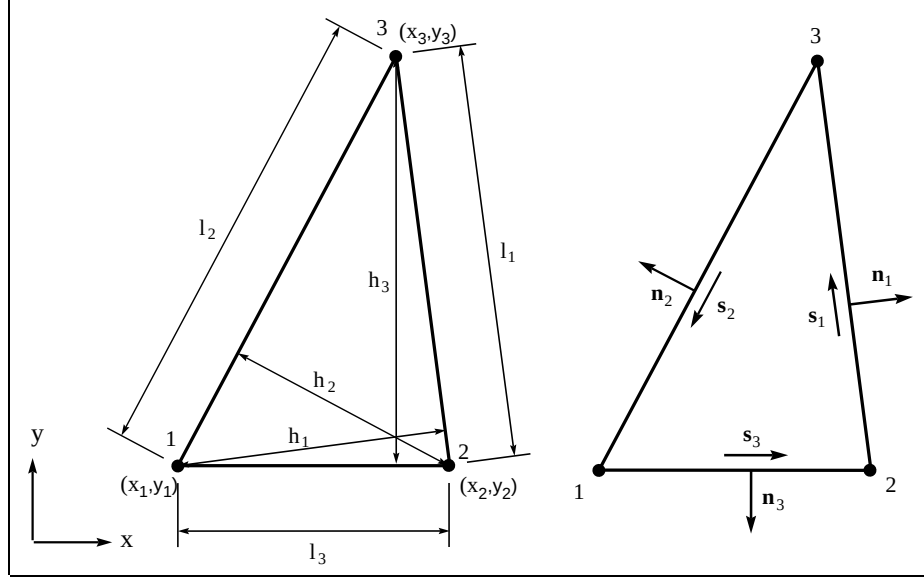


Figure 4.1. Geometric dimensions and unit vector definitions for a triangular element.

The unit vector $\mathbf{s}_i$ along side i and the outward normal vector $\mathbf{n}_i$ at side i can then be defined as

$$\mathbf{s}_i = \begin{Bmatrix} s_{ix} \\ s_{iy} \\ 0 \end{Bmatrix} = \frac{1}{l_i} \begin{Bmatrix} x_{kj} \\ y_{kj} \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{n}_i = \begin{Bmatrix} n_{ix} \\ n_{iy} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -s_{iy} \\ s_{ix} \\ 0 \end{Bmatrix}. \quad (4.2.3)$$

4.3 The triangular membrane element.

The construction of an ANDES triangular membrane element is described by Felippa and Militello in [00]. The present description is adapted to the notation used for the four-node quadrilateral element in Chapter 5.

The nodal degrees of freedom $\mathbf{v}_i$ for the membrane element consists of the in-plane translations u , v and the “drilling” degree of freedom θ_z :

$$\mathbf{v}_i = \begin{Bmatrix} u_i \\ v_i \\ \theta_{zi} \end{Bmatrix}. \quad (4.3.1)$$

4.3.1 Basic stiffness.

The lumping of the constant membrane stresses to a node j is only a function of the neighboring side edges ij and jk . The total lumping matrix can thus be divided into the contributions to the separate nodes as

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \\ \mathbf{L}_3 \end{bmatrix}, \quad (4.3.2)$$

where

$$\mathbf{L}_j = \frac{1}{2} \begin{bmatrix} y_{ki} & 0 & -x_{ki} \\ 0 & -x_{ki} & y_{ki} \\ \frac{\alpha}{6}(y_{ij}^2 - y_{kj}^2) & \frac{\alpha}{6}(x_{ij}^2 - x_{kj}^2) & \frac{\alpha}{3}(x_{kj}y_{kj} - x_{ij}y_{ij}) \end{bmatrix}, \quad (4.3.3)$$

and the nodal indices (i, j, k) take cyclic permutations of $(1, 2, 3)$. The basic stiffness is then computed as

$$\mathbf{K}_b = \frac{t}{A} \mathbf{L} \mathbf{C} \mathbf{L}^T, \quad (4.3.4)$$

where t is the element thickness and A the element area.

4.3.2 Higher order stiffness.

Felippa and Millitello [00] extracted the higher order behavior of the element by defining the higher order degrees of freedom $\tilde{\theta}_i$ as the nodal drilling degrees of freedom minus the rigid body and constant strain rotation θ_0 of the CST element

$$\tilde{\theta}_i = \theta_i - \theta_0, \quad (4.3.5)$$

where

$$\theta_0 = \frac{1}{4A} \begin{bmatrix} -x_{32} & -y_{32} & 0 & -x_{13} & -y_{13} & 0 & -x_{21} & -y_{21} & 0 \end{bmatrix} \mathbf{v}. \quad (4.3.6)$$

By further splitting the hierarchical rotations into mean $\bar{\theta} = (\tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3)/3$ and deviatoric components $\theta'_i = \tilde{\theta}_i - \bar{\theta}$ one gets

$$\begin{Bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \\ \bar{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \frac{2}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{x_{32}}{4A} & \frac{y_{32}}{4A} & \frac{1}{3} & \frac{x_{13}}{4A} & \frac{y_{13}}{4A} & \frac{1}{3} & \frac{x_{21}}{4A} & \frac{y_{21}}{4A} & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} v_{x1} \\ v_{y1} \\ \theta_1 \\ v_{x2} \\ v_{y2} \\ \theta_2 \\ v_{x3} \\ v_{y3} \\ \theta_3 \end{Bmatrix}, \quad (4.3.7)$$

which in matrix form reads

$$\boldsymbol{\theta}_h = \mathbf{H}_{\theta v} \mathbf{v} . \quad (4.3.8)$$

The pure-bending field.

The pure bending field is connected to the deviatoric hierarchical rotations θ'_i as

$$\begin{aligned} \boldsymbol{\epsilon}_{b1} &= \begin{Bmatrix} \epsilon_{b21|1} \\ \epsilon_{b32|1} \\ \epsilon_{b13|1} \end{Bmatrix} = \begin{bmatrix} \rho_1 \chi_{21|1} & -\rho_2 \chi_{21|1} & \rho_4 \chi_{21|1} \\ \rho_5 \chi_{32|1} & \rho_3 \chi_{32|1} & -\rho_3 \chi_{32|1} \\ -\rho_1 \chi_{13|1} & \rho_4 \chi_{13|1} & \rho_2 \chi_{13|1} \end{bmatrix} \begin{Bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{Bmatrix} = \mathbf{Q}_{b1} \boldsymbol{\theta}' \\ \boldsymbol{\epsilon}_{b2} &= \begin{Bmatrix} \epsilon_{b21|2} \\ \epsilon_{b32|2} \\ \epsilon_{b13|2} \end{Bmatrix} = \begin{bmatrix} \rho_2 \chi_{21|2} & -\rho_1 \chi_{21|2} & \rho_4 \chi_{21|2} \\ \rho_4 \chi_{32|2} & \rho_1 \chi_{32|2} & -\rho_2 \chi_{32|2} \\ -\rho_3 \chi_{13|2} & \rho_5 \chi_{13|2} & \rho_3 \chi_{13|2} \end{bmatrix} \begin{Bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{Bmatrix} = \mathbf{Q}_{b2} \boldsymbol{\theta}' \\ \boldsymbol{\epsilon}_{b3} &= \begin{Bmatrix} \epsilon_{b21|3} \\ \epsilon_{b32|3} \\ \epsilon_{b13|3} \end{Bmatrix} = \begin{bmatrix} \rho_3 \chi_{21|3} & -\rho_3 \chi_{21|3} & \rho_5 \chi_{21|3} \\ \rho_4 \chi_{32|3} & \rho_2 \chi_{32|3} & -\rho_1 \chi_{32|3} \\ -\rho_2 \chi_{13|3} & \rho_4 \chi_{13|3} & \rho_1 \chi_{13|3} \end{bmatrix} \begin{Bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{Bmatrix} = \mathbf{Q}_{b3} \boldsymbol{\theta}' \end{aligned} \quad (4.3.9)$$

where

$$\chi_{ij|k} = \frac{4A}{3l_{ij}^2} \quad \text{and} \quad \chi_{ij|i} = \chi_{ij|j} = -\frac{2A}{3l_{ij}^2} \quad (4.3.10)$$

and the ρ_i are numerical coefficients to be chosen. Coefficients ρ_i that optimize in-plane bending behavior of rectangular mesh units are found to be [00]

$$\rho_2 = 1, \quad \rho_3 = \frac{1}{2} \quad \text{and} \quad \rho_1 = \rho_4 = \rho_5 = 0 . \quad (4.3.11)$$

Having defined the matrices $\mathbf{Q}_{bi}$ in (4.3.9), the bending strains over the element can now be interpolated linearly between the nodes:

$$\boldsymbol{\epsilon}_b = (\zeta_1 \mathbf{Q}_{b1} + \zeta_2 \mathbf{Q}_{b2} + \zeta_3 \mathbf{Q}_{b3}) \boldsymbol{\theta}' = \mathbf{B}_b \boldsymbol{\theta}' . \quad (4.3.12)$$

The torsional field.

The torsional field is connected to the mean deviatoric rotation $\bar{\theta}$ and is given in [00] as

$$\boldsymbol{\epsilon}_t = \begin{Bmatrix} \epsilon_{t21} \\ \epsilon_{t32} \\ \epsilon_{t13} \end{Bmatrix} = 3 \begin{Bmatrix} \chi_{21|1} \zeta_{21} \\ \chi_{32|2} \zeta_{21} \\ \chi_{13|3} \zeta_{21} \end{Bmatrix} \bar{\theta} = \mathbf{B}_t \bar{\theta} .$$

The total strain field.

The total natural coordinate strain field is the combination of the pure-bending and torsional strain fields expressed with respect to the visible degrees of freedom

$$\begin{aligned}
\boldsymbol{\epsilon} &= \boldsymbol{\epsilon}_b + \boldsymbol{\epsilon}_t \\
&= \mathbf{B}_b \boldsymbol{\theta}' + \mathbf{B}_t \bar{\boldsymbol{\theta}} \\
&= [\mathbf{B}_b \quad \mathbf{B}_t] \boldsymbol{\theta}_h \\
&= [\mathbf{B}_b \quad \mathbf{B}_t] \mathbf{H}_{\theta v} \mathbf{v} \\
&= \mathbf{B} \mathbf{v} .
\end{aligned} \tag{4.3.13}$$

The stiffness matrix.

The higher order stiffness matrix is computed as

$$\mathbf{K}_h = \int_A \mathbf{B}^T \mathbf{C}_\epsilon \mathbf{B} dA \quad \text{where} \quad \mathbf{C}_\epsilon = \mathbf{T}^T \mathbf{C} \mathbf{T} , \tag{4.3.14}$$

and

$$\mathbf{T}^{-1} = \begin{bmatrix} s_{12x}^2 & s_{12y}^2 & s_{12x}s_{12y} \\ s_{23x}^2 & s_{23y}^2 & s_{23x}s_{23y} \\ s_{31x}^2 & s_{31y}^2 & s_{31x}s_{31y} \end{bmatrix} . \tag{4.3.15}$$

Matrix $\mathbf{T}$ transforms the natural coordinate strains to Cartesian strains, while $\mathbf{T}^{-1}$ does the opposite.

4.4 The triangular bending elements.

The bending component of the triangular shell element is based on the linear three node plate bending element AQR developed by Militello [00]. A higher order stiffness is also developed by sanitizing the BCIZ element [00]. Two basic stiffnesses exist, one based on linear interpolation of normal rotations along a side edge and one based on quadratic variation of the normal rotation. The triangular ANDES bending elements can thus be formed by combining several basic and higher order stiffnesses.

The nodal bending degrees of freedom $\mathbf{v}_i$ consists of the out of plane translation w and the in-plane rotations θ_x and θ_y

$$\mathbf{v}_i = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} . \tag{4.4.1}$$

4.4.1 Basic stiffnesses.

$\mathbf{K}_b$ is one of the basic stiffness matrices described by Militello in [00] as

$$\mathbf{K}_b = \frac{1}{A} \mathbf{L}_l \mathbf{C} \mathbf{L}_l^T \quad \text{or} \quad \mathbf{K}_b = \frac{1}{A} \mathbf{L}_q \mathbf{C} \mathbf{L}_q^T ,$$

where

$$\mathbf{L}_l = \begin{bmatrix} \mathbf{L}_{l1} \\ \mathbf{L}_{l2} \\ \mathbf{L}_{l3} \end{bmatrix} \quad \text{and} \quad \mathbf{L}_q = \begin{bmatrix} \mathbf{L}_{q1} \\ \mathbf{L}_{q2} \\ \mathbf{L}_{q3} \end{bmatrix} .$$

$\mathbf{L}_{q_i}$ and $\mathbf{L}_{l_i}$ are described in equation (0.0.0) and (0.0.0) respectively. The nodal indices (i, j, k) in the equations above takes the cyclic permutations of $(1, 2, 3)$ as in the case of the membrane lumping.

4.4.2 BCIZ higher order stiffness.

The BCIZ element developed by Bazeley et al. [00] is an historically important nonconforming element. However, the element is known not to pass the Patch Test. In fact the puzzling behavior of the element motivated the original development of that test. The use of the BCIZ element as an higher order stiffness for a triangular Free Formulation plate bending element was developed by Felippa, Haugen and Militello [00]. The transverse displacement field of the BCIZ element was given explicitly by Felippa [00] as

$$w = \left\{ \begin{array}{l} \zeta_1^2(3 - 2\zeta_1) + 2\zeta_1\zeta_2\zeta_3 \\ -\zeta_1^2(y_{12}\zeta_2 + y_{13}\zeta_3) - \frac{1}{2}(y_{12} + y_{13})\zeta_1\zeta_2\zeta_3 \\ \zeta_1^2(x_{12}\zeta_2 + x_{13}\zeta_3) + \frac{1}{2}(x_{12} + x_{13})\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(3 - 2\zeta_2) + 2\zeta_1\zeta_2\zeta_3 \\ -\zeta_2^2(y_{23}\zeta_3 + y_{21}\zeta_1) - \frac{1}{2}(y_{23} + y_{21})\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(x_{23}\zeta_3 + x_{21}\zeta_1) + \frac{1}{2}(x_{23} + x_{21})\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(3 - 2\zeta_3) + 2\zeta_1\zeta_2\zeta_3 \\ -\zeta_3^2(y_{31}\zeta_1 + y_{32}\zeta_2) - \frac{1}{2}(y_{31} + y_{32})\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(x_{31}\zeta_1 + x_{32}\zeta_2) + \frac{1}{2}(x_{31} + x_{32})\zeta_1\zeta_2\zeta_3 \end{array} \right\}^T \mathbf{v}$$

The strain displacement matrix $\mathbf{B}_\chi$ giving the natural curvatures from the visible degrees of freedom is obtained by double differentiation of the displacement field with respect to the triangular coordinates and appropriate relations detailed in the Appendix of [00]:

$$\chi = \left\{ \begin{array}{l} \chi_{12} \\ \chi_{23} \\ \chi_{31} \end{array} \right\} = \mathbf{B}_\chi \mathbf{v} = (\mathbf{B}_{\chi 0} + \mathbf{B}_{\chi 1}\zeta_1 + \mathbf{B}_{\chi 2}\zeta_2 + \mathbf{B}_{\chi 3}\zeta_3) \mathbf{v} ,$$

where

$$\mathbf{B}_{\chi^0}^T = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{\chi^1}^T = \begin{bmatrix} -12 & -4 & -12 \\ 4y_{12} & y_{12} + y_{13} & 4y_{13} \\ -4x_{12} & -x_{12} - x_{13} & -4x_{13} \\ 0 & -4 & 0 \\ -2y_{21} & -y_{21} + y_{23} & 0 \\ 2x_{21} & x_{21} - x_{23} & 0 \\ 0 & -4 & 0 \\ 0 & -y_{31} + y_{32} & -2y_{31} \\ 0 & x_{31} - x_{32} & 2x_{31} \end{bmatrix},$$

$$\mathbf{B}_{\chi^2}^T = \begin{bmatrix} 0 & 0 & -4 \\ -2y_{12} & 0 & -y_{12} + y_{13} \\ 2x_{12} & 0 & x_{12} - x_{13} \\ -12 & -12 & -4 \\ 4y_{21} & 4y_{23} & y_{21} + y_{23} \\ -4x_{21} & -4x_{23} & -x_{21} - x_{23} \\ 0 & 0 & -4 \\ 0 & -2y_{32} & y_{31} - y_{32} \\ 0 & 2x_{32} & -x_{31} + x_{32} \end{bmatrix},$$

and

$$\mathbf{B}_{\chi^3}^T = \begin{bmatrix} -4 & 0 & 0 \\ y_{12} - y_{13} & 0 & -2y_{13} \\ -x_{12} + x_{13} & 0 & 2x_{13} \\ -4 & 0 & 0 \\ y_{21} - y_{23} & -2y_{23} & 0 \\ -x_{21} + x_{23} & 2x_{23} & 0 \\ -4 & -12 & -12 \\ y_{31} + y_{32} & 4y_{32} & 4y_{31} \\ -x_{31} - x_{32} & -4x_{32} & -4x_{31} \end{bmatrix}.$$

By using a natural curvature constitutive matrix $\mathbf{C}_\chi = \mathbf{T}^T \mathbf{C} \mathbf{T}$ the higher order stiffness matrix becomes

$$\mathbf{K}_h = \int_A \mathbf{B}_{\chi d}^T \mathbf{C}_\chi \mathbf{B}_{\chi d} dA,$$

where $\mathbf{B}_{\chi d} = \mathbf{B}_\chi - \bar{\mathbf{B}}_\chi$ and $\bar{\mathbf{B}}_\chi = \mathbf{B}_{\chi^0} + \frac{1}{3}(\mathbf{B}_{\chi^1} + \mathbf{B}_{\chi^2} + \mathbf{B}_{\chi^3})$.

4.4.3 ANDES higher order stiffness by direct curvature readings.

The three node ANDES element is based on direct curvature interpolation of the natural curvatures. As reference lines Millitello [00] chose the three side edges, which function as Hermitian beams. The nodal strain gage readings expressed as function of the visible degrees of freedom can be written

$$\mathbf{g} = \mathbf{Q}\mathbf{v} = \mathbf{Q}_F * \mathbf{F}\mathbf{v} , \quad (4.4.2)$$

where

$$\begin{aligned} \mathbf{g}^T &= [\kappa_{31|1} \quad \kappa_{12|1} \quad \kappa_{12|2} \quad \kappa_{23|2} \quad \kappa_{23|3} \quad \kappa_{31|3}] , \\ \mathbf{v}^T &= [v_{z1} \quad \theta_{x1} \quad \theta_{y1} \quad v_{z2} \quad \theta_{x2} \quad \theta_{y2} \quad v_{z3} \quad \theta_{x3} \quad \theta_{y3}] , \end{aligned} \quad (4.4.3)$$

$$\mathbf{Q}_F = \begin{bmatrix} -6 & 4 & 4 & 0 & 0 & 0 & 6 & 2 & 2 \\ -6 & -4 & -4 & 6 & -2 & -2 & 0 & 0 & 0 \\ 6 & 2 & 2 & -6 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & -4 & -4 & 6 & -2 & -2 \\ 0 & 0 & 0 & 6 & 2 & 2 & -6 & 4 & 4 \\ 6 & -2 & -2 & 0 & 0 & 0 & -6 & -4 & -4 \end{bmatrix} \quad (4.4.4)$$

and

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{31} & \mathbf{F}_{31} & \mathbf{F}_{31} \\ \mathbf{F}_{12} & \mathbf{F}_{12} & \mathbf{F}_{12} \\ \mathbf{F}_{12} & \mathbf{F}_{12} & \mathbf{F}_{12} \\ \mathbf{F}_{23} & \mathbf{F}_{23} & \mathbf{F}_{23} \\ \mathbf{F}_{23} & \mathbf{F}_{23} & \mathbf{F}_{23} \\ \mathbf{F}_{31} & \mathbf{F}_{31} & \mathbf{F}_{31} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \mathbf{F}_{12} &= \begin{bmatrix} \frac{1}{l_{12}^2} & \frac{n_{12x}}{l_{12}} & \frac{n_{12y}}{l_{12}} \end{bmatrix} \\ \mathbf{F}_{23} &= \begin{bmatrix} \frac{1}{l_{23}^2} & \frac{n_{23x}}{l_{23}} & \frac{n_{23y}}{l_{23}} \end{bmatrix} \\ \mathbf{F}_{31} &= \begin{bmatrix} \frac{1}{l_{31}^2} & \frac{n_{31x}}{l_{31}} & \frac{n_{31y}}{l_{31}} \end{bmatrix} \end{aligned} \quad (4.4.5)$$

The six curvature gage readings in $\mathbf{g}$ give two curvature gage readings at each node. But three natural coordinate curvature readings are necessary to transform to the Cartesian strains at each node. A third reading is obtained by invoking the following *projection rule* [00]: the natural curvature κ_{ij} is assumed to vary linearly along side ij and constant along lines normal to side ij . Node k is then assigned a κ_{ij} value according to the projection of the node on line ij . This assumption can be expressed as the matrix relationship

$$\boldsymbol{\kappa} = \mathbf{A}_\kappa \mathbf{g} , \quad (4.4.6)$$

where

$$\mathbf{A}_\kappa = \begin{bmatrix} 0 & \zeta_1 + \lambda_{12}\zeta_3 & \zeta_1 + \lambda_{21}\zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_2 + \lambda_{23}\zeta_1 & \zeta_2 + \lambda_{32}\zeta_1 & 0 \\ \zeta_1 + \lambda_{13}\zeta_2 & 0 & 0 & 0 & 0 & \zeta_3 + \lambda_{31}\zeta_2 \end{bmatrix} \quad (4.4.7)$$

and

$$\lambda_{ij} = \frac{-\mathbf{s}_{ki}^T \mathbf{s}_{ij} l_{ki}}{l_{ij}} .$$

The deviatoric parts of the strains are now obtained by subtracting the mean strain:

$$\mathbf{A}_{\kappa d} = \mathbf{A}_\kappa - \int_A \mathbf{A}_\kappa dA ,$$

which gives

$$\mathbf{A}_{\kappa d} = \begin{bmatrix} 0 & \tilde{\zeta}_1 + \lambda_{12}\tilde{\zeta}_3 & \tilde{\zeta}_1 + \lambda_{21}\tilde{\zeta}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\zeta}_2 + \lambda_{23}\tilde{\zeta}_1 & \tilde{\zeta}_2 + \lambda_{32}\tilde{\zeta}_1 & 0 \\ \tilde{\zeta}_1 + \lambda_{13}\tilde{\zeta}_2 & 0 & 0 & 0 & 0 & \tilde{\zeta}_3 + \lambda_{31}\tilde{\zeta}_2 \end{bmatrix} \quad (4.4.8)$$

in which $\tilde{\zeta}_i = \zeta_i - \frac{1}{3}$.

The deviatoric cartesian curvatures.

The deviatoric cartesian strain distribution over the element can now be expressed as

$$\kappa_d = \mathbf{T}\boldsymbol{\kappa} = \mathbf{T}\mathbf{A}_{\kappa d}\mathbf{g} = \mathbf{T}\mathbf{A}_{\kappa d}\mathbf{Q}\mathbf{v} = \mathbf{B}_d\mathbf{v} ,$$

where $\mathbf{T}$ is defined in equation (4.3.15).

The higher order stiffness.

Finally, the higher order stiffness can be computed from the deviatoric strains as

$$\mathbf{K}_h = \int_A \mathbf{B}_d^T \mathbf{C} \mathbf{B}_d dA . \quad (4.4.9)$$

4.5 Nonlinear extensions for a triangular shell element.

The linear triangular shell element is now ready to be incorporated in the co-rotational formulation discussed in Chapter 2. The shadow element C_{0n} is best fit to the deformed element C_n by a rigid body motion of the undeformed initial element C_0 . However this “best fit” is not unique. The rotation gradient matrix $\mathbf{G}$ defined in equation (0.0.0) can be split into contributions from each node as

$$\delta\tilde{\omega}_r = \tilde{\mathbf{G}} \delta\tilde{\mathbf{v}} \quad \text{where} \quad \tilde{\mathbf{G}} = [\tilde{\mathbf{G}}_1 \quad \tilde{\mathbf{G}}_2 \quad \tilde{\mathbf{G}}_3] = \begin{bmatrix} \tilde{\mathbf{G}}_{\theta_x} \\ \tilde{\mathbf{G}}_{\theta_y} \\ \tilde{\mathbf{G}}_{\theta_z} \end{bmatrix}, \quad (4.5.1)$$

and $\delta\tilde{\mathbf{v}}$ is defined as

$$\delta\tilde{\mathbf{v}} = \begin{Bmatrix} \delta\tilde{\mathbf{v}}_1 \\ \delta\tilde{\mathbf{v}}_2 \\ \delta\tilde{\mathbf{v}}_3 \end{Bmatrix} \quad \text{where} \quad \delta\tilde{\mathbf{v}}_i = \begin{Bmatrix} \delta\tilde{\mathbf{u}}_i \\ \delta\tilde{\omega}_i \end{Bmatrix}. \quad (4.5.2)$$

Three techniques for fitting the shadow element are discussed below. Each technique produces different $\tilde{\mathbf{G}}_i$ submatrices.

4.5.1 Aligning a triangle side.

This procedure is similar to Rankin’s alignment of the C_{0n} and C_n elements [00] in that it uses a common side edge direction for those configurations. Whereas Rankin picks side 13 for the unit-vector $\mathbf{e}_2$ and node 1 as the origin of the coordinate system, the current approach aligns the directions of side 12 with the $\mathbf{e}_1$ axis and uses the element nodal average (triangle centroid) as the origin of the coordinate system. This choice of centroid as origin is necessary in order to satisfy the orthogonality of $\mathbf{P}_T$ and $\mathbf{P}$ in equation (0.0.0) .

Through consistent variation of the foregoing choice of local coordinate system, the nodal submatrices $\tilde{\mathbf{G}}_i$ of equation (4.5.1) is obtained as

$$\begin{aligned} \tilde{\mathbf{G}}_1 &= \frac{1}{2A} \begin{bmatrix} 0 & 0 & x_{32} & 0 & 0 & 0 \\ 0 & 0 & y_{32} & 0 & 0 & 0 \\ 0 & -\frac{2A}{l_{12}} & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{\mathbf{G}}_2 &= \frac{1}{2A} \begin{bmatrix} 0 & 0 & x_{13} & 0 & 0 & 0 \\ 0 & 0 & y_{13} & 0 & 0 & 0 \\ 0 & \frac{2A}{l_{12}} & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{\mathbf{G}}_3 &= \frac{1}{2A} \begin{bmatrix} 0 & 0 & x_{32} & 0 & 0 & 0 \\ 0 & 0 & y_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (4.5.3)$$

where A is the area of the triangle and l_{12} is length of side 12. (This variation is carried out in more detail for the four node shell element in Section 0.0.)

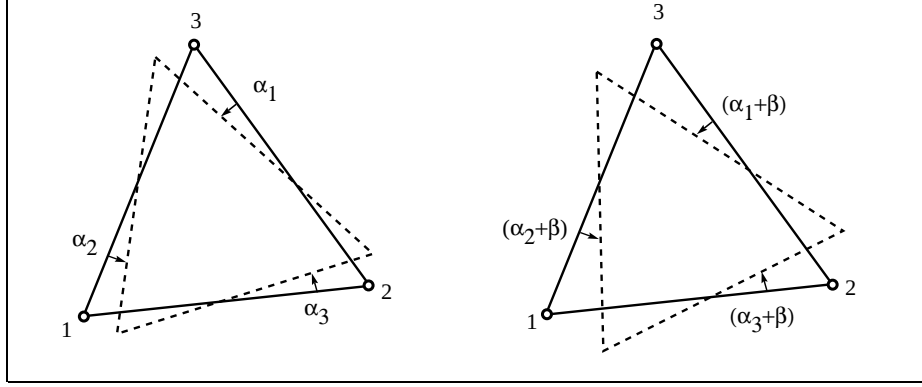


Figure 4.2. Definition of side edge angular errors.

The in-plane rotations can be recognized as $\omega_x = \frac{\partial w}{\partial y}$ and $\omega_y = -\frac{\partial w}{\partial x}$, using the geometric shape functions to interpolate w . This choice of shadow element fit satisfies the required splitting of the rotation gradient matrix as $\tilde{\mathbf{G}} = \tilde{\mathbf{X}}\mathbf{A}$ where only matrix $\mathbf{X}$ is coordinate dependent as required for the consistency condition in equation (0.0.0). On the other hand, this choice does not give an invariant deformational displacement vector for the element in the sense discussed in Section 0.0.0.

4.5.2 Least square fit of side edge angular errors.

Nygård [00] and Bjærum [00] place the C_{0n} element in the plane of the deformed element C_n with node 1 coinciding. The present study utilizes coinciding centroids. The in-plane orientation of the shadow element is then determined by a least square fit of the side edge angular errors. According to Figure 4.2 the squared-error sum is

$$e^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2. \quad (4.5.4)$$

By rotating the shadow element an angle β the square of the errors becomes

$$e^2(\beta) = (\alpha_1 + \beta)^2 + (\alpha_2 + \beta)^2 + (\alpha_3 + \beta)^2. \quad (4.5.5)$$

Minimizing with respect to β :

$$\frac{\partial e^2(\beta)}{\partial \beta} = 0 \quad \Rightarrow \quad \beta = -\frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3). \quad (4.5.6)$$

Consequently, the optimal in-plane position of the shadow element according to this algorithm is given by the mean of the side edge angular errors. This condition yields for the nodal submatrices

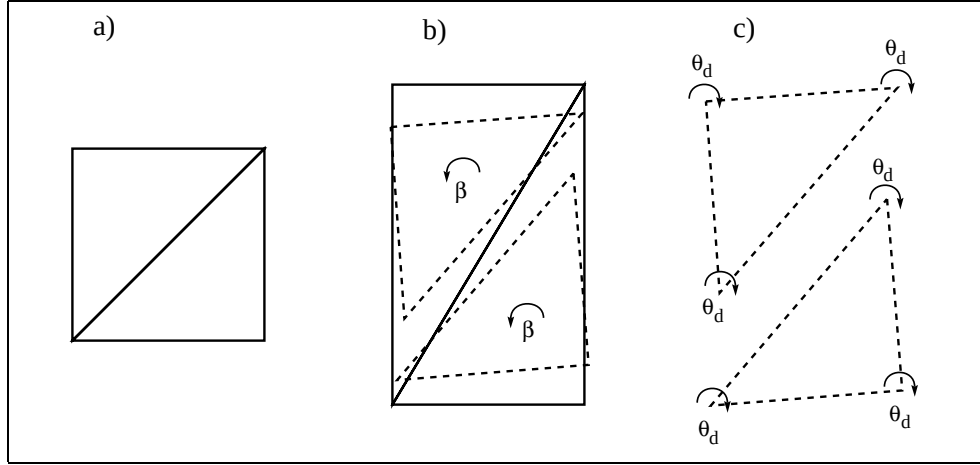


Figure 4.3. Patch of triangle elements subjected to pure stretching.

$$\tilde{\mathbf{G}}_i = \frac{1}{2A} \begin{bmatrix} 0 & 0 & x_{kj} & 0 & 0 & 0 \\ 0 & 0 & y_{kj} & 0 & 0 & 0 \\ \frac{2A}{3}(-\frac{s_{jy}}{l_j} + \frac{s_{ky}}{l_k}) & \frac{2A}{3}(\frac{s_{jx}}{l_j} - \frac{s_{kx}}{l_k}) & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4.5.7)$$

The major advantage of this method is that it gives a unique fit independent of node numbering, which leads to a invariant deformational displacement vector as discussed in section 0.0.0. The rotational gradient matrix cannot be split into a coordinate dependent and independent part in order to be consistent with equation (0.0.0). However, again, this is of minor importance for triangular elements since the shadow element C_{0n} and the deformed element C_n will be close together for small membrane strains.

A more serious disadvantage of this fitting method is that it reintroduces the problem of fictitious normal rotations when an element is subjected to pure stretch.

This difficulty is illustrated in Figure 4.3, where the C_{0n} elements rotate due to the in-plane rotation of the diagonal. The deformational displacement vector is then computed as the difference between C_n and C_{0n} . A deformational normal rotation is thus picked up since the predictor step gives no rotation at the nodes and the deformational rotation is the total rotation minus the rigid body rotation

$$\theta_d = (\theta - \beta) = -\beta. \quad (4.5.8)$$

This problem is similar to that pointed out by Irons and Ahmad [00] when defining drilling degrees of freedom as the mean of the side edge rotations at an

node. This was overcome by Bergan and Felippa [00] when they defined the normal rotation as $\theta_z = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ for the linear FF membrane element. It is seen that the problem of fictitious normal rotations has been thus been re-introduced for the nonlinear case by the choice of shadow element positioning. This problem is even more pronounced with the side edge alignment procedure described in Section 4.5.1.

4.5.3 Fit according to CST-rotation.

As with the least square fit of side edge angular errors the shadow C_{0n} element is chosen to be co-planar with the deformed element C_n , and the centroids coincide.

By using the normal rotation of the CST element as the rigid body rotation β for the in-plane positioning of the shadow element, one avoids the problem of fictitious normal rotations when an element is subjected to pure stretching.

The definition $\theta_z = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ gives an invariant definition of the normal rotation for the infinitesimal case. This also provides the variation of the rigid body rotation with respect to the visible degrees of freedom.

Extending the above definition to finite rotations seems to suggest

$$\theta_z = \frac{1}{2}(\tan^{-1}(\frac{\Delta v}{\Delta x}) - \tan^{-1}(\frac{\Delta u}{\Delta y})). \quad (4.5.9)$$

However this choice gives slightly varying results with respect to the orientation of the (x, y) -coordinate system. In order to obtain a completely invariant positioning with respect to node numbering, the rigid body rotation can be computed as the average of the rotations obtained with the local x -axis along each of the three side edges. The continuum mechanics definition of the normal rotation is $\tilde{\theta}_z = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$. This definition is invariant with respect to the orientation of the x and y coordinate axis, and gives the rotation gradient matrix as

$$\tilde{\mathbf{G}}_i = \frac{1}{2A} \begin{bmatrix} 0 & 0 & x_{kj} & 0 & 0 & 0 \\ 0 & 0 & y_{kj} & 0 & 0 & 0 \\ -\frac{1}{2}x_{kj} & -\frac{1}{2}y_{kj} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.5.10)$$

In the present investigation the three techniques just outlined for choosing the shadow element position were tested in the nonlinear problems reported in Chapters 7 and 8.

Chapter 5

Quadrilateral Shell Elements.

Flat quadrilateral shell elements have use limitations even in linear analysis, since a mesh that consists of strictly flat elements may be impossible to construct over a doubly-curved shell surface. For large deflection nonlinear analysis this deficiency becomes more pronounced. Even if the initial mesh satisfies the flat element restriction, the deformations can become so large that warping of the elements can be significant. Finding ways of handling warped element geometries is thus of fundamental importance for quadrilateral shell elements.

The current research initially set out to develop a non-flat quadrilateral element that was able to satisfy self-equilibrium in a warped configuration. This proved to be difficult. Finding a basic stiffness $\mathbf{K}_b = \frac{1}{A} \mathbf{LCL}^T$ that maintained self equilibrium was especially troublesome. To achieve this goal one must have the lumped node forces from a constant stress state $\mathbf{f} = \mathbf{L}\boldsymbol{\sigma}$ to be in self equilibrium. Two questions arise at this point: What is a constant stress state for a warped shell element? Is a constant stress state for a warped surface an equilibrium state for arbitrary element shapes? These are key questions that remain to be answered in order to develop satisfactory FF and ANDES elements for such element geometries. It follows that those formulations have to be further extended for developing warped shell elements. The construction of the basic stiffness matrix needs special attention because the Individual Element Test is not clearly defined for curved or warped element geometries.

Restoring equilibrium in the warped element geometry can be done *a posteriori* for elements developed with reference to the flat projected “footprint” by using a projector matrix. This is the approach that has been chosen for the current quadrilateral element. It allows independent development of the membrane and bending components since these two stiffness contributions decouple, which greatly simplifies the development of the element. Both linear and nonlinear projectors have been developed for the quadrilateral element. Both versions give identical results for linear static problems, but the linear projector is recommended for linear finite element codes because its formulation is much simpler than that of the nonlinear projector.

5.1 Geometric definitions for a quadrilateral element.

This section describes geometric relationships for a quadrilateral element. The development covers both warped geometry and the flat “best fit” element obtained by setting the z coordinate for each node equal to zero. The flat projection relationships are later used for the development of the membrane and bending stiffness of the shell element. The non-flat relationships are needed for the development of the nonlinear projector for quadrilaterals.

A vector $\mathbf{r}$ to a point on a nonflat quadrilateral element can be parametrized with respect to the natural coordinates ξ and η as

$$\mathbf{r}(\xi, \eta) = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \mathbf{N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{Bmatrix}, \quad (5.1.1)$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \quad \mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}, \quad \mathbf{z} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix}, \quad (5.1.2)$$

and x_i , y_i and z_i denote the global coordinates of node i . Row vector $\mathbf{N}$ contains the “usual” bi-linear isoparametric interpolation for a quadrilateral [00]. These functions and their partial derivatives with respect to ξ and η are

$$\begin{aligned} \mathbf{N} &= \frac{1}{4} [(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta)], \\ \mathbf{N}_{,\xi} &= \frac{1}{4} [-(1-\eta) \quad (1-\eta) \quad (1+\eta) \quad -(1+\eta)], \\ \mathbf{N}_{,\eta} &= \frac{1}{4} [-(1-\xi) \quad -(1+\xi) \quad (1+\xi) \quad (1-\xi)]. \end{aligned} \quad (5.1.3)$$

Using these geometric relations the variation of the position vector $\mathbf{r}$ can be written as

$$\begin{aligned} \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} &= \begin{Bmatrix} \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \\ \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \\ \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} \\ &= \begin{bmatrix} \mathbf{N}_{,\xi} \mathbf{x} & \mathbf{N}_{,\eta} \mathbf{x} \\ \mathbf{N}_{,\xi} \mathbf{y} & \mathbf{N}_{,\eta} \mathbf{y} \\ \mathbf{N}_{,\xi} \mathbf{z} & \mathbf{N}_{,\eta} \mathbf{z} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = [\mathbf{g}_\xi \quad \mathbf{g}_\eta] \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} \end{aligned} \quad (5.1.4)$$

or

$$d\mathbf{r} = \mathbf{J} d\boldsymbol{\xi}.$$

The Jacobian $\mathbf{J}$ introduced above is also used for computing partial derivatives with respect to the natural coordinates ξ and η :

$$\begin{aligned} \begin{Bmatrix} \frac{\partial(\cdot)}{\partial\xi} \\ \frac{\partial(\cdot)}{\partial\eta} \end{Bmatrix} &= \begin{Bmatrix} \frac{\partial(\cdot)}{\partial x} \frac{\partial x}{\partial\xi} + \frac{\partial(\cdot)}{\partial y} \frac{\partial y}{\partial\xi} \\ \frac{\partial(\cdot)}{\partial x} \frac{\partial x}{\partial\eta} + \frac{\partial(\cdot)}{\partial y} \frac{\partial y}{\partial\eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial(\cdot)}{\partial x} \\ \frac{\partial(\cdot)}{\partial y} \end{Bmatrix} \\ &= \begin{bmatrix} \mathbf{N}_{,\xi}\mathbf{x} & \mathbf{N}_{,\xi}\mathbf{y} \\ \mathbf{N}_{,\eta}\mathbf{x} & \mathbf{N}_{,\eta}\mathbf{y} \end{bmatrix} \begin{Bmatrix} \frac{\partial(\cdot)}{\partial x} \\ \frac{\partial(\cdot)}{\partial y} \end{Bmatrix} \end{aligned} \quad (5.1.5)$$

or

$$\frac{\partial(\cdot)}{\partial\boldsymbol{\xi}} = \mathbf{J}^T \frac{\partial(\cdot)}{\partial\mathbf{x}} . \quad (5.1.6)$$

It should be noted that only the x and y components are included in the above relationship for the partial derivatives. This is a consequence of assuming that the element represents a best fit of the warped quadrilateral in the (x, y) -plane.

If the z -coordinate is neglected the relationship between (x, y) and (ξ, η) is isoparametric and the inverse relation can be formed as

$$\begin{Bmatrix} \frac{\partial(\cdot)}{\partial x} \\ \frac{\partial(\cdot)}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{x\xi}^{-T} & J_{x\eta}^{-T} \\ J_{y\xi}^{-T} & J_{y\eta}^{-T} \end{bmatrix} \begin{Bmatrix} \frac{\partial(\cdot)}{\partial\xi} \\ \frac{\partial(\cdot)}{\partial\eta} \end{Bmatrix} . \quad (5.1.7)$$

5.1.1 Geometric quantities for a flat quadrilateral element.

By defining l_{ij} to be length of side edge ij one obtains

$$l_{ij} = \sqrt{x_{ji}^2 + y_{ji}^2} . \quad (5.1.8)$$

The unit vector $\mathbf{s}_{ij}$ along side ij and the outward normal vector $\mathbf{n}_{ij}$ of side ij can then be defined as

$$\mathbf{s}_{ij} = \begin{Bmatrix} s_{ix} \\ s_{iy} \\ 0 \end{Bmatrix} = \frac{1}{l_{ij}} \begin{Bmatrix} x_{ji} \\ y_{ji} \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{n}_{ij} = \begin{Bmatrix} n_{ix} \\ n_{iy} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -s_{iy} \\ s_{ix} \\ 0 \end{Bmatrix} . \quad (5.1.9)$$

5.2 The quadrilateral membrane element.

Nygård [00] developed a 4-node membrane element with drilling degrees of freedom based on the Free Formulation, which is called the FFQ element. The element has given accurate results for plane membrane problems. Unfortunately, the element is computationally expensive because the formation of each element stiffness requires the numerical inversion of a 12×12 matrix. The goal of the present development is to construct a 4-node membrane element with the same freedom configuration and similar accuracy as the FFQ, but that avoids those expensive matrix inversion.

Recall that element stiffness of the ANDES element is the sum of the basic and higher order contributions:

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_h = \frac{1}{A} \mathbf{LCL}^T + \int_A \mathbf{B}_h^T \mathbf{C} \mathbf{B}_h dA. \quad (5.2.1)$$

These matrices are now developed for the membrane component.

5.2.1 Basic stiffness.

The basic stiffness for the membrane element is developed by lumping the constant stress state over side edges to consistent nodal forces at the neighboring nodes according to a boundary displacement field. When the boundary displacement field is defined so that interelement compatibility is satisfied, pair-wise cancelation of nodal forces for a constant stress state is assured, and thus satisfaction of the Individual Element Test [00]. In turn, satisfaction of the Individual Element Test ensures that the conventional multi-element Patch Test is passed.

A very successful lumping scheme for membrane stresses was first introduced by Bergan and Felippa [00] in the paper describing the triangular membrane FF element with drilling degrees of freedoms. This procedure has since been used by Nygård [00] and Militello [00].

The presentation here rewrites the lumping matrix, in terms of nodal submatrices. The expressions of the lumping matrix thus becomes valid for elements of arbitrary number of corner nodes.

It is convenient to order the visible degrees of freedom as translations along x , y and drilling rotation about z -axis for each node. This gives the lumping of the constant stress $\boldsymbol{\sigma}$ state to nodal forces $\mathbf{f}$ as

$$\mathbf{f} = \mathbf{L}\boldsymbol{\sigma} \quad \text{where} \quad \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}, \quad (5.2.2)$$

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \\ \mathbf{L}_3 \\ \mathbf{L}_4 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{Bmatrix} \quad \text{where} \quad \mathbf{f}_i = \begin{Bmatrix} f_{x_i} \\ f_{y_i} \\ m_{z_i} \end{Bmatrix}. \quad (5.2.3)$$

By using the Hermitian beam shape function for a side edge one obtains a boundary displacement field that is compatible between adjacent elements because the displacements along an edge are only functions of the end nodes freedoms. This again gives a lumping matrix $\mathbf{L}$ where each nodal contribution $\mathbf{L}_j$ is only a function of its adjoining side edges ij and jk :

$$\mathbf{L}_j = \frac{1}{2} \begin{bmatrix} y_{ki} & 0 & -x_{ki} \\ 0 & -x_{ki} & y_{ki} \\ \frac{\alpha}{6}(y_{ij}^2 - y_{kj}^2) & \frac{\alpha}{6}(x_{ij}^2 - x_{kj}^2) & \frac{\alpha}{3}(x_{kj}y_{kj} - x_{ij}y_{ij}) \end{bmatrix}. \quad (5.2.4)$$

The nodal indices (i, j, k, l) for a four node element undergo cyclic permutations of $(1, 2, 3, 4)$ in the equation above. Factor α represents a scaling of the contributions of the drilling freedom to the normal boundary displacements; see [00] for details.

5.2.2 Higher order stiffness.

To construct $\mathbf{K}_h$ a set of higher order degrees of freedoms that vanish for rigid body and constant strain states is constructed.

Higher order degrees of freedom.

The 12 visible nodal degrees of freedom v_x , v_y and θ for each node are ordered in an element displacement vector $\mathbf{v}$ as

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \boldsymbol{\theta} \end{Bmatrix}, \quad \text{where} \quad \begin{aligned} \mathbf{v}_x^T &= [v_{x1} \quad v_{x2} \quad v_{x3} \quad v_{x4}], \\ \mathbf{v}_y^T &= [v_{y1} \quad v_{y2} \quad v_{y3} \quad v_{y4}], \\ \boldsymbol{\theta}^T &= [\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4]. \end{aligned} \quad (5.2.5)$$

The correct rank of the element stiffness matrix for the quadrilateral membrane element is 9, coming from 12 degrees of freedom minus 3 rigid body modes. The basic stiffness gives is rank 3 from the 3 constant strain modes. The higher order stiffness must therefore be a rank 6 matrix. This can be conveniently achieved by introducing 6 higher order “intrinsic” degrees of freedom, which are collected in a vector $\tilde{\mathbf{v}}$ defined below.

Experience from the 3-node ANDES membrane element with drilling degrees of freedom [00] and the 4-node ANDES tetrahedron solid element with

rotational degrees of freedom [00], suggests using the hierarchical rotations $\tilde{\boldsymbol{\theta}}$ as higher order degrees of freedom:

$$\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \boldsymbol{\theta}_0, \quad \boldsymbol{\theta}_0^T = [\theta_0 \quad \theta_0 \quad \theta_0 \quad \theta_0], \quad (5.2.6)$$

where the subtracted θ_0 represents the overall or mean rotation of the element, associated with rigid body and constant strain rotation motions. Furthermore, splitting the hierarchical rotations into their mean and deviatoric parts as

$$\tilde{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}} + \boldsymbol{\theta}', \quad \bar{\boldsymbol{\theta}}^T = [\bar{\theta} \quad \bar{\theta} \quad \bar{\theta} \quad \bar{\theta}], \quad (5.2.7)$$

has the advantage of singling out the often troublesome “drilling mode”, or “torsional mode”, where all the drilling node rotations take the same value with all the other degrees of freedom being zero.

Unfortunately, the hierarchical rotations give only 4 higher order degrees of freedom and at most a rank 4 update of the stiffness matrix. Two more higher order degrees of freedom must be found. The element has 8 translational degrees of freedom represented by $\mathbf{v}_x$ and $\mathbf{v}_y$. The 3 rigid body and 3 constant strain modes can all be described by the translational degrees of freedom. There are still 2 higher order modes which can be described by the translational degrees of freedom. These two modes must be recognized so as to associate two higher order degrees of freedom with them. The amplitudes of the six higher order degrees of freedom are then represented as

$$\tilde{\mathbf{v}}^T = [\theta'_1 \quad \theta'_2 \quad \theta'_3 \quad \theta'_4 \quad \bar{\theta} \quad \alpha_1 \quad \alpha_2], \quad (5.2.8)$$

where α_1 and α_2 are associated with the two higher order translational modes. Although 7 degrees of freedom appear in this vector, the hierarchical rotation constraint

$$\sum_{i=1}^4 \theta'_i = 0 \quad (5.2.9)$$

reduces (5.2.8) effectively to six independent components.

Higher order rotational degrees of freedom.

θ_0 is evaluated as the rotation of the bi-linear quadrilateral computed at the element center given by $(\xi = 0, \eta = 0)$, and is function of the translational degrees of freedom only:

$$\theta_0 = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} \begin{bmatrix} -\mathbf{N}_{,y} & \mathbf{N}_{,x} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{Bmatrix}. \quad (5.2.10)$$

By using the partial derivative expressions in equation (5.1.7) one obtains the expressions for the rigid body and constant strain rotation as

$$\begin{aligned} -\mathbf{N}_{,y} &= -(J_{y\xi}^{-T} \mathbf{N}_{,\xi} + J_{y\eta}^{-T} \mathbf{N}_{,\eta}) = \frac{1}{16|\mathbf{J}|} \begin{bmatrix} x_{24} & x_{31} & x_{42} & x_{13} \end{bmatrix}, \\ \mathbf{N}_{,x} &= -(J_{x\xi}^{-T} \mathbf{N}_{,\xi} + J_{x\eta}^{-T} \mathbf{N}_{,\eta}) = \frac{1}{16|\mathbf{J}|} \begin{bmatrix} y_{24} & y_{31} & y_{42} & y_{13} \end{bmatrix}, \end{aligned} \quad (5.2.11)$$

where

$$|\mathbf{J}| = \frac{1}{8} ((x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_4 - x_4 y_3) + (x_4 y_1 - x_1 y_4)). \quad (5.2.12)$$

The higher order rotational degrees of freedom θ_h can be expressed in terms of the visible degrees of freedoms as

$$\boldsymbol{\theta}_h = \mathbf{H}_{\theta v} \mathbf{v}, \quad \boldsymbol{\theta}_h^T = [\theta'_1 \quad \theta'_2 \quad \theta'_3 \quad \theta'_4 \quad \bar{\theta}], \quad (5.2.13)$$

$$\mathbf{H}_{\theta v} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ \frac{x_{42}}{f} & \frac{x_{13}}{f} & \frac{x_{24}}{f} & \frac{x_{31}}{f} & \frac{y_{42}}{f} & \frac{y_{13}}{f} & \frac{y_{24}}{f} & \frac{y_{31}}{f} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad (5.2.14)$$

and $f = 16|\mathbf{J}|$.

Higher order translational degrees of freedom.

The translational degrees of freedom can be split into rigid body and constant strain displacements and higher order displacements:

$$\mathbf{v} = \mathbf{v}_{rc} + \mathbf{v}_h \quad \text{where} \quad \mathbf{v} = \begin{Bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{Bmatrix}. \quad (5.2.15)$$

$\mathbf{v}_{rc}$ can be written as a linear combination of the rc -modes as

$$\mathbf{v}_{rc} = \mathbf{R}\mathbf{a} \quad \text{with} \quad \mathbf{R} = [\mathbf{r}_x \quad \mathbf{r}_y \quad \mathbf{r}_{\theta_z} \quad \mathbf{c}_{\epsilon_{xx}} \quad \mathbf{c}_{\epsilon_{yy}} \quad \mathbf{c}_{\epsilon_{xy}}], \quad (5.2.16)$$

where $\mathbf{r}_x$, $\mathbf{r}_y$ and $\mathbf{r}_{\theta_z}$ are the rigid translations in the x and y directions, and the rigid rotation about the z axis, respectively. $\mathbf{c}_{\epsilon_{xx}}$, $\mathbf{c}_{\epsilon_{yy}}$ and $\mathbf{c}_{\epsilon_{xy}}$ are the constant strain displacement modes. By combining equations (5.2.15) and (5.2.16), and requiring that the higher order displacement vector be orthogonal to the rc -modes, that is $\mathbf{R}^T \mathbf{v}_h = \mathbf{0}$, one obtains

$$\mathbf{R}^T \mathbf{v} = \mathbf{R}^T \mathbf{R}\mathbf{a} + \mathbf{R}^T \mathbf{v}_h \quad \Rightarrow \quad \mathbf{a} = (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T \mathbf{v}. \quad (5.2.17)$$

On the basis of this relation two projector matrices $\mathbf{P}_{rc}$ and $\mathbf{P}_h$ that project the displacement vector $\mathbf{v}$ on the rc and h subspaces, respectively, can be constructed:

$$\begin{aligned} \mathbf{v}_{rc} &= \mathbf{P}_{rc} \mathbf{v}, & \mathbf{P}_{rc} &= \mathbf{R}(\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T, \\ \mathbf{v}_h &= \mathbf{P}_h \mathbf{v}, & \mathbf{P}_h &= \mathbf{I} - \mathbf{R}(\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T. \end{aligned} \quad (5.2.18)$$

The higher order translational modes can now be found either as the null-space of $\mathbf{P}_{rc}$, or as eigenvectors of $\mathbf{P}_h$ with associated eigenvalues equal to one, that is $\mathbf{P}_h \mathbf{v}_h = \mathbf{v}_h$. The latter scheme is the simpler one because the projector matrices enjoy the property $\mathbf{P}\mathbf{P} = \mathbf{P}$. Every column in $\mathbf{P}_h$ is thus its own eigenvector with eigenvalue one, and a higher order mode.

To write down these higher order modes in compact form it is convenient to express the vector that goes from the element centroid to the nodes $\mathbf{r}_i$ with respect to the local coordinate system $(\mathbf{g}_\xi, \mathbf{g}_\eta)$. These base vectors are the ξ - and η - gradient vectors computed at the element center according to equation (5.1.4). The nodal vector $\mathbf{r}_i$ can thus be expressed as

$$\mathbf{r}_i = \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} = [\mathbf{g}_\xi \quad \mathbf{g}_\eta] \begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix}. \quad (5.2.19)$$

The nodal coordinates ξ_i and η_i in the $(\mathbf{g}_\xi, \mathbf{g}_\eta)$ coordinate system are then obtained from

$$\begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix} = \mathbf{T}_\xi \begin{Bmatrix} x_i \\ y_i \end{Bmatrix}, \quad \mathbf{T}_\xi = \mathbf{J}^{-1} = [\mathbf{g}_\xi \quad \mathbf{g}_\eta]^{-1}. \quad (5.2.20)$$

The higher order translational mode is then given by

$$\mathbf{v}_h = \begin{Bmatrix} \varsigma_3 \\ \varsigma_4 \\ \varsigma_1 \\ \varsigma_2 \end{Bmatrix} = \begin{Bmatrix} \xi_3\eta_3 - a \\ \xi_4\eta_4 - a \\ \xi_1\eta_1 - a \\ \xi_2\eta_2 - a \end{Bmatrix}, \quad a = \frac{1}{4} \sum_{i=1}^4 \xi_i\eta_i. \quad (5.2.21)$$

The higher order translational degrees of freedom are the components along this higher order mode for the x and y nodal displacements $\mathbf{v}_x$ and $\mathbf{v}_y$ respectively. Expressed in term of the visible degrees of freedom this becomes

$$\tilde{\mathbf{v}}_t = \mathbf{H}_{tv} \mathbf{v}, \quad (5.2.22)$$

$$\begin{Bmatrix} \tilde{v}_x \\ \tilde{v}_y \end{Bmatrix} = \begin{bmatrix} \varsigma_3 & \varsigma_4 & \varsigma_1 & \varsigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varsigma_3 & \varsigma_4 & \varsigma_1 & \varsigma_2 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{v}. \quad (5.2.23)$$

If one defines the higher order translational degrees of freedom to be the higher order translational components along the ξ and η directions the relationship becomes

$$\begin{Bmatrix} \tilde{v}_\xi \\ \tilde{v}_\eta \end{Bmatrix} = \begin{bmatrix} \varsigma_3 s_{\xi x} & \varsigma_4 s_{\xi x} & \varsigma_1 s_{\xi x} & \varsigma_2 s_{\xi x} & \varsigma_3 s_{\xi y} & \varsigma_4 s_{\xi y} & \varsigma_1 s_{\xi y} & \varsigma_2 s_{\xi y} & \mathbf{0} \\ \varsigma_3 s_{\eta x} & \varsigma_4 s_{\eta x} & \varsigma_1 s_{\eta x} & \varsigma_2 s_{\eta x} & \varsigma_3 s_{\eta y} & \varsigma_4 s_{\eta y} & \varsigma_1 s_{\eta y} & \varsigma_2 s_{\eta y} & \mathbf{0} \end{bmatrix} \mathbf{v} \quad (5.2.24)$$

where $\mathbf{0} = [0 \ 0 \ 0 \ 0]$, and $s_{\xi i}$ and $s_{\eta i}$ denotes the unit vectors along the ξ - and η -directions, respectively:

$$\mathbf{s}_\xi = \begin{Bmatrix} s_{\xi x} \\ s_{\xi y} \end{Bmatrix}, \quad \mathbf{s}_\eta = \begin{Bmatrix} s_{\eta x} \\ s_{\eta y} \end{Bmatrix}. \quad (5.2.25)$$

The total higher order degrees of freedom vector $\tilde{\mathbf{v}}$ can then be obtained from the visible degrees of freedom $\mathbf{v}$ as

$$\tilde{\mathbf{v}} = \mathbf{H} \mathbf{v} \quad \text{where} \quad \mathbf{H} = \begin{bmatrix} \mathbf{H}_{\theta v} \\ \mathbf{H}_{vt} \end{bmatrix} \quad \text{and} \quad \begin{aligned} \tilde{\mathbf{v}}^T &= [\theta'_1 \quad \theta'_2 \quad \theta'_3 \quad \theta'_4 \quad \bar{\theta} \quad \tilde{v}_\xi \quad \tilde{v}_\eta] \\ \mathbf{v}^T &= [\mathbf{v}_x^T \quad \mathbf{v}_y^T \quad \boldsymbol{\theta}^T]. \end{aligned} \quad (5.2.26)$$

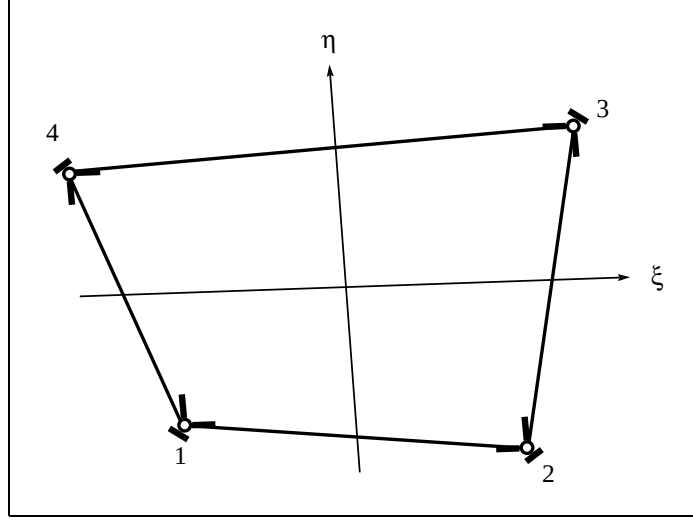


Figure 5.1. Nodal strain gages for membrane.

Higher order strains.

The distribution of higher order strains is expressed in terms of natural strain gage readings. The strain gage locations are placed at the 4 nodes (quadrilateral corners). Readings along three directions are required. These directions are: the ξ and η axis (quadrilateral medians) and the diagonal passing through the neighboring nodes. See Figure 4.1.

The nodal natural strain readings are thus defined as

$$\epsilon_1 = \begin{Bmatrix} \epsilon_\xi \\ \epsilon_\eta \\ \epsilon_{24} \end{Bmatrix}, \quad \epsilon_2 = \begin{Bmatrix} \epsilon_\xi \\ \epsilon_\eta \\ \epsilon_{13} \end{Bmatrix}, \quad \epsilon_3 = \begin{Bmatrix} \epsilon_\xi \\ \epsilon_\eta \\ \epsilon_{24} \end{Bmatrix}, \quad \epsilon_4 = \begin{Bmatrix} \epsilon_\xi \\ \epsilon_\eta \\ \epsilon_{13} \end{Bmatrix}. \quad (5.2.27)$$

The next step is to connect these readings to the higher order degrees of freedom. This can be done by defining a generic template

$$\epsilon_i = \mathbf{Q}_i \tilde{\mathbf{v}}, \quad (5.2.28)$$

where $\mathbf{Q}_i$ are 3×7 matrices. These templates are worked out below.

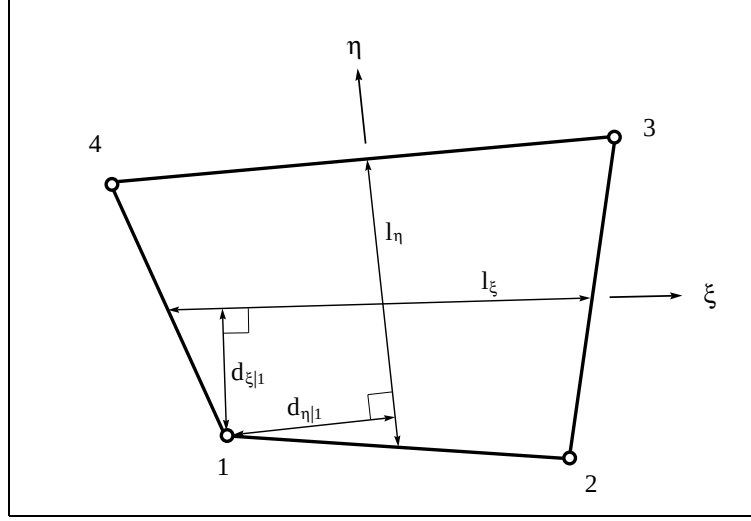


Figure 5.2. Geometric dimensions for a quadrilateral element.

Higher order bending strain field.

The main displacement and strain mode that the field is trying to match is the pure bending of the element along an arbitrary direction. The bending strain field is associated with the higher order degrees of freedom θ'_i , $\tilde{v}_\xi$ and $\tilde{v}_\eta$. If one considers pure bending of the element along the ξ direction, it seems intuitive that the ξ strain should be proportional to the distance d_ξ from the ξ -axis. The ξ strain should also be proportional to the curvature along the ξ -axis. In terms of the rotational degrees of freedoms this curvature will have the form $\Delta\theta/l_\xi$ where l_ξ is the element length along the ξ -axis. The ξ strain thus gets coefficients of the form d_ξ/l_ξ associated with the rotational degrees of freedom. Following a similar reasoning for the η strains the strain distribution factors associated with the ξ and η strains are established to be

$$\chi_{\xi|i} = \frac{d_{\xi|i}}{l_\xi}, \quad \chi_{\eta|i} = \frac{d_{\eta|i}}{l_\eta}, \quad (5.2.29)$$

where

$$d_{\xi|i} = \sqrt{(\mathbf{r}_i \times \mathbf{s}_\xi) \cdot (\mathbf{r}_i \times \mathbf{s}_\xi)}, \quad l_\xi = \sqrt{\mathbf{r}_\xi \cdot \mathbf{r}_\xi}, \quad \mathbf{r}_\xi = \frac{1}{2}(\mathbf{r}_2 + \mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_4),$$

$$d_{\eta|i} = \sqrt{(\mathbf{r}_i \times \mathbf{s}_\eta) \cdot (\mathbf{r}_i \times \mathbf{s}_\eta)}, \quad l_\eta = \sqrt{\mathbf{r}_\eta \cdot \mathbf{r}_\eta}, \quad \mathbf{r}_\eta = \frac{1}{2}(\mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_1 - \mathbf{r}_2).$$

The quantities $d_{\xi|i}$, $d_{\eta|i}$, l_ξ and l_η are illustrated in Figure 4.2.

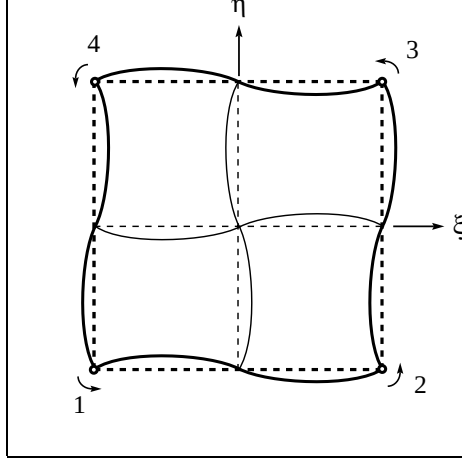


Figure 5.3. Torsional mode for four node membrane element.

The strain distribution sensed by the diagonal strain gages are similarly assumed to be proportional to the curvature along the diagonal, and proportional to the distance from the diagonal as

$$\chi_{24} = \frac{d_{24}}{2l_{24}}, \quad \chi_{13} = \frac{d_{13}}{2l_{13}}, \quad (5.2.30)$$

where

$$d_{24} = \sqrt{(\mathbf{r}_{31} \times \mathbf{e}_{24}) \cdot (\mathbf{r}_{13} \times \mathbf{e}_{24})}, \quad l_{24} = \sqrt{\mathbf{r}_{24} \cdot \mathbf{r}_{24}}, \quad \mathbf{r}_{24} = (\mathbf{r}_2 - \mathbf{r}_4),$$

$$d_{13} = \sqrt{(\mathbf{r}_{31} \times \mathbf{e}_{24}) \cdot (\mathbf{r}_{13} \times \mathbf{e}_{24})}, \quad l_{13} = \sqrt{\mathbf{r}_{13} \cdot \mathbf{r}_{13}}, \quad \mathbf{r}_{13} = (\mathbf{r}_1 - \mathbf{r}_3).$$

Torsional strain field.

The torsional strain field is associated to the $\bar{\theta}$ higher order degree of freedom. As a guide for the construction of this strain field one can use the torsional displacement mode illustrated in Figure 4.3. This figure indicates that this displacement mode should not induce shear strains, and that ϵ_ξ should be positive in 1st and 3rd quadrants and negative in 2nd and 4th. Similarly, ϵ_η should be positive in 2nd and 4th, and negative in 1st and 3rd quadrants. A simplified strain distribution function for the strains ϵ_ξ and ϵ_η can thus be $N_t = \xi\eta$.

With a unit rotation at all the nodes, the maximum displacement in the ξ direction, u_ξ , will be proportional to the length l_η . Since the strain ϵ_ξ is the gradient of the displacement u_ξ in the ξ direction this strain will be proportional to $1/l_\xi$. The torsional strain field is thus assumed to be

$$\epsilon_\xi = \alpha \frac{l_\eta}{l_\xi} \xi\eta = \alpha \chi_{\xi t} \xi\eta, \quad \epsilon_\eta = -\alpha \frac{l_\xi}{l_\eta} \xi\eta = -\alpha \chi_{\eta t} \xi\eta. \quad (5.2.31)$$

Generic nodal strain templates.

With the strain assumptions just described, the nodal strain gage readings can be written down as

$$\mathbf{Q}_1 = \begin{bmatrix} \rho_1 \chi_{\xi|1} & \rho_2 \chi_{\xi|1} & \rho_3 \chi_{\xi|1} & \rho_4 \chi_{\xi|1} & \alpha \chi_{\xi t} & -\beta_1 \frac{\chi_{\xi|1}}{\bar{\chi}_{\xi} l_{\xi}} & 0 \\ -\rho_1 \chi_{\eta|1} & -\rho_4 \chi_{\eta|1} & -\rho_3 \chi_{\eta|1} & -\rho_2 \chi_{\eta|1} & -\alpha \chi_{\eta t} & 0 & -\beta_1 \frac{\chi_{\eta|1}}{\bar{\chi}_{\eta} l_{\eta}} \\ \rho_5 \chi_{24} & \rho_6 \chi_{24} & \rho_7 \chi_{24} & \rho_8 \chi_{24} & 0 & \beta_2 \frac{c_{24\xi}}{l_{24}} & -\beta_2 \frac{c_{24\eta}}{l_{24}} \end{bmatrix}, \quad (5.2.32)$$

$$\mathbf{Q}_2 = \begin{bmatrix} -\rho_2 \chi_{\xi|2} & -\rho_1 \chi_{\xi|2} & -\rho_4 \chi_{\xi|2} & -\rho_3 \chi_{\xi|2} & -\alpha \chi_{\xi t} & -\beta_1 \frac{\chi_{\xi|2}}{\bar{\chi}_{\xi} l_{\xi}} & 0 \\ \rho_4 \chi_{\eta|2} & \rho_1 \chi_{\eta|2} & \rho_2 \chi_{\eta|2} & \rho_3 \chi_{\eta|2} & \alpha \chi_{\eta t} & 0 & \beta_1 \frac{\chi_{\eta|2}}{\bar{\chi}_{\eta} l_{\eta}} \\ \rho_8 \chi_{13} & \rho_5 \chi_{13} & \rho_6 \chi_{13} & \rho_7 \chi_{13} & 0 & -\beta_2 \frac{c_{13\xi}}{l_{13}} & \beta_2 \frac{c_{13\eta}}{l_{13}} \end{bmatrix}, \quad (5.2.33)$$

$$\mathbf{Q}_3 = \begin{bmatrix} \rho_3 \chi_{\xi|3} & \rho_4 \chi_{\xi|3} & \rho_1 \chi_{\xi|3} & \rho_2 \chi_{\xi|3} & \alpha \chi_{\xi t} & \beta_1 \frac{\chi_{\xi|3}}{\bar{\chi}_{\xi} l_{\xi}} & 0 \\ -\rho_3 \chi_{\eta|3} & -\rho_2 \chi_{\eta|3} & -\rho_1 \chi_{\eta|3} & -\rho_4 \chi_{\eta|3} & -\alpha \chi_{\eta t} & 0 & \frac{\beta_1 \chi_{\eta|3}}{\bar{\chi}_{\eta} l_{\eta}} \\ \rho_7 \chi_{13} & \rho_8 \chi_{13} & \rho_5 \chi_{13} & \rho_6 \chi_{13} & 0 & -\beta_2 \frac{c_{13\xi}}{l_{13}} & \beta_2 \frac{c_{13\eta}}{l_{13}} \end{bmatrix}, \quad (5.2.34)$$

$$\mathbf{Q}_4 = \begin{bmatrix} -\rho_4 \chi_{\xi|4} & -\rho_3 \chi_{\xi|4} & -\rho_2 \chi_{\xi|4} & -\rho_1 \chi_{\xi|4} & -\alpha \chi_{\xi t} & \beta_1 \frac{\chi_{\xi|4}}{\bar{\chi}_{\xi} l_{\xi}} & 0 \\ \rho_2 \chi_{\eta|4} & \rho_3 \chi_{\eta|4} & \rho_4 \chi_{\eta|4} & \rho_1 \chi_{\eta|4} & \alpha \chi_{\eta t} & 0 & -\beta_1 \frac{\chi_{\eta|4}}{\bar{\chi}_{\eta} l_{\eta}} \\ \rho_6 \chi_{13} & \rho_7 \chi_{13} & \rho_8 \chi_{13} & \rho_5 \chi_{13} & 0 & \beta_2 \frac{c_{13\xi}}{l_{13}} & -\beta_2 \frac{c_{13\eta}}{l_{13}} \end{bmatrix}, \quad (5.2.35)$$

where $c_{13\xi} = \mathbf{s}_{13}^T \mathbf{s}_{\xi}$, $c_{13\eta} = \mathbf{s}_{13}^T \mathbf{s}_{\eta}$, $c_{24\xi} = \mathbf{s}_{24}^T \mathbf{s}_{\xi}$ and $c_{24\eta} = \mathbf{s}_{24}^T \mathbf{s}_{\eta}$.

The cartesian strain displacement matrices at the nodes are obtained by the transformations

$$\begin{aligned} \mathbf{B}_{h1} &= \mathbf{T}_{13} \mathbf{Q}_1, \\ \mathbf{B}_{h3} &= \mathbf{T}_{13} \mathbf{Q}_3, \end{aligned} \quad \text{where} \quad \mathbf{T}_{13}^{-1} = \begin{bmatrix} s_{\xi x}^2 & s_{\xi y}^2 & s_{\xi x} s_{\xi y} \\ s_{\eta x}^2 & s_{\eta y}^2 & s_{\eta x} s_{\eta y} \\ s_{24x}^2 & s_{24y}^2 & s_{24x} s_{24y} \end{bmatrix}, \quad (5.2.36)$$

$$\begin{aligned} \mathbf{B}_{h2} &= \mathbf{T}_{24} \mathbf{Q}_2, \\ \mathbf{B}_{h4} &= \mathbf{T}_{24} \mathbf{Q}_4, \end{aligned} \quad \text{where} \quad \mathbf{T}_{24}^{-1} = \begin{bmatrix} s_{\xi x}^2 & s_{\xi y}^2 & s_{\xi x} s_{\xi y} \\ s_{\eta x}^2 & s_{\eta y}^2 & s_{\eta x} s_{\eta y} \\ s_{13x}^2 & s_{13y}^2 & s_{13x} s_{13y} \end{bmatrix}. \quad (5.2.37)$$

A higher order strain field over the element can now be obtained by interpolating the nodal Cartesian strains by use of the bi-linear shape functions defined in (5.1.3).

$$\begin{aligned} \mathbf{B}_h(\xi, \eta) = & (1 - \xi)(1 - \eta)\mathbf{B}_{h1} + (1 + \xi)(1 - \eta)\mathbf{B}_{h2} \\ & + (1 + \xi)(1 + \eta)\mathbf{B}_{h3} + (1 - \xi)(1 + \eta)\mathbf{B}_{h4} . \end{aligned} \quad (5.2.38)$$

Interpolation of these nodal strains does not automatically give a deviatoric higher order strain field. Such a condition can be achieved by subtracting the mean strain values:

$$\mathbf{B}_d(\xi, \eta) = \mathbf{B}_h(\xi, \eta) - \bar{\mathbf{B}}_h \quad \text{where} \quad \bar{\mathbf{B}}_h = \int_A \mathbf{B}(\xi, \eta) dA . \quad (5.2.39)$$

Optimal coefficients for the strain computation.

When computing the strain displacement expressions symbolically using Mathematica, the contributions of the different coefficients ρ_i and β_i were evaluated with respect to certain higher order strain modes. Based on pure bending of rectangular element shapes the following dependencies between the coefficients were obtained:

$$\begin{aligned} \rho_2 &= -\rho_1 , & \rho_3 &= \rho_2 , & \rho_4 &= \rho_1 , \\ \rho_6 &= \beta_1 - \rho_1 , & \beta_1 &= \frac{1}{2} + \rho_1 , \\ \rho_8 &= -\rho_6 , & \rho_5 &= \rho_7 = \beta_2 = 0 . \end{aligned} \quad (5.2.40)$$

As seen this makes all the coefficients a function of ρ_1 . Optimizing ρ_1 with respect to irregular meshes for the cantilever described in the numerical section suggests $\rho_1 = 0.1$, and the following set of optimal coefficients:

$$\begin{array}{cccc} \rho_1 = 0.1 & \rho_2 = -0.1 & \rho_3 = -0.1 & \rho_4 = 0.1 \\ \rho_5 = 0.0 & \rho_6 = 0.5 & \rho_7 = 0.0 & \rho_8 = -0.5 \\ & \beta_1 = 0.6 & \beta_2 = 0.0 & \end{array} \quad (5.2.41)$$

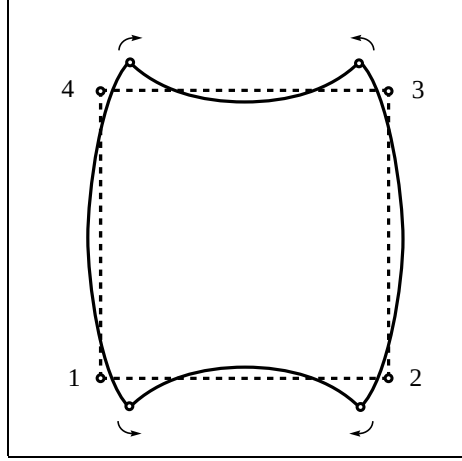


Figure 5.4. Spurious membrane mode for the four node ANDES element.

Stiffness computation for the membrane element.

According to the ANDES formulation the element stiffness is computed as

$$\mathbf{K} = \frac{1}{A} \mathbf{LCL}^T + \mathbf{H}^T \tilde{\mathbf{K}}_d \mathbf{H} \quad \text{where} \quad \tilde{\mathbf{K}}_d = \int_A \mathbf{B}_d^T \mathbf{C} \mathbf{B}_d dA . \quad (5.2.42)$$

Numerical experiments, however, indicate that the element performs better when the element stiffness is computed as

$$\mathbf{K} = \frac{1}{A} \mathbf{LCL}^T + \mathbf{H}^T \tilde{\mathbf{K}}_h \mathbf{H} \quad \text{where} \quad \tilde{\mathbf{K}}_h = \int_A \mathbf{B}_h^T \mathbf{C} \mathbf{B}_h dA , \quad (5.2.43)$$

that is when the non-deviatoric higher order strains are used. This is not strictly justified according to the standard ANDES formulation since the higher order strains displacement matrix $\mathbf{B}_h$ is not energy orthogonal with respect to the constant strain modes for arbitrary element geometries. However, both of the above element stiffness matrices satisfy the Individual Element Test and thus also the conventional Patch Test.

Rank of the stiffness matrix.

Performing an eigenvalue analysis of the element stiffness matrices given in equations (5.2.42) and (5.2.43) it was found that the element has one spurious zero energy mode in addition to the correct three rigid body modes. This spurious mode occurred using a 2×2 Gauss integration rule. It is expected that this spurious mode would disappear with a 3×3 integration rule. For a square element

as shown in Figure 5.4 this spurious mode is defined by the nodal displacement pattern

$$\begin{aligned}\mathbf{v}^T &= [v_{x1} \quad v_{x2} \quad v_{x3} \quad v_{x4} \quad v_{y1} \quad v_{y2} \quad v_{y3} \quad v_{y4} \quad \theta_{z1} \quad \theta_{z2} \quad \theta_{z3} \quad \theta_{z4}] \\ &= [1 \quad -1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1 \quad 4 \quad -4 \quad 4 \quad -4].\end{aligned}\tag{5.2.44}$$

Analysis of a mesh of two elements shows that this the pattern (5.2.44) can not occur in a mesh of more than one element. The spurious mode is then not practically significant for the performance of the element.

5.3 The quadrilateral bending element.

The current approach to deriving the quadrilateral plate bending element utilizes reference lines. Hrennikoff [00] first used this concept for plate modeling where the goal was to come up with a beam framework useful as a model for bending of flat plates.

Park and Stanley [00, 00] used the reference line concept in their development of several plate and shell elements based on the ANS formulation. The reference lines were used to find beam-like curvatures; these curvatures were then used to find the plate curvatures through various Assumed Natural Strain distributions. These plate and shell elements were of Mindlin-Reissner type, and the reference lines were treated as Timoshenko beams.

The present element is a Kirchhoff type plate and the reference lines are thus treated like Euler-Bernoulli (or Hermitian) beams.

5.3.1 Basic stiffness.

The basic stiffness for a flat quadrilateral bending element has been developed by extending the triangle element lumping matrices $\mathbf{L}_l$ and $\mathbf{L}_q$ of Militello to four node elements. $\mathbf{L}_l$ and $\mathbf{L}_q$ denotes lumping with respect to a linear and quadratic variation in the normal side rotation respectively.

By ordering the element degrees of freedom as rotation about x and y axis and translation in z direction for each node one obtains the lumped forces from bending as

$$\mathbf{f} = \mathbf{L}_l \boldsymbol{\sigma} \quad \text{or} \quad \mathbf{f} = \mathbf{L}_q \boldsymbol{\sigma} \quad \text{where} \quad \boldsymbol{\sigma} = \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix}, \tag{5.3.1}$$

$$\mathbf{L}_l = \begin{bmatrix} \mathbf{L}_{l1} \\ \mathbf{L}_{l2} \\ \mathbf{L}_{l3} \\ \mathbf{L}_{l4} \end{bmatrix}, \quad \mathbf{L}_q = \begin{bmatrix} \mathbf{L}_{q1} \\ \mathbf{L}_{q2} \\ \mathbf{L}_{q3} \\ \mathbf{L}_{q4} \end{bmatrix} \tag{5.3.2}$$

and

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_1 \\ \mathbf{f}_1 \\ \mathbf{f}_1 \end{Bmatrix} \quad \text{where} \quad \mathbf{f}_i = \begin{Bmatrix} m_x \\ m_y \\ f_z \end{Bmatrix}. \quad (5.3.3)$$

The lumped node forces at a node j given by lumping matrix $\mathbf{L}_j$, receive contributions from the moments from adjoining sides ij and jk . The lumped force vector at a node j is thus a function of the coordinates of the sides ij and jk only. With linear interpolation of the normal and tangential rotations along a side the lumping matrix becomes

$$\mathbf{L}_{lj} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x_{ki} & y_{ki} \\ -y_{ki} & 0 & -x_{ki} \end{bmatrix}, \quad (5.3.4)$$

where superscript l denotes linear variation of normal rotation. If the normal rotation is assumed to vary quadratically in accordance to Hermitian interpolation whereas and the tangential rotation still varies linearly the lumping matrix becomes

$$\mathbf{L}_{qj} = \begin{bmatrix} -c_{jk}s_{jk} + c_{ij}s_{ij} & c_{jk}s_{jk} - c_{ij}s_{ij} & -(s_{jk}^2 - c_{jk}^2) + (s_{ij}^2 + s_{ij}^2) \\ \frac{1}{2}(s_{jk}^2 x_{jk} + s_{ij}^2 x_{ij}) & \frac{1}{2}(c_{jk}^2 x_{jk} + c_{ij}^2 x_{ij}) & -c_{jk}^2 y_{jk} - c_{ij}^2 y_{ij} \\ \frac{1}{2}(s_{jk}^2 y_{jk} + s_{ij}^2 y_{ij}) & \frac{1}{2}(c_{jk}^2 y_{jk} + c_{ij}^2 y_{ij}) & -s_{jk}^2 x_{jk} - s_{ij}^2 x_{ij} \end{bmatrix} \quad (5.3.5)$$

where superscript q is used to denote quadratic variation of normal rotations. The nodal indices (i, j, k, l) in the equations above undergo cyclic permutations of $(1, 2, 3, 4)$ as for the membrane lumping.

5.3.2 Higher order stiffness

The higher order stiffness is computed as the deviatoric part of an ANS type element using the Euler-Bernoulli beam as a reference line strain guide.

Nodal curvatures of a Euler-Bernoulli beam.

The transverse displacement of a Euler-Bernoulli beam, written as a function of the nodal displacements and rotations is

$$w = \mathbf{N}_w \mathbf{v}_{bij} , \quad (5.3.6)$$

where

$$\mathbf{N}^T = \frac{1}{8} \begin{Bmatrix} 2(2+\xi)(-1+\xi)^2 \\ l(1+\xi)(-1+\xi)^2 \\ 2(2-\xi)(1+\xi)^2 \\ l(-1+\xi)(1+\xi)^2 \end{Bmatrix} \quad \text{and} \quad \mathbf{v}_{bij} = \begin{Bmatrix} w_i \\ \theta_{ni} \\ w_j \\ \theta_{nj} \end{Bmatrix} .$$

The beam curvatures are

$$\kappa = \frac{\partial^2 w}{\partial x^2} = \frac{1}{l^2} \begin{Bmatrix} 6\xi \\ l(-1+3\xi) \\ -6\xi \\ l(-1+3\xi) \end{Bmatrix} \mathbf{v}_{bij} , \quad (5.3.7)$$

The nodal curvatures are then

$$\begin{Bmatrix} \kappa_{ij|i} \\ \kappa_{ij|j} \end{Bmatrix} = \frac{1}{l^2} \begin{bmatrix} -6 & -4l & 6 & -2l \\ 6 & 2l & -6 & 4l \end{bmatrix} \mathbf{v}_{bij} \quad (5.3.8)$$

The nodal displacements of a reference-line from node i to j can be expressed in terms of the visible degrees of freedom at those nodes as

$$\mathbf{v}_{bij} = \mathbf{T}_{vij} \mathbf{v}_{ij} \quad (5.3.9)$$

$$\begin{Bmatrix} w_i \\ \theta_{ni} \\ w_j \\ \theta_{nj} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & n_{ijx} & n_{ijy} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_{ijx} & n_{ijy} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \\ w_j \\ \theta_{yj} \\ \theta_{yj} \end{Bmatrix} . \quad (5.3.10)$$

The nodal curvatures expressed in terms of the visible dofs at node i and j then become

$$\begin{Bmatrix} \kappa_{ij|i} \\ \kappa_{ij|j} \end{Bmatrix} = \frac{1}{l^2} \begin{bmatrix} -6 & -4l n_{ijx} & -4l n_{ijy} & 6 & -2l n_{ijx} & -2l n_{ijy} \\ 6 & 2l n_{ijx} & 2l n_{ijy} & -6 & 4l n_{ijx} & 4l n_{ijy} \end{bmatrix} \mathbf{v} . \quad (5.3.11)$$

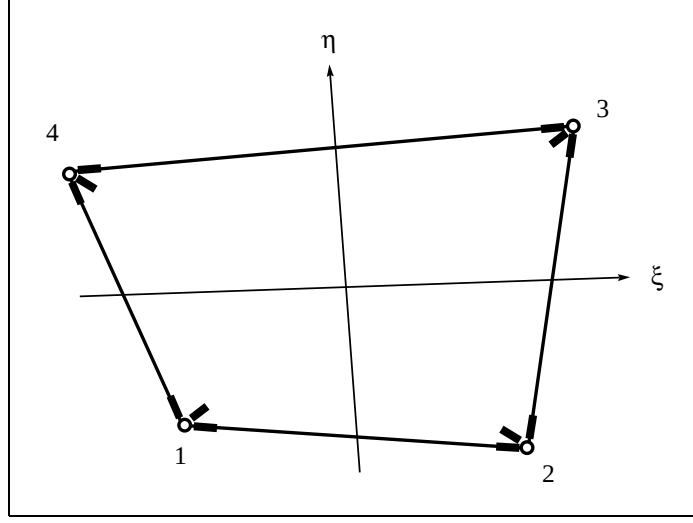


Figure 5.5. Nodal curvature gages for bending.

Nodal natural coordinate curvatures for a quadrilateral.

When one collects all the nodal straingages in a vector $\mathbf{g}$, the strain-gage displacement relationship becomes

$$\mathbf{g} = \mathbf{Q}\mathbf{v} = \mathbf{Q}_F * \mathbf{F}\mathbf{v} , \quad (5.3.12)$$

where $*$ denotes entry by entry matrix multiplication, and

$$\mathbf{g}^T = [\kappa_{41|1} \quad \kappa_{12|1} \quad \kappa_{13|1} \quad \kappa_{12|2} \quad \kappa_{23|2} \quad \kappa_{24|2} \quad \kappa_{23|3} \quad \kappa_{34|3} \quad \kappa_{13|3} \quad \kappa_{34|4} \quad \kappa_{41|4} \quad \kappa_{24|4}] , \quad (5.3.13)$$

$$\mathbf{Q}_F = \begin{bmatrix} -6 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 2 \\ -6 & -4 & -4 & 6 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -4 & -4 & 0 & 0 & 0 & 6 & -2 & -2 & 0 & 0 & 0 \\ \\ 6 & 2 & 2 & -6 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & -4 & -4 & 6 & -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & -4 & -4 & 0 & 0 & 0 & 6 & -2 & -2 \\ \\ 0 & 0 & 0 & 6 & 2 & 2 & -6 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & -4 & -4 & 6 & -2 & -2 \\ 6 & 2 & 2 & 0 & 0 & 0 & -6 & 4 & 4 & 0 & 0 & 0 \\ \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -6 & 4 & 4 \\ 6 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -4 & -4 \\ 0 & 0 & 0 & 6 & 2 & 2 & 0 & 0 & 0 & -6 & 4 & 4 \end{bmatrix} , \quad (5.3.14)$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{41} & \mathbf{F}_{41} & \mathbf{F}_{41} & \mathbf{F}_{41} \\ \mathbf{F}_{12} & \mathbf{F}_{12} & \mathbf{F}_{12} & \mathbf{F}_{12} \\ \mathbf{F}_{13} & \mathbf{F}_{13} & \mathbf{F}_{13} & \mathbf{F}_{13} \\ \mathbf{F}_{12} & \mathbf{F}_{12} & \mathbf{F}_{12} & \mathbf{F}_{12} \\ \mathbf{F}_{23} & \mathbf{F}_{23} & \mathbf{F}_{23} & \mathbf{F}_{23} \\ \mathbf{F}_{24} & \mathbf{F}_{24} & \mathbf{F}_{24} & \mathbf{F}_{24} \\ \mathbf{F}_{23} & \mathbf{F}_{23} & \mathbf{F}_{23} & \mathbf{F}_{23} \\ \mathbf{F}_{34} & \mathbf{F}_{34} & \mathbf{F}_{34} & \mathbf{F}_{34} \\ \mathbf{F}_{13} & \mathbf{F}_{13} & \mathbf{F}_{13} & \mathbf{F}_{13} \\ \mathbf{F}_{34} & \mathbf{F}_{34} & \mathbf{F}_{34} & \mathbf{F}_{34} \\ \mathbf{F}_{41} & \mathbf{F}_{41} & \mathbf{F}_{41} & \mathbf{F}_{41} \\ \mathbf{F}_{24} & \mathbf{F}_{24} & \mathbf{F}_{24} & \mathbf{F}_{24} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \mathbf{F}_{12} &= \begin{bmatrix} \frac{1}{l_{12}^2} & \frac{n_{12x}}{l_{12}} & \frac{n_{12y}}{l_{12}} \end{bmatrix} \\ \mathbf{F}_{23} &= \begin{bmatrix} \frac{1}{l_{23}^2} & \frac{n_{23x}}{l_{23}} & \frac{n_{23y}}{l_{23}} \end{bmatrix} \\ \mathbf{F}_{34} &= \begin{bmatrix} \frac{1}{l_{34}^2} & \frac{n_{34x}}{l_{34}} & \frac{n_{34y}}{l_{34}} \end{bmatrix} \\ \mathbf{F}_{41} &= \begin{bmatrix} \frac{1}{l_{41}^2} & \frac{n_{41x}}{l_{41}} & \frac{n_{41y}}{l_{41}} \end{bmatrix} \\ \mathbf{F}_{13} &= \begin{bmatrix} \frac{1}{l_{13}^2} & \frac{n_{13x}}{l_{13}} & \frac{n_{13y}}{l_{13}} \end{bmatrix} \\ \mathbf{F}_{24} &= \begin{bmatrix} \frac{1}{l_{24}^2} & \frac{n_{24x}}{l_{24}} & \frac{n_{24y}}{l_{24}} \end{bmatrix} \end{aligned} \quad (5.3.15)$$

Cartesian curvatures for a quadrilateral.

The cartesian curvatures $\boldsymbol{\kappa}^T = [\kappa_{xx} \quad \kappa_{yy} \quad \kappa_{xy}]$ at the nodes can now be obtained as

$$\mathbf{g}_C = \mathbf{Q}_C \mathbf{v} \quad (5.3.16)$$

or

$$\begin{Bmatrix} \boldsymbol{\kappa}_{|1} \\ \boldsymbol{\kappa}_{|2} \\ \boldsymbol{\kappa}_{|3} \\ \boldsymbol{\kappa}_{|4} \end{Bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \mathbf{B}_4 \end{bmatrix} \mathbf{v} = \begin{bmatrix} \mathbf{T}_{\kappa 1} \mathbf{Q}_1 \\ \mathbf{T}_{\kappa 2} \mathbf{Q}_2 \\ \mathbf{T}_{\kappa 3} \mathbf{Q}_3 \\ \mathbf{T}_{\kappa 4} \mathbf{Q}_4 \end{bmatrix} \mathbf{v},$$

where

$$\begin{aligned} \mathbf{T}_{\kappa 1}^{-1} &= \begin{bmatrix} s_{41x}^2 & s_{41y}^2 & s_{41x}s_{41y} \\ s_{12x}^2 & s_{12y}^2 & s_{12x}s_{12y} \\ s_{13x}^2 & s_{13y}^2 & s_{13x}s_{13y} \end{bmatrix}, \\ \mathbf{T}_{\kappa 2}^{-1} &= \begin{bmatrix} s_{12x}^2 & s_{12y}^2 & s_{12x}s_{12y} \\ s_{23x}^2 & s_{23y}^2 & s_{23x}s_{23y} \\ s_{24x}^2 & s_{24y}^2 & s_{24x}s_{24y} \end{bmatrix}, \\ \mathbf{T}_{\kappa 3}^{-1} &= \begin{bmatrix} s_{23x}^2 & s_{23y}^2 & s_{23x}s_{23y} \\ s_{34x}^2 & s_{34y}^2 & s_{34x}s_{34y} \\ s_{13x}^2 & s_{13y}^2 & s_{13x}s_{13y} \end{bmatrix}, \\ \mathbf{T}_{\kappa 4}^{-1} &= \begin{bmatrix} s_{34x}^2 & s_{34y}^2 & s_{34x}s_{34y} \\ s_{41x}^2 & s_{41y}^2 & s_{41x}s_{41y} \\ s_{24x}^2 & s_{24y}^2 & s_{24x}s_{24y} \end{bmatrix}. \end{aligned}$$

The Cartesian curvatures over the element can then be obtained by interpolation of the nodal values as

$$\boldsymbol{\kappa} = \mathbf{B}(\xi, \eta) \mathbf{v}, \quad (5.3.17)$$

where

$$\begin{aligned} \mathbf{B}(\xi, \eta) = & (1 - \xi)(1 - \eta)\mathbf{B}_1 + (1 + \xi)(1 - \eta)\mathbf{B}_2 \\ & + (1 + \xi)(1 + \eta)\mathbf{B}_3 + (1 - \xi)(1 + \eta)\mathbf{B}_4 . \end{aligned} \quad (5.3.18)$$

Higher order stiffness for the element.

The ANDES higher order stiffness is computed as

$$\mathbf{K}_d = \int_A \mathbf{B}_d^T \mathbf{C} \mathbf{B}_d dA \quad \text{where} \quad \mathbf{B}_d = \mathbf{B} - \frac{1}{A} \int_A \mathbf{B} dA . \quad (5.3.19)$$

5.3.3 The ANS quadrilateral plate bending element.

Clearly one can form an ANS type element by

$$\mathbf{K} = \int_A \mathbf{B}^T \mathbf{C} \mathbf{B} dA \quad (5.3.20)$$

i.e. without extracting the mean part of the strain displacement matrix and not including the basic stiffness described in Section 5.3.1.

5.4 The linear non-flat quadrilateral shell element.

The objective of this section is to develop a technique that allows the use of the flat quadrilateral membrane and bending element as parts of a non-flat shell element for linear problems. This is obtained by formulating a linear projector matrix, which for the linear case restores equilibrium at the undeformed element geometry. This can also be obtained by using the nonlinear projector with respect to the initial geometry. In fact the linear and nonlinear projector gives identical results for linear problems. However the linear projector is recommended for linear finite element codes due to its greater simplicity.

The four node shell element is obtained by assembling the membrane element and bending element to the appropriate degrees of freedom. This is sufficient as long as the shell element is strictly flat since both the membrane and bending elements are developed as flat elements. Unfortunately, four node shell elements on a “real” structure quite often end up being warped. To restore or improve the behavior of the warped element one can use a projection technique similar to that developed by Rankin and coworkers [00, 00].

The element stiffness matrix does not have the correct rigid body modes if the element geometry is warped since the element stiffness has been developed using the projected flat positions of the element nodes. This causes two deficiencies of the element stiffness:

1. The element picks up strains and thus forces from a rigid body displacement vector i.e. $\mathbf{f}_r = \mathbf{K} \mathbf{v}_r \neq \mathbf{0}$.

2. The element forces are not in self equilibrium and the force vector will thus pick up energy for a rigid body motion. $\mathbf{v}_r^T \mathbf{f} = \mathbf{v}_r^T \mathbf{K} \mathbf{v} \neq \mathbf{0}$.

These two statements are equivalent for a symmetric element stiffness matrix. If an element stiffness has columns that are in self equilibrium the element has the correct rigid body modes and vice versa.

The foregoing deficiencies lead to the investigation of the element internal energy

$$\begin{aligned} \Phi &= \frac{1}{2} \mathbf{v}^T \mathbf{K} \mathbf{v} = \frac{1}{2} (\mathbf{v}_r^T + \mathbf{v}_d^T) \mathbf{K} (\mathbf{v}_r + \mathbf{v}_d) \\ &= \frac{1}{2} (\mathbf{v}_d^T \mathbf{K} \mathbf{v}_d + \mathbf{v}_d^T \mathbf{K} \mathbf{v}_r + \mathbf{v}_r^T \mathbf{K} \mathbf{v}_d + \mathbf{v}_r^T \mathbf{K} \mathbf{v}_r). \end{aligned} \quad (5.4.1)$$

If the element fails the equilibrium and rigid-body conditions;

$$\mathbf{v}_r^T \mathbf{K} \mathbf{v}_r \neq 0, \quad \mathbf{v}_r^T \mathbf{K} \mathbf{v}_d \neq 0 \quad \text{and} \quad \mathbf{v}_d^T \mathbf{K} \mathbf{v}_r \neq 0. \quad (5.4.2)$$

To extract the deformational energy, the total displacements are split into deformational and rigid body motions, the latter being spanned by the matrix $\mathbf{R}$:

$$\mathbf{v} = \mathbf{v}_d + \mathbf{v}_r = \mathbf{v}_d + \mathbf{R} \mathbf{a}. \quad (5.4.3)$$

By requiring that the deformational displacement vector be orthogonal to the rigid body modes one must have $\mathbf{R}^T \mathbf{v}_d = \mathbf{0}$. On pre-multiplying the equation above with $\mathbf{R}^T$ the rigid body amplitudes can be solved for:

$$\mathbf{R}^T \mathbf{v} = \mathbf{R}^T \mathbf{R} \mathbf{a} \quad \Rightarrow \quad \mathbf{a} = (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T \mathbf{v}, \quad (5.4.4)$$

from which the deformational displacement vector can be extracted as

$$\mathbf{v}_d = \mathbf{v} - \mathbf{v}_r = (\mathbf{I} - \mathbf{R} (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T) \mathbf{v} = \mathbf{P}_d \mathbf{v}. \quad (5.4.5)$$

If $\mathbf{R}$ is orthonormal the foregoing expression simplifies to

$$\mathbf{P}_d = \mathbf{I} - \mathbf{R} \mathbf{R}^T. \quad (5.4.6)$$

Applying this projection to gain invariance of the internal energy with respect to rigid body motion $\Phi(\mathbf{v}) = \Phi(\mathbf{v}_d)$ yields

$$\Phi(\mathbf{v}_d) = \frac{1}{2} \mathbf{v}_d^T \mathbf{K} \mathbf{v}_d = \frac{1}{2} \mathbf{v}^T \mathbf{P}_d^T \mathbf{K} \mathbf{P}_d \mathbf{v} = \frac{1}{2} \mathbf{v}^T \mathbf{K}_d \mathbf{v}. \quad (5.4.7)$$

5.4.1 Linear projector matrix for a general quad.

In order to express the rigid body modes one defines the vector $\mathbf{r}_i$ from the element centroid to node i as

$$\tilde{\mathbf{r}}_i = \mathbf{r}_i - \bar{\mathbf{r}}, \quad \text{where} \quad \mathbf{r}_i = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \quad \text{and} \quad \bar{\mathbf{r}} = \frac{1}{4} \sum_{i=1}^4 \mathbf{r}_i. \quad (5.4.8)$$

By ordering the element degrees of freedom as

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{Bmatrix} \quad \text{where} \quad \mathbf{v}_i = \begin{Bmatrix} v_{xi} \\ v_{yi} \\ v_{zi} \\ \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} \quad (5.4.9)$$

the rigid body modes can be expressed as

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \mathbf{R}_4 \end{bmatrix}, \quad \mathbf{R}_i = \begin{bmatrix} \mathbf{I} & -\text{Spin}(\tilde{\mathbf{r}}_i) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \tilde{z}_i & -\tilde{y}_i \\ 0 & 1 & 0 & -\tilde{z}_i & 0 & \tilde{x}_i \\ 0 & 0 & 1 & \tilde{y}_i & -\tilde{x}_i & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5.4.10)$$

The projector matrix becomes

$$\mathbf{P}_d = \mathbf{I} - \mathbf{R}(\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T, \quad (5.4.11)$$

where

$$\mathbf{R}^T \mathbf{R} = \begin{bmatrix} 4\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix} \quad \text{with} \quad \mathbf{S} = 4\mathbf{I} - \sum_{i=1}^4 \text{Spin}(\tilde{\mathbf{r}}_i) \text{Spin}(\tilde{\mathbf{r}}_i).$$

This simplifies the computation of the projector matrix because only the lowest 3×3 submatrix of $\mathbf{R}^T \mathbf{R}$ is non-diagonal, and $(\mathbf{R}^T \mathbf{R})^{-1}$ can be efficiently formed.

5.5 Nonlinear extensions for quadrilateral shell element.

The nonlinear extensions for an element consists of defining a procedure that aligns the shadow element C_{0n} as close as possible to the deformed element C_n . This defines the element deformational displacement vector $\mathbf{v}_d$.

One also needs to form the rotational gradient of the shadow element with respect to the visible degrees of freedom of the deformed element, as stated in equation (0.0.0). In the local coordinate system this relation is

$$\delta \tilde{\omega}_r = \frac{\partial \tilde{\omega}_r}{\partial \tilde{v}_i} \delta v_i = \tilde{\mathbf{G}} \delta \tilde{\mathbf{v}} . \quad (5.5.1)$$

The local coordinate relationship is sought since this is needed in forming the geometric stiffness of the element as expressed in equation (0.0.0) .

The rotation of the shadow element is most easily obtained from the rotation of the shared or common local frame for the C_{0n} and C_n configurations. This orthogonal element coordinate frame with unit axis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ is rigidly attached to the shadow element C_{0n} , since this element only moves as a rigid body, and elastically attached to the deformed and elastic element C_n . This local coordinate system for a quadrilateral element can be defined in various ways. Most researchers select the element z -axis unit vector as the cross product of the diagonals vectors $\mathbf{d}_{13}$ and $\mathbf{d}_{24}$

$$\mathbf{e}_3 = \frac{\mathbf{d}_{13} \times \mathbf{d}_{24}}{A_p} \quad \text{where} \quad A_p = \sqrt{(\mathbf{d}_{13} \times \mathbf{d}_{24})^T (\mathbf{d}_{13} \times \mathbf{d}_{24})} \quad (5.5.2)$$

This defines A_p as the area of the element projection on the local $x - y$ plane.

The positioning of the x and y axis unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ differs among researchers. Rankin and Brogan [00] chooses $\mathbf{e}_2$ to coincide with the projection of the side edge 24 on the plane normal to $\mathbf{e}_3$. This effectively lets only one of the side edges determine the rigid rotation of the element about the local z axis. The origin of the element coordinate system is chosen to coincide with node 1. When this procedure is performed for both the C_0 and C_n element configurations the net result is that the shadow element C_{0n} will be positioned relative to C_n so that nodes 1 coincide and the projections of side edge 24 on the (x, y) plane coincide. A consequence of this choice is that the element deformational displacement vector $\mathbf{v}_d$, which is the difference between the coordinate between the C_n and C_{0n} coordinates, is not invariant with respect to the element node numbering.

Bergan and Nygård [00] choose vector $\mathbf{e}_1$ and $\mathbf{e}_2$ to coincide with the directions of side edge 12 and 14 for a rectangle that is positioned relative to the quadrilateral element so that the sum of the angles between the side edges of the quadrilateral and rectangle is zero. The origin of the coordinate system

is chosen at node 1. By applying this to both the C_0 and C_n configurations the shadow element C_{0n} is positioned relative to the deformed element C_n so that the element centroids coincide and so that the sum of the square of the angles between the side edges of C_{0n} and C_n is minimized. This represents a least square fit with respect to the side edge angular errors. This procedure gives a element deformational displacement vector $\mathbf{v}_d$ which produces an internal force vector $\mathbf{f}_e = \mathbf{K}_e \mathbf{v}_d$ that is invariant with respect to the node numbering of the element, provided that the element stiffness matrix $\mathbf{K}_e$ satisfies the correct rigid body translations.

5.5.1 Aligning side 12 of C_{0n} and C_n .

The element frame is positioned at the element centroid. This change from Rankin's positioning at node 1 has been done in order to satisfy the orthogonality condition for $\mathbf{P}_T \mathbf{P}_R = \mathbf{0}$ as expressed in equation (0.0.0). Rankin's formulation did not contain $\mathbf{P}_T$ so this requirement was ignored.

By expressing the nodal coordinates of the element in the local coordinate system equation (5.5.2) gives

$$\tilde{\mathbf{e}}_3 = \begin{Bmatrix} \tilde{e}_{3x} \\ \tilde{e}_{3y} \\ \tilde{e}_{3z} \end{Bmatrix} = \frac{1}{A_p} \begin{Bmatrix} \tilde{y}_{31}\tilde{z}_{42} - \tilde{y}_{42}\tilde{z}_{31} \\ -\tilde{x}_{31}\tilde{z}_{42} + \tilde{x}_{42}\tilde{z}_{31} \\ \tilde{x}_{31}\tilde{y}_{42} - \tilde{x}_{42}\tilde{y}_{31} \end{Bmatrix}, \quad (5.5.3)$$

where

$$A_p = \sqrt{(\tilde{y}_{31}\tilde{z}_{42} - \tilde{y}_{42}\tilde{z}_{31})^2 + (-\tilde{x}_{31}\tilde{z}_{42} + \tilde{x}_{42}\tilde{z}_{31})^2 + (\tilde{x}_{31}\tilde{y}_{42} - \tilde{x}_{42}\tilde{y}_{31})^2}. \quad (5.5.4)$$

These expressions simplify, but the full expressions has to be kept in order to obtain the correct variation with respect to the nodal coordinates. The $\tilde{\omega}_x$ and $\tilde{\omega}_y$ variation can now be obtained from the variation of e_{3y} and e_{3x} respectively

$$\begin{aligned} \delta\tilde{\omega}_x &= -\left(\frac{\partial\tilde{e}_{3y}}{\partial\tilde{x}_i}\delta\tilde{x}_i + \frac{\partial\tilde{e}_{3y}}{\partial\tilde{y}_i}\delta\tilde{y}_i + \frac{\partial\tilde{e}_{3y}}{\partial\tilde{z}_i}\delta\tilde{z}_i\right), \\ \delta\tilde{\omega}_y &= \left(\frac{\partial\tilde{e}_{3x}}{\partial\tilde{x}_i}\delta\tilde{x}_i + \frac{\partial\tilde{e}_{3x}}{\partial\tilde{y}_i}\delta\tilde{y}_i + \frac{\partial\tilde{e}_{3x}}{\partial\tilde{z}_i}\delta\tilde{z}_i\right). \end{aligned} \quad (5.5.5)$$

The variation of $\tilde{\omega}_x$ and $\tilde{\omega}_y$ with respect to the in-plane coordinate components of the nodes $\tilde{x}_i$ and $\tilde{y}_i$ is zero since

$$\frac{\partial\tilde{e}_{3x}}{\partial\tilde{x}_i} = \frac{\partial\tilde{e}_{3x}}{\partial\tilde{y}_i} = \frac{\partial\tilde{e}_{3y}}{\partial\tilde{x}_i} = \frac{\partial\tilde{e}_{3y}}{\partial\tilde{y}_i} = 0. \quad (5.5.6)$$

This gives the variation of the in-plane rotations $\tilde{\omega}_x$ and $\tilde{\omega}_y$ as function of the out of plane displacements only:

$$\begin{aligned}\delta\tilde{\omega}_x &= -\frac{\partial\tilde{e}_{3x}}{\partial\tilde{z}_i}\delta\tilde{z}_i = \sum_{i=1}^4 \frac{\tilde{x}_{lj}}{A_p}\delta\tilde{z}_i, \\ \delta\tilde{\omega}_y &= \frac{\partial\tilde{e}_{3y}}{\partial\tilde{z}_i}\delta\tilde{z}_i = \sum_{i=1}^4 \frac{\tilde{y}_{lj}}{A_p}\delta\tilde{z}_i,\end{aligned}\quad \text{with } A_p = \tilde{x}_{31}\tilde{y}_{42} - \tilde{x}_{42}\tilde{y}_{31}, \quad (5.5.7)$$

where the nodal indices (i, j, k, l) takes cyclic permutations of $(1, 2, 3, 4)$.

The $\mathbf{e}_1$ vector is chosen to lie along the projection of side 12 in the x - y plane. This gives the y axis unit vector as

$$\mathbf{e}_2 = \frac{\mathbf{e}_3 \times \mathbf{r}_{12}}{l_{12}}, \quad (5.5.8)$$

where l_{12} is the projected length of side 12 in the x - y plane. By expressing $\mathbf{e}_2$ in the local coordinate system the variation of $\tilde{\omega}_z$ can be obtained as

$$\delta\tilde{\omega}_z = -\left(\frac{\partial\tilde{e}_{2x}}{\partial\tilde{x}_i}\delta\tilde{x}_i + \frac{\partial\tilde{e}_{2x}}{\partial\tilde{y}_i}\delta\tilde{y}_i + \frac{\partial\tilde{e}_{2x}}{\partial\tilde{z}_i}\delta\tilde{z}_i\right). \quad (5.5.9)$$

Carrying out the derivations gives

$$\delta\tilde{\omega}_z = \frac{1}{l_{12}}(\delta\tilde{y}_1 + \delta\tilde{y}_2) - \sum_{i=1}^4 \frac{\tilde{x}_{lj}z_{21}}{A_p}\delta\tilde{z}_i, \quad (5.5.10)$$

where A_p is defined in equation (5.5.7).

The rotation gradient matrix in equation (5.5.1) can now be expressed as

$$\tilde{\mathbf{G}} = [\tilde{\mathbf{G}}_1 \quad \tilde{\mathbf{G}}_2 \quad \tilde{\mathbf{G}}_3] = \begin{bmatrix} \tilde{\mathbf{G}}_{\omega_x} \\ \tilde{\mathbf{G}}_{\omega_y} \\ \tilde{\mathbf{G}}_{\omega_z} \end{bmatrix} \quad (5.5.11)$$

where

$$\begin{aligned}\tilde{\mathbf{G}}_1 &= \frac{1}{A_p} \begin{bmatrix} 0 & 0 & x_{42} & 0 & 0 & 0 \\ 0 & 0 & y_{42} & 0 & 0 & 0 \\ 0 & -\frac{A_p}{l_{12}} & x_{42}z_{21} & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{\mathbf{G}}_2 &= \frac{1}{A_p} \begin{bmatrix} 0 & 0 & x_{13} & 0 & 0 & 0 \\ 0 & 0 & y_{13} & 0 & 0 & 0 \\ 0 & \frac{A_p}{l_{12}} & x_{13}z_{21} & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{\mathbf{G}}_3 &= \frac{1}{A_p} \begin{bmatrix} 0 & 0 & x_{24} & 0 & 0 & 0 \\ 0 & 0 & y_{24} & 0 & 0 & 0 \\ 0 & 0 & x_{24}z_{21} & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{\mathbf{G}}_4 &= \frac{1}{A_p} \begin{bmatrix} 0 & 0 & x_{31} & 0 & 0 & 0 \\ 0 & 0 & y_{31} & 0 & 0 & 0 \\ 0 & 0 & x_{42}z_{31} & 0 & 0 & 0 \end{bmatrix}.\end{aligned} \quad (5.5.12)$$

The rotation gradient matrix as expressed here can be split as $\tilde{\mathbf{G}} = \tilde{\mathbf{X}}\mathbf{A}$ where $\tilde{\mathbf{X}}$ contains all the coordinate dependencies and $\mathbf{A}$ is constant matrix. This is possible since none of the vector components of $\delta\tilde{\boldsymbol{\omega}}$ is a function of more than three distinct coordinate expressions. This $\tilde{\mathbf{G}}$ thus satisfies the consistency requirement set forth in equation (0.0.0).

5.5.2 Least square fit of side edge angular errors.

The orientation of $\mathbf{e}_3$ is the same as with Rankin's procedure, which gives identical expressions for the rotations $\tilde{\omega}_x$ and $\tilde{\omega}_y$. The positioning procedure of Bergan and Nygård [00] positioning procedure can be viewed as a least square fit of the side edge angular errors between the C_{0n} and C_n configurations. This gives different expressions for the variation of the angle $\tilde{\omega}_z$ with respect to the visible degrees of freedom. The nodal submatrices $\tilde{\mathbf{G}}_i$ of $\tilde{\mathbf{G}}$ can then be defined as

$$\mathbf{G}_i = \frac{1}{A_p} \begin{bmatrix} 0 & 0 & x_{lj} \\ 0 & 0 & y_{lj} \\ \frac{A_p}{4}(\frac{n_{ijx}}{l_{ij}} - \frac{n_{kix}}{l_{ki}}) & \frac{A_p}{4}(\frac{n_{ijy}}{l_{ij}} - \frac{n_{kiy}}{l_{ki}}) & (x_{lj}f_x + y_{lj}f_y) \end{bmatrix}, \quad (5.5.13)$$

where

$$\begin{aligned} f_x &= \frac{1}{4} \left(\frac{z_{21}x_{21}}{l_{21}} + \frac{z_{32}x_{32}}{l_{32}} + \frac{z_{43}x_{43}}{l_{43}} + \frac{z_{14}x_{14}}{l_{14}} \right), \\ f_y &= \frac{1}{4} \left(\frac{z_{21}y_{21}}{l_{21}} + \frac{z_{32}y_{32}}{l_{32}} + \frac{z_{43}y_{43}}{l_{43}} + \frac{z_{14}y_{14}}{l_{14}} \right), \end{aligned} \quad (5.5.14)$$

and $\tilde{\mathbf{n}}_{ij}$ and l_{ij} is the outward normal and length of side edge ij respectively:

$$\mathbf{n}_{ij} = \frac{1}{l_{ij}} \begin{Bmatrix} y_{ji} \\ -x_{ji} \\ 0 \end{Bmatrix} \quad \text{and} \quad l_{ij} = \sqrt{x_{ij}^2 + y_{ij}^2}. \quad (5.5.15)$$

The nodal indices (i, j, k, l) undergo cyclic permutations of $(1, 2, 3, 4)$.

The $\tilde{\mathbf{G}}$ derived above can not be expressed as $\tilde{\mathbf{G}} = \tilde{\mathbf{X}}\mathbf{A}$ where the 3×3 matrix $\tilde{\mathbf{X}}$ contains all the coordinate dependencies of $\tilde{\mathbf{G}}$, since the expressions for $\tilde{\omega}_z$ contains more than three distinct coordinate expressions. This will give a loss in convergence rate if the deformed configuration C_n and shadow configuration C_{0n} are far apart since the present tangent stiffness expressions have omitted terms containing the unbalanced element forces. See Section 0.0.0.

Chapter 6

Numerical examples for linear analysis.

This Chapter presents numerical examples of several linear test problems. These are used to validate the formulation of the linear ANDES elements that represents the “kernel” of the co-rotational formulation.

6.1 Patch tests.

The Patch Test has become a standard test for evaluation of new finite elements. Though neither a necessary or sufficient condition for convergence, it has a strong following who considers the test “necessary” for an element to be considered reliable. However, there is little disagreement about the tests value as a debugging tool when an element is implemented in an actual finite element code.

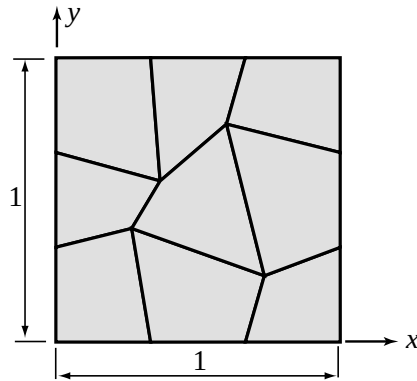


Figure 6.1. Patch test for quadrilateral elements.

The patch shown on Figure 6.1 has been used to perform the patch test for the new ANDES4 element by giving the boundary nodes displacements according to a constant strain pattern. The patch test requires that the internal nodes get displacements that satisfies this constant strain displacement mode exactly.

Membrane tests.

$$u = x \quad \Rightarrow \quad \epsilon_{xx} = \frac{\partial u}{\partial x} = 1 \quad : \quad \text{Identically satisfied.}$$

$$v = y \quad \Rightarrow \quad \epsilon_{yy} = \frac{\partial v}{\partial y} = 1 \quad : \quad \text{Identically satisfied.}$$

$$u = y, v = x \quad \Rightarrow \quad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 2 \quad : \quad \text{Identically satisfied.}$$

Bending tests.

$$w = x^2 \quad \Rightarrow \quad \kappa_{xx} = \frac{\partial^2 w}{\partial x^2} = 2 \quad : \quad \text{Identically satisfied.}$$

$$w = y^2 \quad \Rightarrow \quad \kappa_{yy} = \frac{\partial^2 w}{\partial y^2} = 2 \quad : \quad \text{Identically satisfied.}$$

$$w = xy \quad \Rightarrow \quad \kappa_{xy} = 2 \frac{\partial^2 w}{\partial x \partial y} = 2 \quad : \quad \text{Identically satisfied.}$$

The membrane patch test for the ANDES4 element are satisfied regardless of whether the higher order strain displacement matrix $\mathbf{B}_h$ or the deviatoric higher order strain displacement matrix $\mathbf{B}_d = \mathbf{B}_h - \bar{\mathbf{B}}_h$ is used for the higher order membrane stiffness according to equation (0.0.0) and (0.0.0).

6.2 Membrane problems.

6.2.1 Shear-loaded cantilever beam.

A shear loaded cantilever beam is defined according to Figure 8.1. This Figure also shows the 16×4 regular and irregular element meshes.

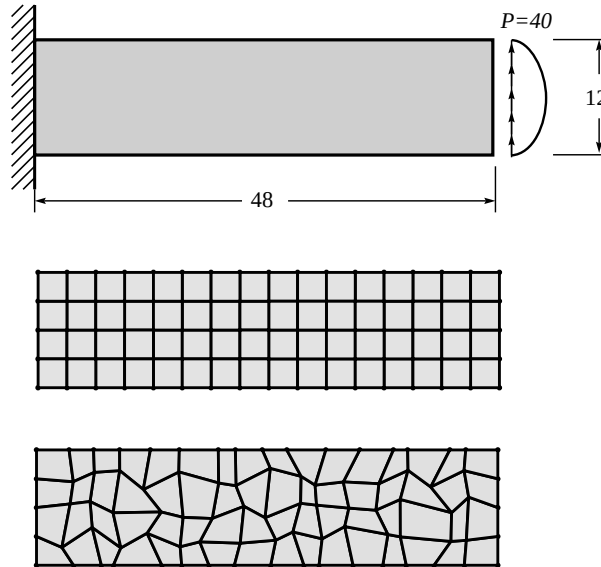


Figure 6.2. Cantilever under end shear. $E = 30000$ $\nu = 0.25$

The test has been run using a totally clamped boundary at the fixed end, and the applied nodal forces on the end cross-section are consistent lumping of a shear load with parabolic variation in the y -direction.

The comparison value is the tip deflection of 0.35583 at the end of the beam. This number is the exact solution of the two-dimensional plane stress as given in [28]. The numerical results have been scaled so that the analytical displacement of 0.35583 corresponds to 100 in Table 6.1. Table 6.1 also includes numerical results for the quadrilateral FFQ and the triangular FFT as described by Nygård in [47].

Table 6.1. Tip deflection of cantilever beam.

Element	$x \times y$ -subdivisions				
	4×1	8×2	16×4	32×8	64×16
Regular element mesh					
CST	25.48	55.24	82.66	94.96	98.71
QSHELL3	111.78	101.18	100.03	99.97	100.01
QSHELL4	97.72	98.86	99.54	99.87	100.00
Irregular element mesh					
CST	27.86	55.84	81.47	94.27	98.41
QSHELL3	98.16	100.12	99.66	99.87	99.97
QSHELL4	103.93	98.60	99.45	99.90	100.01

Table 6.1 shows very similar convergence rates for the QSHELL3 and QSHELL4 element compared to Nygård's FFT and FFQ element. However, the ANDES elements tend to be more flexible for very coarse meshes.

6.3 Bending problems.

6.3.1 Centrally loaded square plate.

A square plate subjected to a central load of $P = 40.0$. The test has been run with both simply supported and fully clamped boundary conditions.

Plate dimensions are 100.0×100.0 with thickness $t = 2.0$ and material properties $E = 1500.0$ and $\nu = 0.2$. Due to symmetry only a quarter of the plate have been modeled.

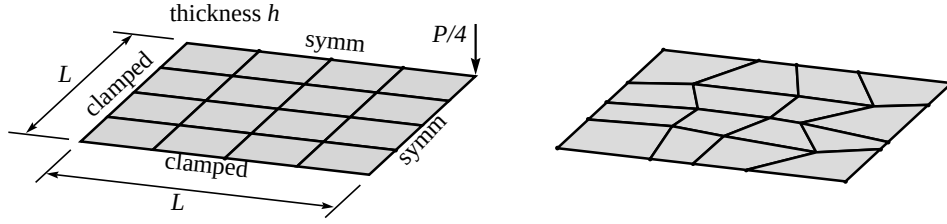


Figure 6.3. 4×4 quarter model of centrally loaded square plate.

Table 6.2. Central deflection of square plate with clamped boundary. Displacement 2.1552 is scaled to 100.00 .

Element type	Mesh over quarter plate					
	1×1	2×2	4×4	8×8	16×16	32×32
Regular mesh.						
BCIZ-SQ	88.223	102.71	101.18	100.37	100.10	100.02
ANDES3	92.798	103.75	101.57	100.50	100.15	100.04
ANDES4	88.944	100.07	100.18	100.07	100.02	100.00
Irregular mesh.						
BCIZ-SQ	88.223	99.805	101.27	99.781	99.983	99.938
ANDES3	92.798	102.48	102.38	100.40	100.23	100.02
ANDES4	88.944	99.793	101.71	100.19	100.14	100.02

6.4 Shell problems.

6.4.1 Pinched cylinder problem.

An open cylinder is subjected to two diametrically opposite point loads. Due to symmetry only $1/8$ of the problem is modeled. The geometry of the $1/8$ model is shown in Figure 6.4.

The mesh has been given an increasing distortion angle θ . As θ increases the four node elements are no longer flat elements. This induced warping gives different results for the projected and unprojected versions of the ANDES4 element. The ANDES3 element is invariant under projection because the element always possesses the correct rigid body modes.

The improvement with the projected stiffness matrix for the ANDES4 element is dramatic. This example displays the importance of correct rigid body modes, as well as showing the robustness of the stiffness projection procedure.

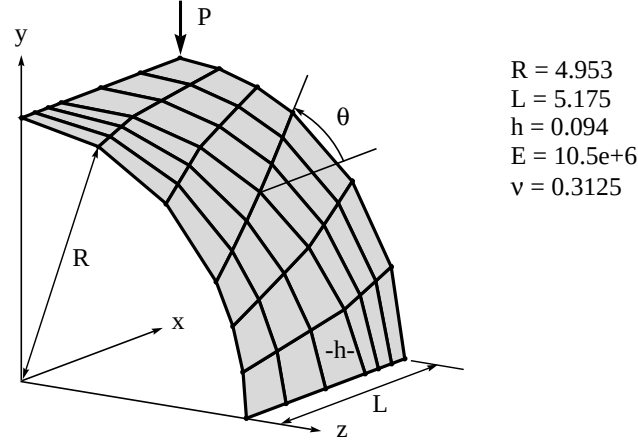


Figure 6.4. Pinched cylinder problem.

Table 6.3. Vertical displacement under load for pinched cylinder. Displacement 4.5301×10^{-3} is scaled to 100.00.

Element type	Distortion angle				
	$\theta = 0^\circ$	$\theta = 10^\circ$	$\theta = 20^\circ$	$\theta = 30^\circ$	$\theta = 40^\circ$
ANDES3 Unproj	99.578	99.136	98.737	98.296	97.608
ANDES3 Proj	99.578	99.136	98.737	98.296	97.608
ANDES4 Unproj	99.430	30.594	12.652	9.381	7.960
ANDES4 Proj	99.430	99.334	99.255	99.125	98.789

6.4.2 Pinched hemisphere problem.

A hemispherical shell is subjected to two pairs of diametrically opposite loads along the x and y axis respectively. Due to symmetry only a 1/4 model is used according to Figure 6.5.

This problem is often modelled with a hole at the top of the hemisphere. This allows using a mesh of strictly flat quadrilateral elements, which makes the test much less demanding for quadrilateral elements. One has chosen to model the hemisphere without a hole since this gives warped elements for the quadrilateral element meshes, and thus poses a much more severe test for those elements.

For the triangular ANDES3 two discretizations are used. Mesh 1 in Table 6.4 refers to a mesh where two triangular elements join at the loaded nodes, whereas Mesh 2 has one element attached to the loaded nodes.

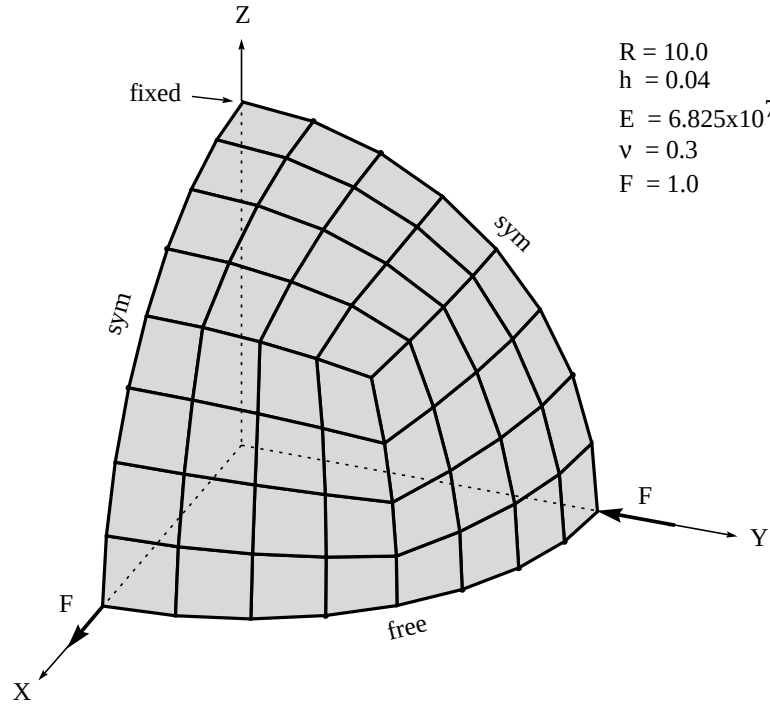


Figure 6.5. Pinched hemisphere problem.

Table 6.4. Displacements under loads for pinched hemisphere. Displacement 9.1898×10^{-2} is scaled to 100.00.

Element type	Elements per side					
	2	4	8	12	16	20
ANDES3 Mesh 1	0.18	2.59	25.30	61.44	83.23	92.52
ANDES3 Mesh 2	0.42	4.07	31.91	68.23	86.85	94.23
ANDES4 Unproj	13.57	6.13	12.85	19.49	25.17	30.18
ANDES4 Proj	67.55	23.73	85.53	97.24	99.39	100.00

This test again shows the dramatic improvement of the performance of the ANDES4 element when the stiffness projection is used.

Chapter 7

Numerical examples for linearized buckling analysis.

7.1 Buckling analysis of square plate compressed in one direction.

The buckling of a square plate subjected to in-plane uniaxial compression is considered. The geometry and material constants of the plate are given in Figure 7.1. The plate is simply supported along all edges with the in-plane deformations being unconstrained.

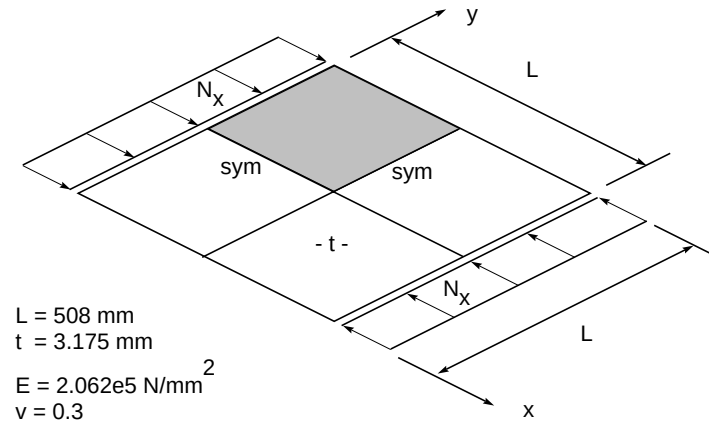


Figure 7.1. Square plate subjected to uniaxial compression.

The plate is compressed in its middle plane by a uniform load N_x along the edges $x = 0$ and $x = L$. Timoshenko [63] gives the analytical critical value of the compressive force per unit length as

$$(N_x)_{cr} = \frac{\pi^2 D}{L^2} \left(m + \frac{1}{m}\right)^2 \quad \text{where} \quad D = \frac{Et^3}{12(1 - \nu^2)}. \quad (7.1.1)$$

where L is plate side length, t is the plate thickness, ν is the Poisson's ratio and m is the number of half-waves in the compressive direction. The buckling modes are associated with odd values of m .

The geometric stiffness is based on the incremental solution at a load level of 1% of the critical load level. The results from the numerical analysis is tabulated in Table 7.2 and compared to results obtained by Bj  rum [17] with

the QSEL and FFQC elements. The QSEL and FFQC elements performs better than ANDES3 and ANDES4 for the higher order buckling modes. This is due to the “tuned” higher order geometric stiffness matrix used for those elements. However the geometric stiffness matrices used for the QSEL and FFQC element do not give a consistent tangent stiffness for nonlinear continuation analysis.

Table 7.1. Numerical results of square plate subjected to compression, normalized by the analytical solution of the first mode ($m = 1$).

Element Type	Analytical Solution	Mesh used for quarter of plate			
		4×4	8×8	16×16	32×32
ANDES3	$m = 1$: 1.000	1.008	1.002	1.000	1.000
	$m = 3$: 2.778	3.070	2.854	2.797	2.783
	$m = 5$: 6.760	8.342	7.270	6.893	6.798
ANDES4	$m = 1$: 1.000	1.043	1.011	1.002	1.001
	$m = 3$: 2.778	3.278	2.898	2.808	2.786
	$m = 5$: 6.760	10.11	7.522	6.950	6.815
QSEL	$m = 1$: 1.000	1.010	1.002	1.001	1.000
	$m = 3$: 2.778	3.032	2.840	2.793	2.782
	$m = 5$: 6.760	8.751	7.265	6.884	6.791
FFQC	$m = 1$: 1.000	0.973	0.993	0.998	1.000
	$m = 3$: 2.778	2.673	2.745	2.769	2.778
	$m = 5$: 6.760	6.750	6.694	6.738	6.754

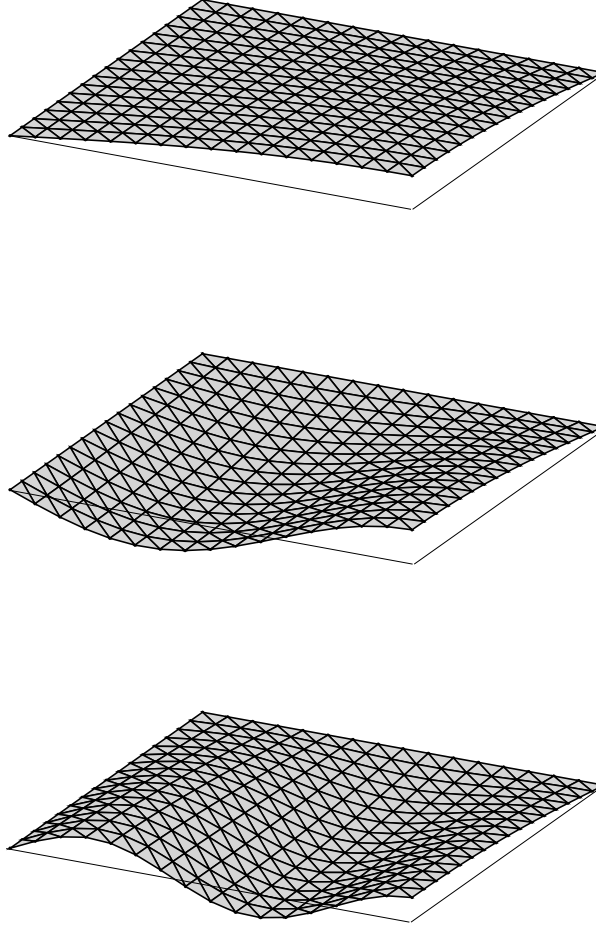


Figure 7.2. Buckling modes for $m = 1, 3$ and 5 according to equation (7.1.1) for square plate subjected to uniaxial compression.

7.2 Buckling analysis of shear loaded square plate.

A simply supported square plate with geometry and material constants given in Figure 7.3 is subjected to shear loads uniformly applied along the edges. The out-of-plane displacements and rotations are constrained whereas the in-plane rotations and translations along the boundaries are left free.

The critical shear force associated with the first buckling mode is give analytically by Timoshenko [63] as

$$(N_{xy})_{cr} = 9.34 \frac{\pi^2 D}{L^2} \quad \text{where} \quad D = \frac{Et^3}{12(1 - \nu^2)}. \quad (7.2.1)$$

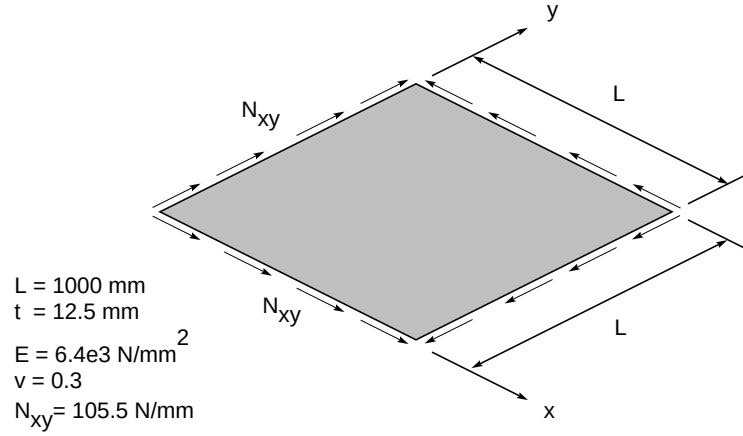


Figure 7.3. Square plate subjected to shear load.

Table 7.2. Numerical results of square plate subjected to shear, normalized by the analytical solution $(N_{xy})_{cr} = 105.5 \frac{N}{mm}$.

Element		Mesh used for the plate				
Type		4×4	8×8	16×16	32×32	64×64
ANDES3	$m = 1:$	1.446	1.145	1.057	1.013	
	$m = 2:$	2.174	1.293	1.240	1.238	
	$m = 3:$	8.631	2.974	2.709	2.675	
ANDES4	$m = 1:$	2.175	1.297	1.088	1.022	
	$m = 2:$	3.079	1.528	1.298	1.250	
	$m = 3:$		4.300	3.013	2.739	
QSEL	$m = 1:$	1.387	1.065	1.008	0.997	0.993
	$m = 2:$	2.096	1.385	1.268	1.242	1.233
	$m = 3:$	207.3	3.582	2.840	2.691	2.642
FFQC	$m = 1:$	0.781	0.908	0.967	0.987	0.994
	$m = 2:$	1.060	1.144	1.206	1.227	1.233
	$m = 3:$	2.073	2.301	2.518	2.614	2.643

This problem again shows that the higher order geometric stiffness matrix for the QSEL and FFQC element outperforms the consistent geometric stiffness matrix of the ANDES elements for linearized buckling analysis. The ANDES3 element performs better than the ANDES4 element simply because

meshes of $N \times N$ elements contains twice as many ANDES3 elements as ANDES4 elements since two triangles are required to fill one quadrilateral. This gives a “finer” discretization with respect to the rigid body displacements and thus a better global geometric stiffness matrix.

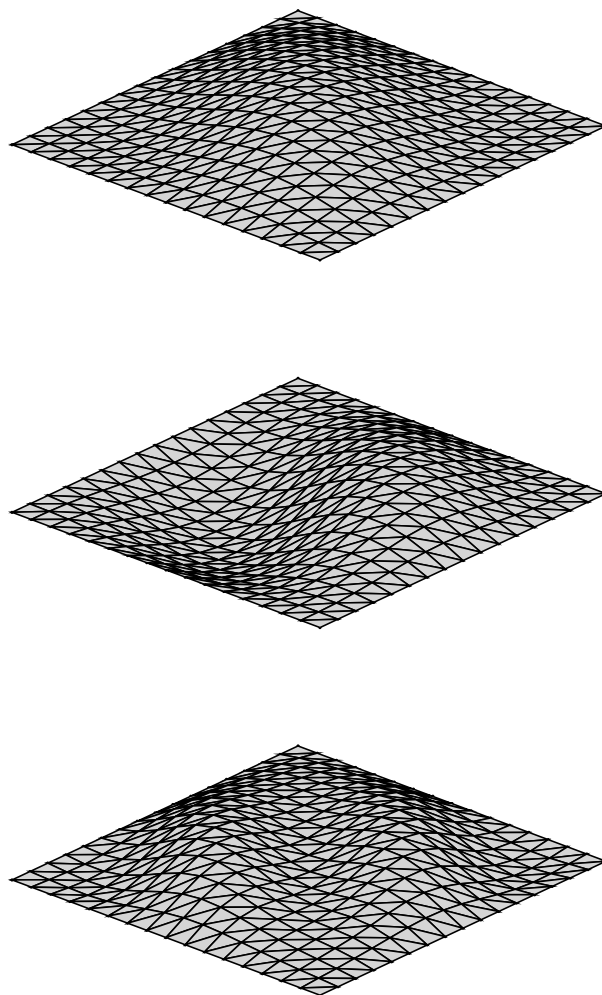


Figure 7.4. First three buckling modes for shear loaded square plate.

Chapter 8

Numerical examples for nonlinear analysis.

8.1 Smooth path following problems.

8.1.1 Cantilever beam subjected to end moment.

The cantilever in Figure 6.2 is used to demonstrate the large rotation and large displacement capabilities. The problem is usually formulated so that bending about one of the global axes takes place. The cantilever here is oriented arbitrarily in space so that the example becomes more demanding as regards the treatment of finite rotations as non-vectorial quantities. The position of the cantilever is defined by the auxiliary coordinate system

$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = \frac{1}{7} \begin{bmatrix} 6 & -2 & -3 \\ 3 & 6 & 2 \\ 2 & -3 & 6 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \quad (8.1.1)$$

and letting one side of the cantilever follow the $\bar{x}$ -axis. This auxiliary system is used for modeling purposes only. The actual computations take place in the global (x, y, z) -system.

The exact solution is obtained from Bernoulli-Euler beam theory, and gives the deflected shape of the beam as a circle with radius

$$R = \frac{EI}{M_{\bar{y}}}. \quad (8.1.2)$$

This gives the tip deflections as

$$\frac{\bar{w}}{L} = \frac{EI}{M_{\bar{y}}L} \left(1 - \cos \frac{M - \bar{y}L}{EI}\right) \quad \text{and} \quad \frac{\bar{v}}{L} = -\frac{EI}{M_{\bar{y}}L} \sin \frac{M - \bar{y}L}{EI} \quad (8.1.3)$$

where $\bar{w}$ and $\bar{v}$ are the deflections in the $\bar{x}$ - and $\bar{z}$ - directions.

The problem was solved using load control with the incremental load steps $\Delta M_{\bar{y}} = \frac{2\pi}{40} \frac{ML}{EI}$, thus requiring 40 steps to bend the beam into a full circle. This small load step was required for the triangular element due to the large unbalanced membrane forces that are introduced during the iterations. The triangular element does not give symmetry about the center line of the cantilever, and thus converges more slowly than a comparable four node element,

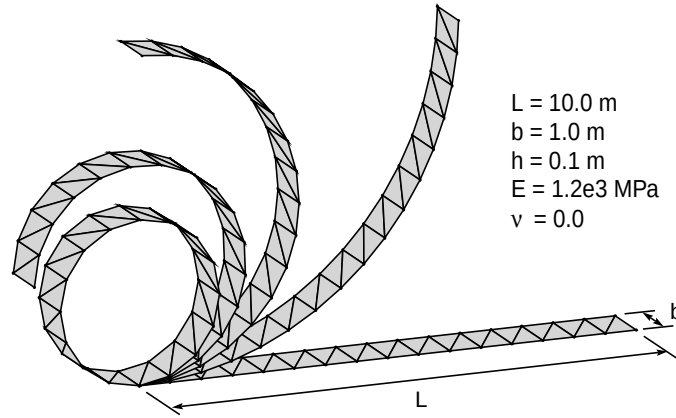


Figure 8.1. Cantilever oriented arbitrarily in space subjected to end moment.

as described by Nygård [47]. For the ANDES4 element the beam was bent into a full circle using 10 load steps.

Table 8.1 gives convergence rates and number of iterations for the cantilever modeled with 20 ANDES3 elements. The label “Fit type” refers to the $\mathbf{G}$ matrix used for fitting the shadow element, with 1, 2 and 3 signifying side 12 alignment, least square fit of side edge angular errors and CST rotation, respectively. “1 diag” means that side 12 is directed along the diagonal of a rectangle assembly of two elements whereas “1 edge” means that side 12 is directed along the beam axis. Table 8.1 illustrates two points: Side 12 alignment is not orientation invariant since it doubles the number of iterations of the “1 diag” mesh compared to “1 edge”. Also, the applied load is a moment load, which gives a geometric stiffness that does not go towards symmetry as equilibrium is approached. The symmetrized stiffness will thus give poor convergence compared to the nonsymmetric tangent stiffness.

Table 8.1. Iterations for the first 10 steps.

Symmetric stiffness											
Fit type	Num. of iterations										Conv.
1 edge.	4	4	5	5	6	7	8	8	8	9	L
1 diag.	9	9	9	9	9	10	11	9	10	10	L
2	5	5	5	6	6	7	8	8	8	8	L
3	4	4	5	5	6	7	8	8	8	9	L
Non-symmetric stiffness											
Fit type	Num. of iterations										Conv.
1 edge.	4	4	4	4	4	4	4	4	4	4	Q
1 diag.	8	8	8	9	8	8	9	9	9	9	Q
2	5	5	5	5	5	5	5	5	5	5	Q
3	4	4	4	4	4	4	4	4	4	4	Q

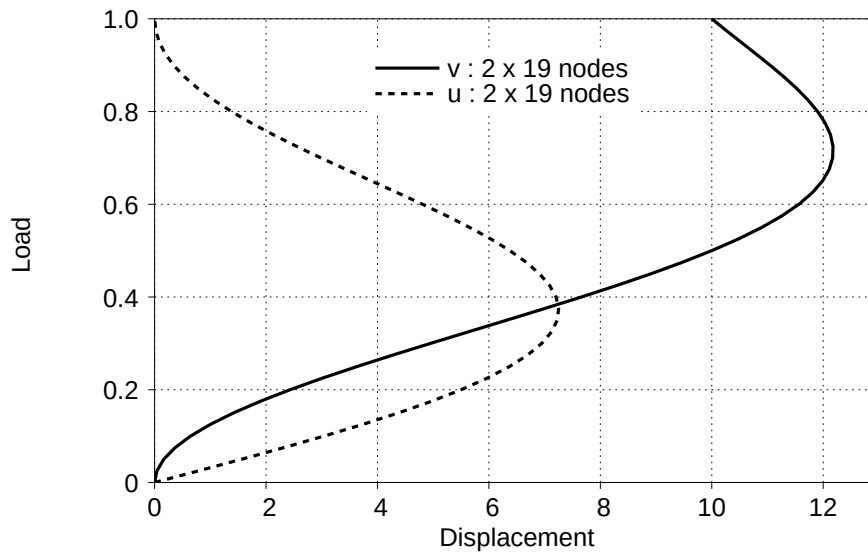


Figure 8.2. Tip displacements of the cantilever beam subjected to end moment.

8.1.2 Hinged cylindrical shell under concentrated load.

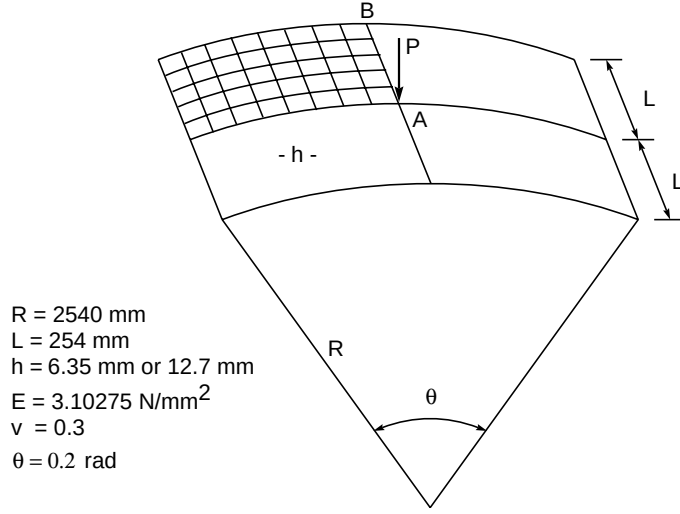


Figure 8.3. Geometry and material properties for hinged cylindrical shell.

The hinged cylindrical shell subjected to concentrated load is a common test case for geometrical nonlinearities. Nygård [47] performs an extensive comparison with other elements. Bjærum [17] gives a thorough comparison of different path-following algorithms for this example. The thinner shell of thickness $h = 6.35 \text{ mm}$ has a dramatic snap-back behavior well suited for verifying path following capabilities of the solution algorithm.

One quarter of the shell has been modeled using 8×8 rectangular units of two three node elements. The actual equilibrium path shown in Figures 8.4 and 8.5 agrees well with results obtained by Nygård [47].

The arc-length algorithm did not evidence convergence difficulties. With the thin shell the path has been followed using 26 steps and convergence was obtained within 5 iterations even at the snap-back section of the equilibrium path.

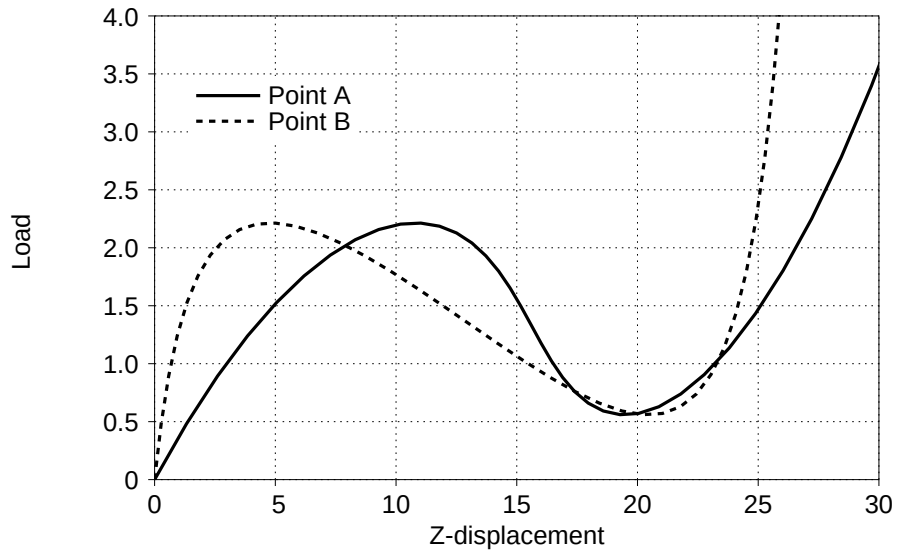


Figure 8.4. Vertical deflection at points A and B for hinged cylindrical panel with $t = 12.7mm$.

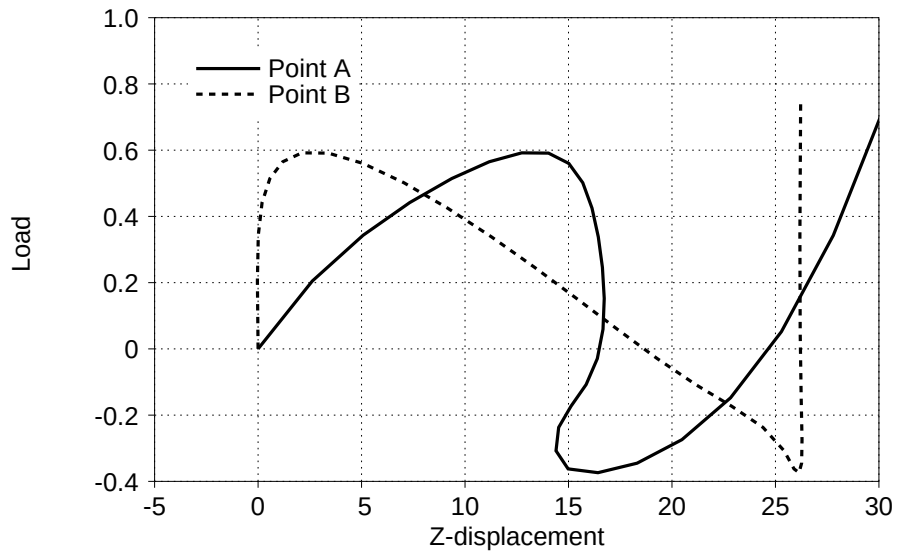


Figure 8.5. Vertical deflection at points A and B for hinged cylindrical panel with $t = 6.35mm$.

8.1.3 Pinching of a clamped cylinder.

A cantilevered cylinder is subjected to two diametrically opposite forces of magnitude F at the open end. Due to symmetry only a quarter of the structure is modelled. The problem has been studied by Stander et al. [60] and Parisch [48] using uniform element meshes. Mathisen, Kvamsdal and Okstad [43] have studied this problem using adaptive mesh techniques. The present analysis was performed using displacement control of the loaded point with 16 equal step up to a total displacement of 1.6 times the radius of the cylinder.

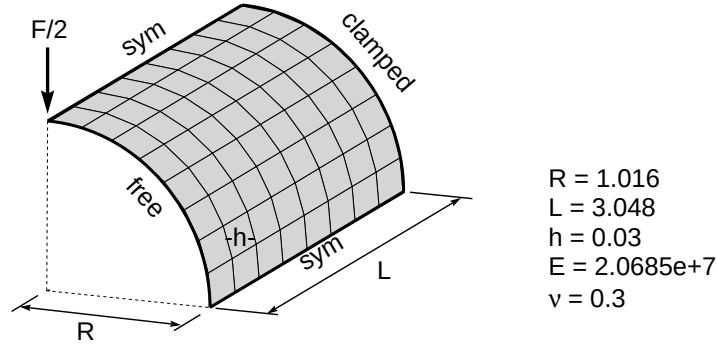


Figure 8.6. Geometry and material properties for the pinched cylinder problem.

This example shows that the ANDES4 element is better than the ANDES3 element with respect to membrane strain gradients. A mesh of 16×16 ANDES3 elements diverges before the analysis is completed, whereas a mesh of 16×16 ANDES4 elements is sufficient to complete the analysis as shown in Figure 8.7. The results in Figure 8.7 shows good agreement with the results obtained by Stander et al. [60] and Parish [48], both of whom used quadrilateral elements.

8.1.4 Stretched cylinder with free ends.

An open cylinder with free ends are subjected to two diametrically opposite forces at the halflength. Geometric data for the cylinder are given in Figure 8.9. This problem was first discussed by Gruttman et al. [32], and later by Perić and Owen [50]. Figure 8.10 shows the results of the present analysis with a mesh of 8×16 ANDES4 elements compared with Perić and Owens analysis using a mesh of 10×20 elements. The displacement of the loaded point agrees well with the results reported in [50] .

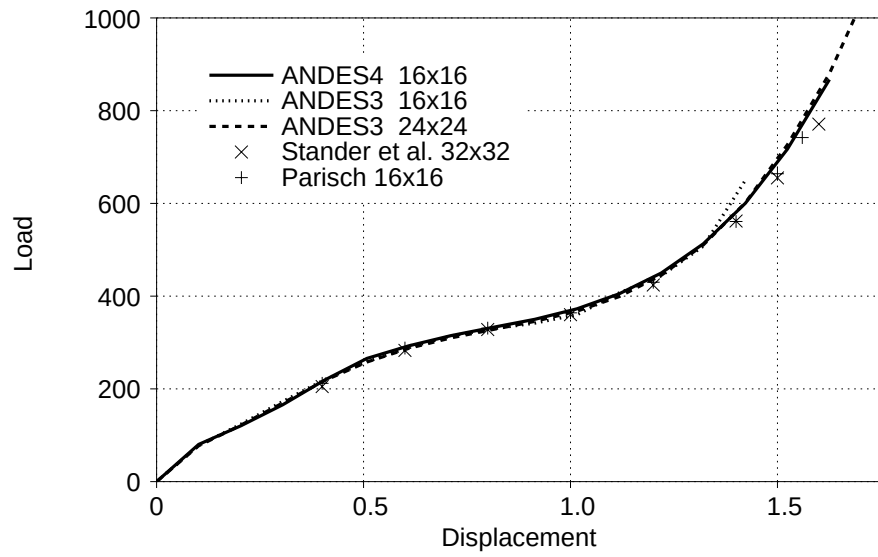


Figure 8.7. Vertical displacement at loading point for the pinched cylinder problem.

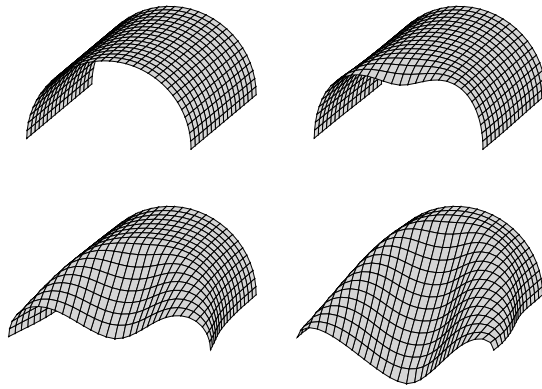


Figure 8.8. Deformed finite element mesh at various loads.

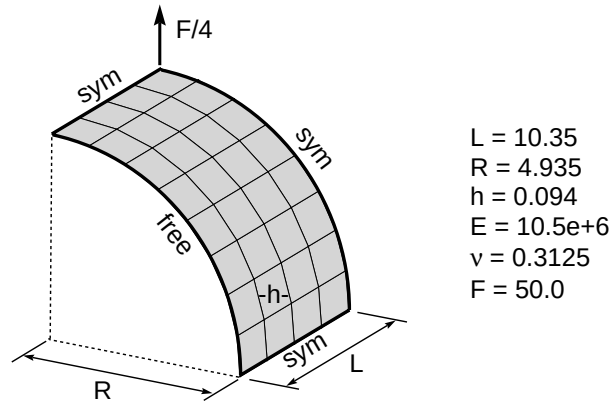


Figure 8.9. Geometry and material properties for the stretch cylinder problem.

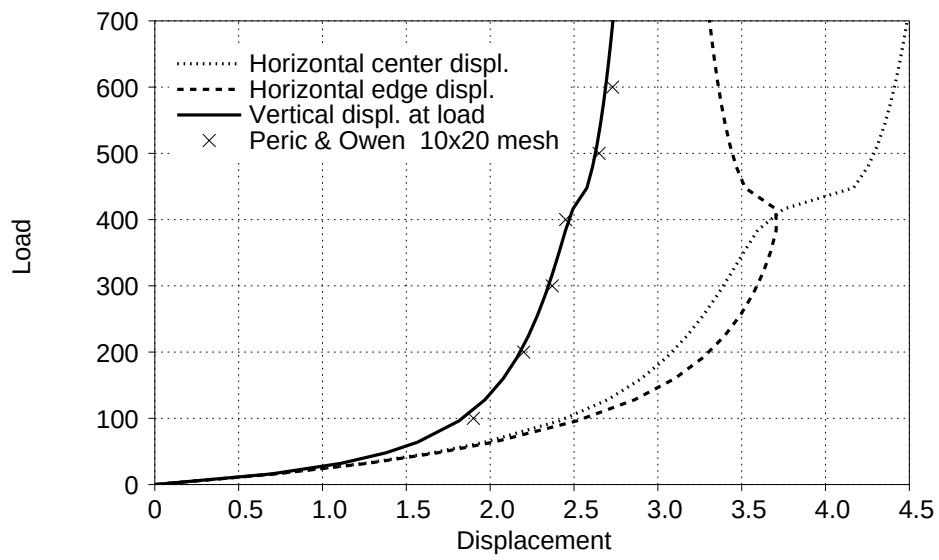


Figure 8.10. Load displacement curves for the stretched cylinder problem.

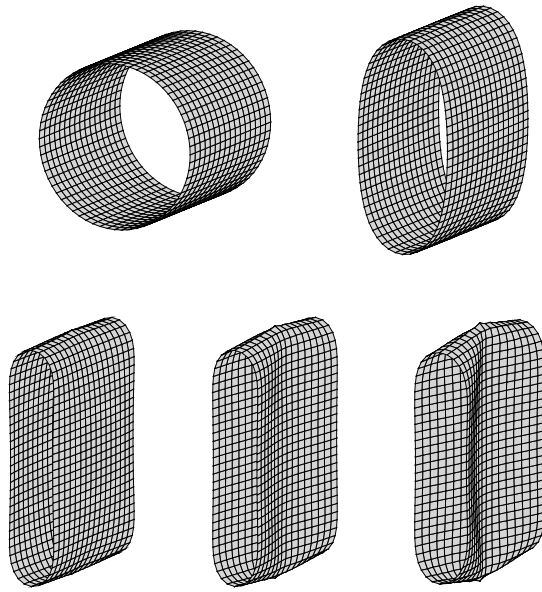


Figure 8.11. Deformed finite element mesh at various loads.

8.2 Path-following problems with bifurcation.

8.2.1 Post-buckling analysis of square plate compressed in one direction.

The geometry and material properties of the square plate are presented in Figure 7.1. The applied load is normalized with respect to the analytical buckling load for this problem given in equation (7.1.1) .

The post-buckling analysis is performed in order to evaluate the stiffness properties of the plate after bifurcation is encountered. One has used a 8×8 mesh of ANDES4 elements over the quarter model. This mesh gave about 1% error in determining the linearized buckling load.

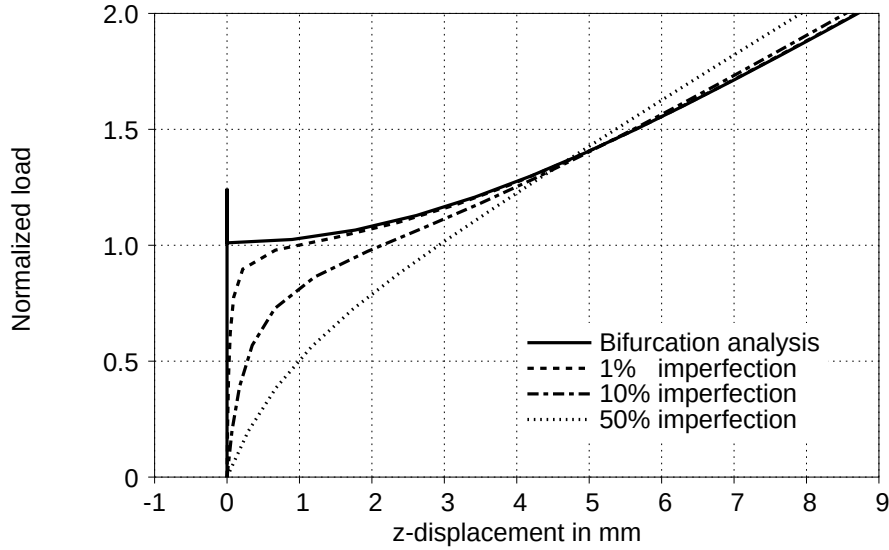


Figure 8.12. Out of plane displacement at plate center for square plate subjected to compression.

As seen in the load displacements curves in Figure 8.12 the structure shows a stable post-buckling response where it can withstand increased load after bifurcation. Figure 8.12 also shows the response of the plate with various geometric imperfection levels. The buckling mode of the structure has been scaled so that the largest out of plane imperfection is equal to 1%, 10% and 50% of the plate thickness.

8.2.2 Post-buckling analysis of shear loaded square plate.

The geometry and material properties of the square plate are described in Figure 7.3. The applied load is normalized with respect to the analytical buckling load $N_{xy_{cr}} = 105.5 \frac{N}{mm}$.

A 16×16 mesh of ANDES3 elements over the quarter model is used. This mesh gave about 5% error in determining the linearized buckling load. It should be noted that the shear loaded plate has traditionally been a difficult problem for triangular elements.

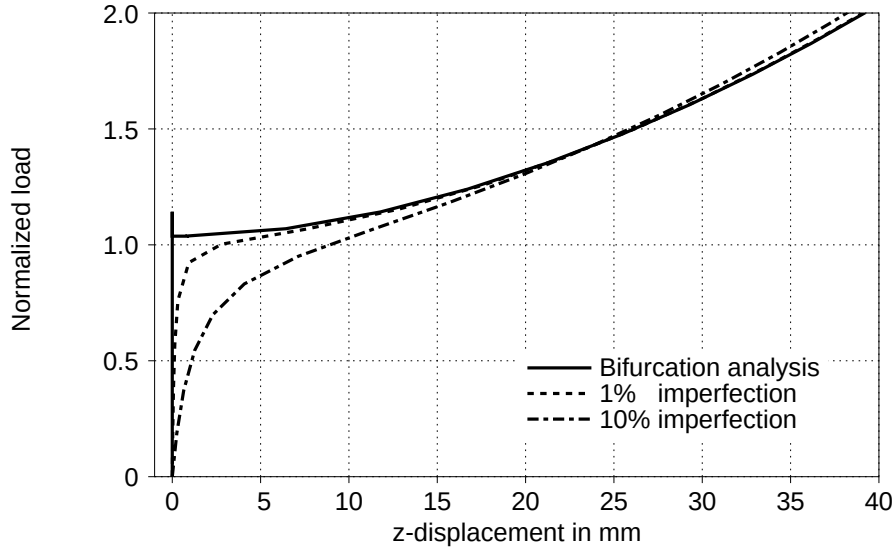


Figure 8.13. Out of plane displacement at plate center for square plate subjected to compression.

As seen in the load displacements curves in Figure 8.13 the structure shows a stable post-buckling response where it can withstand increased load after bifurcation. Figure 8.13 also shows the response of the plate with various geometric imperfection levels. The buckling mode of the structure has been scaled so that the largest out of plane imperfection is equal to 1% and 10% of the plate thickness.

8.2.3 Buckling of a deep circular arch.

This problem of snap-through of a deep circular arch has been investigated by Huddleston [37]. The asymmetric displacement path has been studied by Simons et al. [59] and Feenstra and Schellekens [23] using a small geometric imperfection to induce the buckling mode. Bjærnum [17] analyzed the problem using branch switching to follow the secondary path. The present analysis follows Bjærnum's in that no imperfection is used, and the branch switching algorithm has been used to traverse the bifurcation and continuing along the secondary path.

The dimensions of the arch are given in Figure 8.14. The arch has hinged boundary conditions at both ends and is modelled using 20 quadrilateral shell elements with the z displacements constrained.

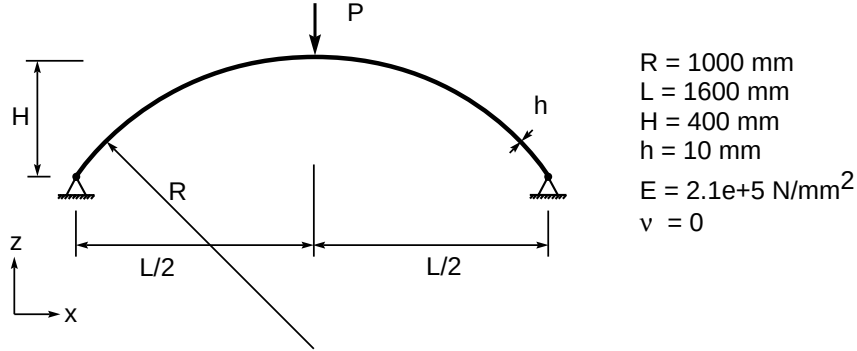


Figure 8.14. Geometry and material properties for the deep circular arch.

Figure 8.15 shows the response of the structure for the primary path, and the secondary path obtained by doing a branch switching at the first bifurcation point at $\frac{PR^2}{EI} = 13.2$. Bjærnum reports a bifurcation point at $\frac{PR^2}{EI} = 12.0$, and from his plot of the secondary path one can see that the load then jumps to approximately 13.0. Such a gap between detected and converged bifurcation points as reported by Bjærnum can indicate an inconsistent tangent stiffness matrix. The deflections of these paths are also illustrated in Figure 8.16

The problem displays some puzzling behaviour. For instance, the primary path appears not to intersect with the secondary path. The primary path keeps doing spiraling motions for the vertical displacement versus load as plotted in Figure 8.17. For each spiraling motion another wavelike deformation is

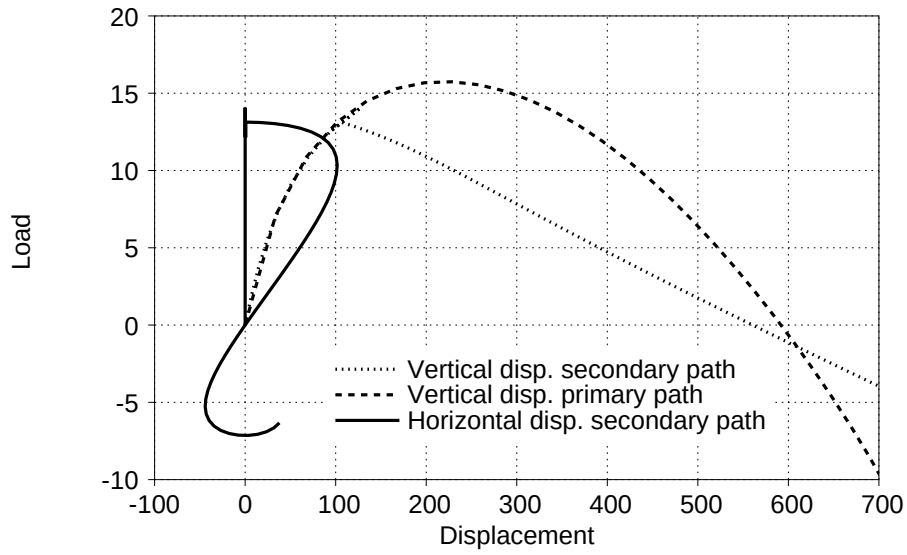


Figure 8.15. Displacements for the primary and secondary paths.

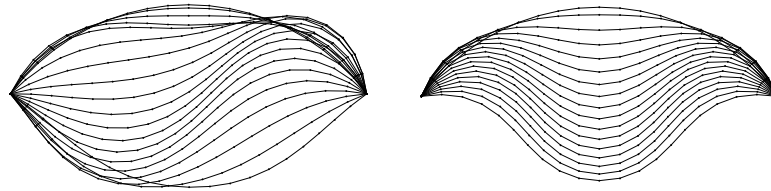


Figure 8.16. Deformations for the secondary and primary paths.

fed into the arch as shown in Figure 8.18. The fact that the primary and secondary path do not intersect can be discerned from the fact that the primary path never achieves the same vertical deflection for the midpoint of the arch as the secondary path.

The secondary path keeps doing figure-of-eight like motions for the midpoint of the arch. The branch switching algorithm does not pick up a new bifurcation point at the bottom point of the secondary path as would be expected.

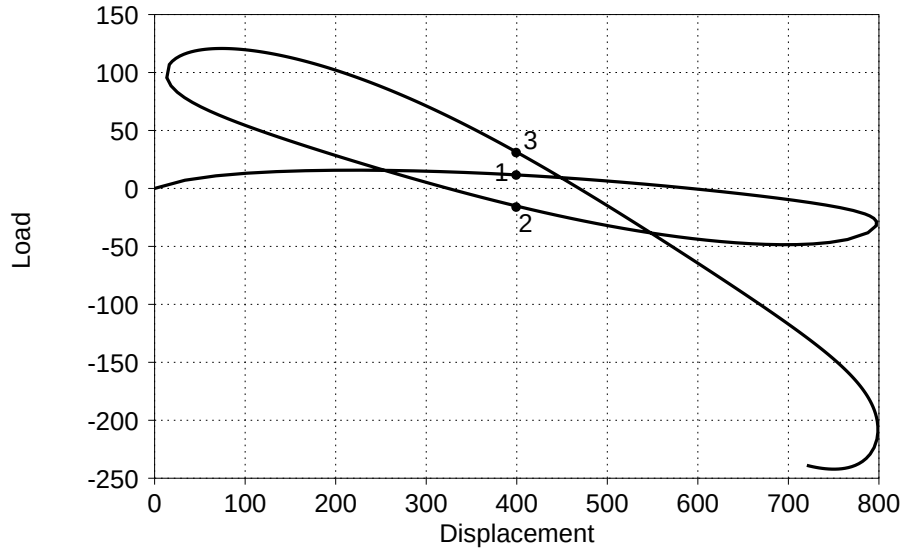


Figure 8.17. Vertical displacement for the primary path. Numbers 1 2 and 3 show the location of the deformed element geometries in Figure 8.18.

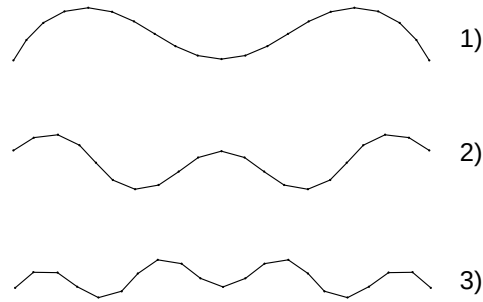


Figure 8.18. Displacements for the primary path. Numbers refer to the load-displacement curves in Figure 8.17.

This conclusion seems to agree with Bjærø, since no such bifurcation point is reported. The “mismatch” is a surprise since one expects the structure to be able to pick up additional load once it hits bottom. But detecting and switching to the new stable path seems to be computationally difficult.

8.2.4 Right angle frame subjected to in-plane load.

The right angle frame in Figure 8.19 is subjected to a in-plane load. The applied load acts on the lower corner of the tip. This problem has been studied by Nour-Omid and Rankin [46] in the post-critical domain.

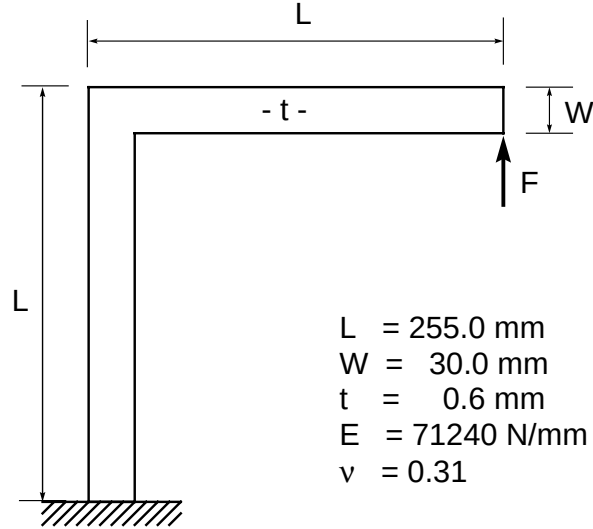


Figure 8.19. Geometry and material properties for the right angle frame.

Table 8.2 lists the critical loads given by different element types and mesh refinements. Results from Nour-Omid and Rankin are included in Table 8.2 for comparison. The postcritical response for the structure is shown in Figure 8.20 for different element meshes with ANDES3 and ANDES4 elements.

Table 8.3 gives the number of steps and iterations for this problem with the various consistent formulations. The convergence rates are measured at the stable “upswing” section of the equilibrium path. For large sections of the analysis this convergence rate is not obtained due to ill-conditioning at the “flat” section of the equilibrium path.

Table 8.2. Critical load for the right angle frame.

Element type	Num. of elements	F_{cr}
ANDES3	17×2	1.164
ANDES3	68×2	1.142
ANDES3	153×2	1.135
ANDES4	17	1.146
ANDES4	68	1.134
ANDES4	153	1.130
Nour-Omid & Rankin [46]	17	1.138
Nour-Omid & Rankin [46]	64	1.130

Table 8.3. Convergence of the 17×2 ANDES3 mesh.

Formulation	Symmetric stiff.			Non-symmetric stiff.		
	N.steps	N.iter.	C.rate.	N.steps	N.iter.	C.rate.
C	18	179	L.	18	152	Q.
CSE	18	174	Sl.	18	145	Q.
CSSE	18	145	Q.	18	145	Q.

Figure 8.20. Post buckling response for the right angle frame.

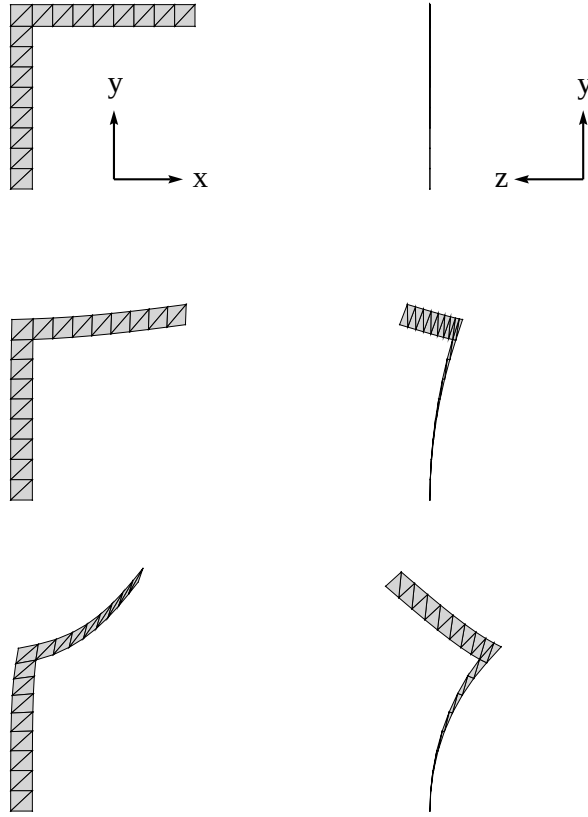


Figure 8.21. Deformations for $F = 1.164$, $F = 1.26$ and $F = 2.0$ with a mesh of 17×2 ANDES3 elements.

8.2.5 Right angle frame subjected to end moments.

The following two problems were modeled with the beam elements described in Appendix 1. The problems were included in the present work because of the numerical challenges they offer as regards branch switching and continuation algorithms.

The right-angle frame subjected to end moments was first introduced by Argyris [3] and later studied by Nour-Omid and Rankin [46]. The frame has been modeled using 10 Timoshenko beam elements for a half model. Beam elements have been chosen to model the frame since the moment loads are impossible to introduce for shell elements without using follower forces that are non-conservative and hence introduce follower-load stiffness matrix. Such a contribution has not been implemented for the shell elements developed in this work.

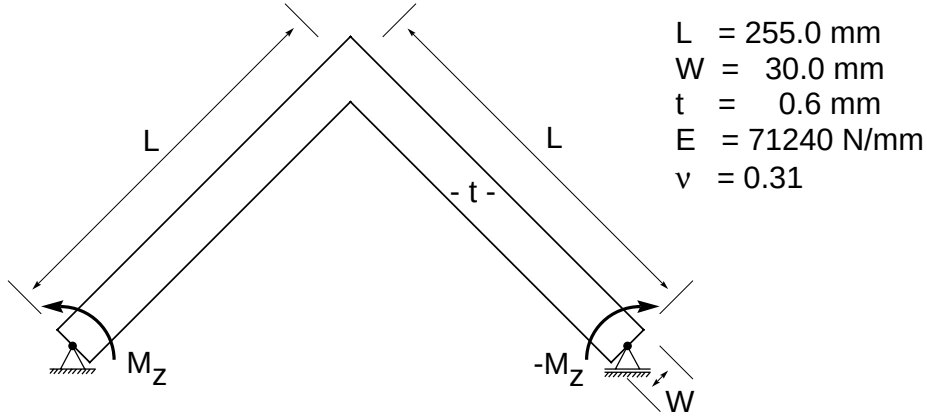


Figure 8.22. Geometry and material properties for the symmetric frame.

The response of the structure displays two distinct equilibrium paths. The primary path has only displacements in the x - y plane, whereas the secondary path switches to out of plane displacements after bifurcation. The response shows that the frame rotates a full 360° as the frame ends rotate through a full circle about the z axis. Finally the frame rotates back to the x - y plane with the load reversed. The out of plane bifurcation happens at $M_z = \pm M_{cr} = 6.464 \text{ Nmm}$. The analysis can be run repeated indefinitely. If the bifurcation starts at $M_z = +M_{cr}$ a full out of plane revolution will be obtained at $M_z = -M_{cr}$. A second revolution will then take place after which the structure finally returns to the same configuration of the first bifurcation with load $M_z = +M_{cr}$.

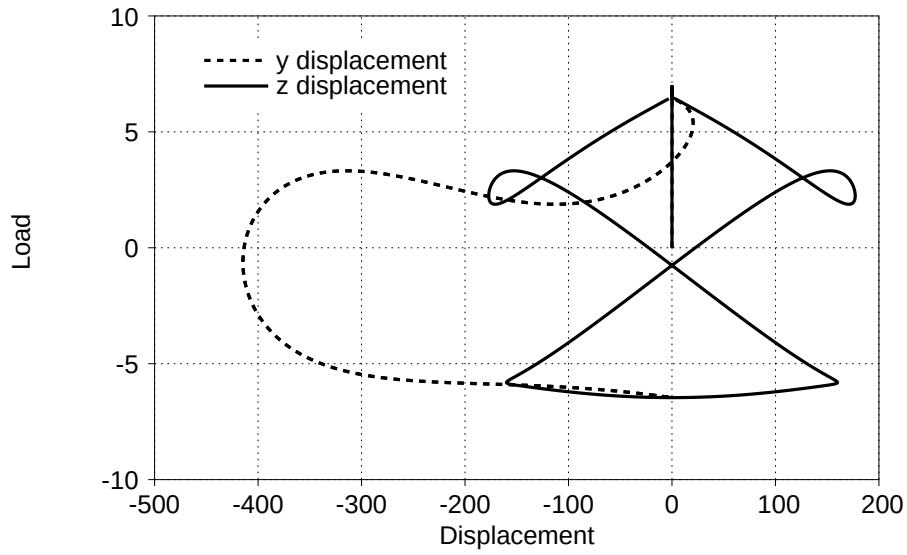


Figure 8.23. y and z displacements for the apex of the right angle frame.

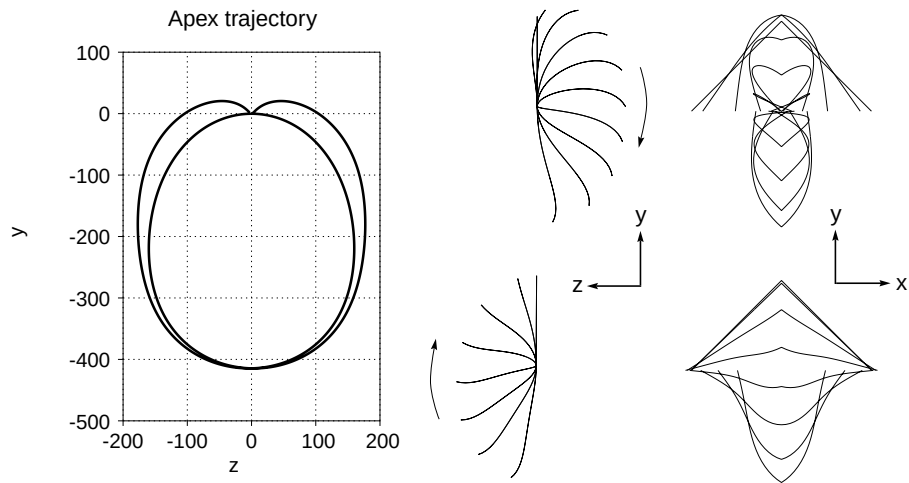


Figure 8.24. Deformations for the for the frame subjected to end moments. Arrows indicate direction of motion.

8.2.6 Cable Hockling.

An initially straight cable is subjected to a tip torsional moment. One end of the cable is fully clamped, whereas the loaded tip is free to rotate about the

longitudinal x axis, and moves along it. No rotation is allowed about the y and z axes at the loaded end. The material and geometrical properties of the cable are defined in Figure 8.25. The Euler-Bernoulli beam element described in Appendix 1 is used to discretize the cable.

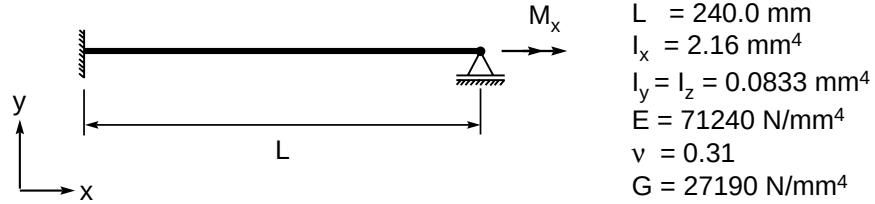


Figure 8.25. Cable geometry and material properties.

This problem was first studied in the postbuckling regime by Nour-Omid and Rankin [46]. The cable exhibits linear response with twisting and no lateral displacement up to the bifurcation point. After bifurcation the cable forms a loop with the loaded end moving towards the clamped end. Finally a full circular loop is formed after the path has traversed a second bifurcation point and the applied load returns back to zero.

The analysis is made more stable by restricting the midpoint of the cable from moving out of the x - y plane. The position of the loop is otherwise undetermined in the y - z plane. The equilibrium path has been followed without this restriction, but the convergence rate is impaired. This additional boundary condition is consistent with that used by Nour-Omid and Rankin.

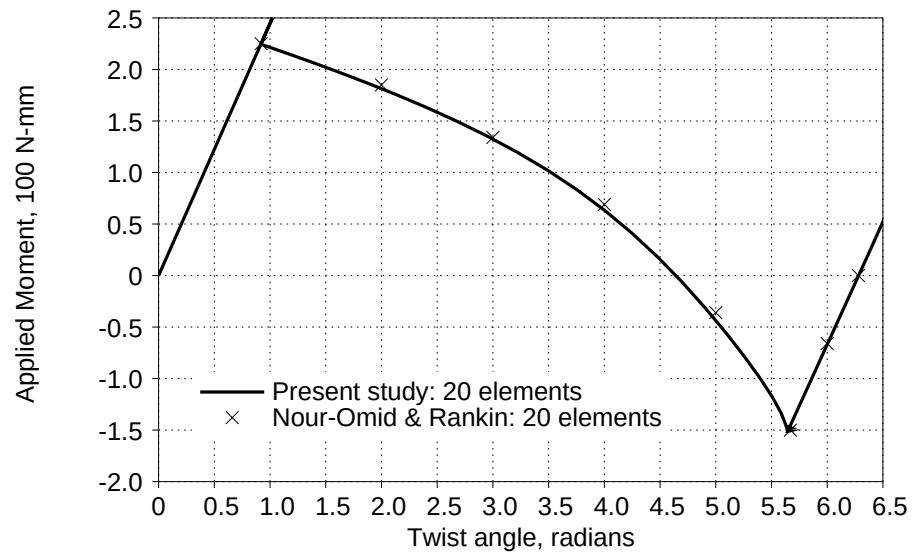


Figure 8.26. Cable hocking. Moment versus tip rotation.

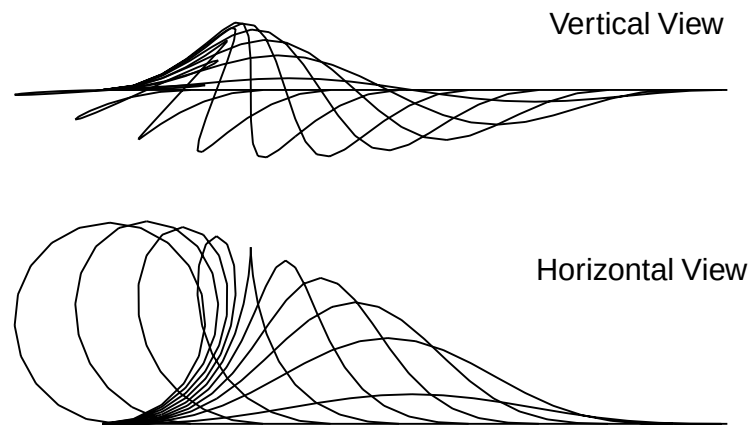


Figure 8.27. Deformations for the for the cable subjected to end moment.

36

A High Performance Thin Shell Triangle

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§36.1. Introduction

The HPSHEL3 thin shell element is formed by combining the AQR high performance plate bending triangle [182] with the optimal ANDES (Assumed Natural DEviatoric Strains) formulation of a plane stress (membrane) triangle with drilling degrees of freedom presented in [6,75,90].¹ The element has 3 nodes and 18 degrees of freedom. The implementation of this element is described in this Chapter to support final AFEM projects that rely on that element.

This document describes the computation of the HPSHEL3 element stiffness as implemented in both *Mathematica* and Fortran 77. It also reports simple numeric tests that may be used to verify the element. The description is oriented towards computer implementation, avoiding theoretical derivations.

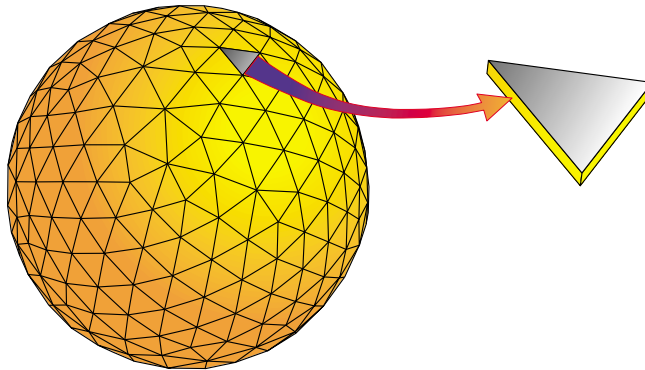


FIGURE 36.1. The HPSHEL thin shell element provides a faceted discretization of a shell structure.

§36.2. Element Overview

The HPSHEL3 shell element is geometrically a flat triangle that can be used to discretize shell structures as illustrated in Figure 36.1. The element geometry is defined by its position of its three corners in a global, right-handed rectangular Cartesian coordinate (RCC) system denoted by (x, y, z) , as shown in Figure 36.1.

The element has three nodes located at the corners. It has six degrees of freedom at each node n : three translations (u_{xn}, u_{yn}, u_{zn}) and three rotations: $(\theta_{xn}, \theta_{yn}, \theta_{zn})$, $n = 1, 2, 3$, for a total of 18 degrees of freedom. The freedom configuration is pictured in Figure 36.2. The element is obtained by combining a membrane (plane stress) and a Kirchhoff plate bending component.

¹ Reference [90] is reproduced in Chapter 29 of these Notes.

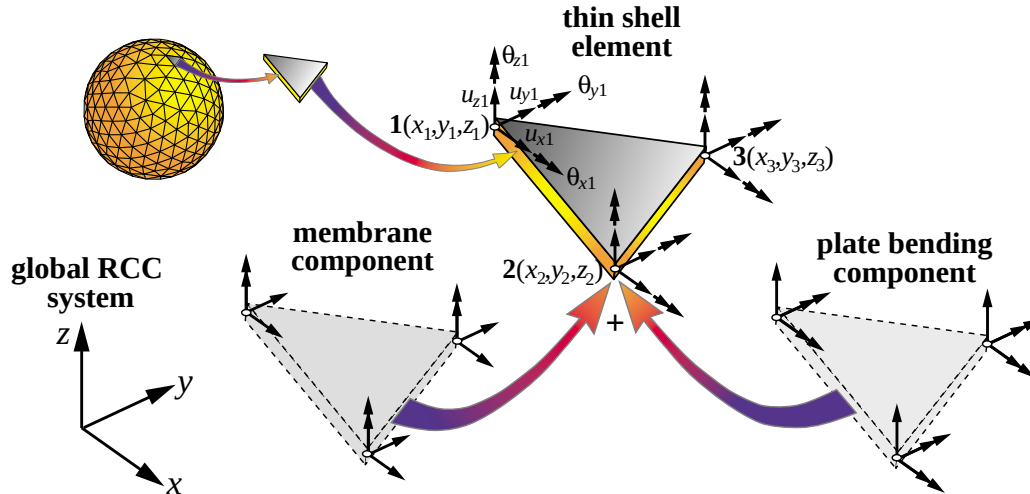


FIGURE 36.2. The HPSHEL3 triangular element, showing geometry and freedom configuration. The element is constructed by combining a membrane (plane stress) and a Krichhoff plate bending component.

§36.3. Element Implementation in *Mathematica*

§36.3.1. Localization

The first operation carried out on the shell element is transformation to a *local* coordinate system $\{\bar{x}, \bar{y}, \bar{z}\}$. For this the corners of the individual triangle are given the local numbers 1,2,3. The triangle geometry is fully defined by giving the corner global coordinates $\{x_i, y_i, z_i\}$, $i = 1, 2, 3$. Assuming that those points are not collinear, they will define the element midsurface plane.

The local frame is selected as illustrated in Figure 36.3:

1. The $\bar{x}$ axis is chosen parallel to side 1→2, with origin at the triangle centroid 0.
2. The $\bar{z}$ axis is normal to the plane of the triangle. Its direction is chosen so that the signed area of the triangle is positive when traversed 1→2→3.
3. The $\bar{y}$ axis is normal to $\bar{x}$ and $\bar{z}$ and forms a right-handed RCC system $\{\bar{x}, \bar{y}, \bar{z}\}$; thus $\bar{z} = \bar{x} \times \bar{y}$. It follows that $\{\bar{x}, \bar{y}\}$ span the triangle plane.

The *Mathematica* module SM3LocalSystem, listed in Figure 36.4, performs the global-to-local transformation.² The module is invoked as

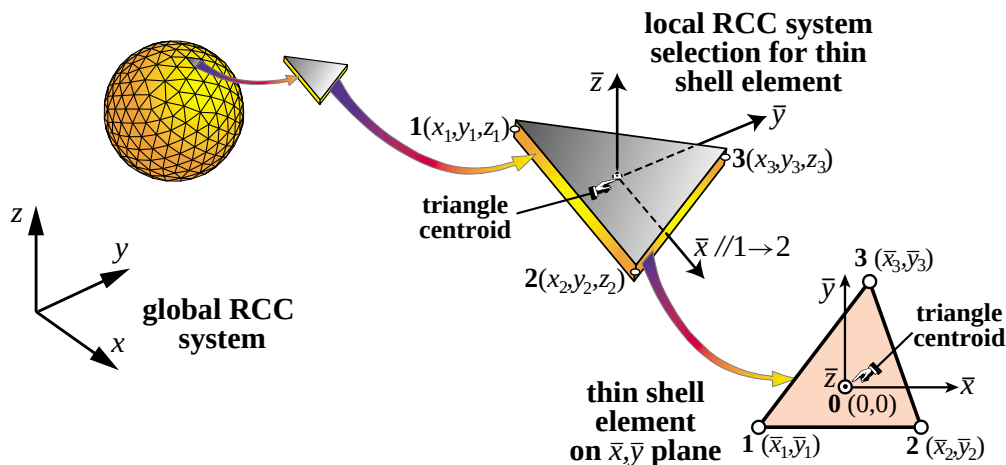
$$\{\text{ncoorbar}, \text{dcm}, \text{status}\} = \text{SM3LocalSystem}[\text{ncoor}] \quad (36.1)$$

The arguments are

ncoor Global node coordinates of element corners stored as the two-dimensional list $\{\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}, \{x_3, y_3, z_3\}\}$.

The function returns are

² The commented-out red-colored statements use built-in *Mathematica* functions forms of vector operators such as `Cross[]` for cross product and `.` (period) for dot product. These were replaced by inlined code to speed up the computations.

FIGURE 36.3. Selecting the local frame $\{\bar{x}, \bar{y}, \bar{z}\}$.

- ncoorbar** Local node coordinates of element corners stored as the two-dimensional list $\{\{\bar{x}1, \bar{y}1, 0\}, \{\bar{x}2, \bar{y}2, 0\}, \{\bar{x}3, \bar{y}3, 0\}\}$.
- dcm** Direction cosine matrix containing the partials of the local system axes with respect to the global ones. More precisely, $dcm[[i, j]]$ contains $\partial \bar{x}_i / \partial x_j$, $i, j = 1, 2, 3$.
- status** A status character string. Blank if no error detected, else error message.

For a simple numerical test of the module, take the triangle corners 1,2,3 as being located at $\{1, 4, 5\}$, $\{2, 6, 7\}$ and $\{3, 3, 11\}$, respectively. Hence **ncoor** is $\{\{1, 4, 5\}, \{2, 6, 7\}, \{3, 3, 11\}\}$. The call `SM3LocalSystem[ncoor]` returns the local coordinate array

ncoorbar = $\{\{-7/3, -5/3, 0\}, \{2/3, -5/3, 0\}, \{5/3, 10/3, 0\}\}$; thus $\bar{x}_1 = -7/3$, etc. The returned direction cosine matrix is

dcm = $\{\{1/3, 2/3, 2/3\}, \{2/15, -11/15, 2/3\}, \{14/15, -2/15, -1/3\}\}$. The **status** flag returns blank. Figure 36.5 shows the test in which the foregoing **ncoor** are supplied as floating point format, and so is the output printed in the bottom cell.

(Chapter in progress)

```

SM3ShellLocalSystem[ncoor_] := Module[{x1, y1, z1, x2, y2, z2, x3, y3, z3,
  x21, y21, z21, x32, y32, z32, xlr, ylr, zlr, x0, y0, z0, xyz10, xyz20, xyz30,
  dx=dy=dz={0, 0, 0}, ncoorbar=dcM=Table[0, {3}, {3}], status=" "},
  {{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}}=ncoor;
  {x0, y0, z0}={x1+x2+x3, y1+y2+y3, z1+z2+z3}/3;
  {x21, y21, z21, x32, y32, z32}={x2-x1, y2-y1, z2-z1, x3-x2, y3-y2, z3-z2};
  xlr=Sqrt[x21^2+y21^2+z21^2]; If [xlr<=0,
    status="Nodes 1-2 coincide"; Return[{ncoorbar, dcM, status}]];
  dx={x21, y21, z21}/xlr;
  (*dz=Cross[{x21, y21, z21}, {x32, y32, z32}]; too slow: inlined *)
  dz={y21*z32-z21*y32, z21*x32-x21*z32, x21*y32-y21*x32};
  zlr=Sqrt[dz[[1]]^2+dz[[2]]^2+dz[[3]]^2];
  If [zlr<=0, status="Nodes 1-2-3 are collinear";
    Return[{ncoorbar, dcM, status}]];
  dz=dz/zlr; (* dy=Cross[dz, dx]; too slow: inlined *)
  dy[[1]]=dz[[2]]*dx[[3]]-dz[[3]]*dx[[2]];
  dy[[2]]=dz[[3]]*dx[[1]]-dz[[1]]*dx[[3]];
  dy[[3]]=dz[[1]]*dx[[2]]-dz[[2]]*dx[[1]];
  ylr=Sqrt[dy[[1]]^2+dy[[2]]^2+dy[[3]]^2]; dy=dy/ylr;
  xyz10={x1-x0, y1-y0, z1-z0}; xyz20={x2-x0, y2-y0, z2-z0};
  xyz30={x3-x0, y3-y0, z3-z0}; dcM={dx, dy, dz};
  (*ncoorl={{dx.xyz10, dy.xyz10, dz.xyz10}, {dx.xyz20, dy.xyz20, dz.xyz20},
    {dx.xyz30, dy.xyz30, dz.xyz30}}; too slow: inlined *)
  ncoorbar={
    {dx[[1]]*xyz10[[1]]+dx[[2]]*xyz10[[2]]+dx[[3]]*xyz10[[3]],
     dy[[1]]*xyz10[[1]]+dy[[2]]*xyz10[[2]]+dy[[3]]*xyz10[[3]], 0},
    {dx[[1]]*xyz20[[1]]+dx[[2]]*xyz20[[2]]+dx[[3]]*xyz20[[3]],
     dy[[1]]*xyz20[[1]]+dy[[2]]*xyz20[[2]]+dy[[3]]*xyz20[[3]], 0},
    {dx[[1]]*xyz30[[1]]+dx[[2]]*xyz30[[2]]+dx[[3]]*xyz30[[3]],
     dy[[1]]*xyz30[[1]]+dy[[2]]*xyz30[[2]]+dy[[3]]*xyz30[[3]], 0}};
  Return[{ncoorbar, dcM, status}]];

```

FIGURE 36.4. Global-to-local system transformation module.

```

ncoor=N[{{1, 4, 5}, {2, 6, 7}, {3, 3, 11}}];
Print["ncoor=", ncoor//MatrixForm];
{ncoorbar, dcM, status}=SM3ShellLocalSystem[ncoor];
Print["ncoorbar=", ncoorbar//MatrixForm];
Print["dcM=", dcM//MatrixForm];

```

$$\text{ncoor} = \begin{pmatrix} 1. & 4. & 5. \\ 2. & 6. & 7. \\ 3. & 3. & 11. \end{pmatrix}$$

$$\text{ncoorbar} = \begin{pmatrix} -2.33333 & -1.66667 & 0 \\ 0.666667 & -1.66667 & 0 \\ 1.66667 & 3.33333 & 0 \end{pmatrix}$$

$$\text{dcM} = \begin{pmatrix} 0.333333 & 0.666667 & 0.666667 \\ 0.133333 & -0.733333 & 0.666667 \\ 0.933333 & -0.133333 & -0.333333 \end{pmatrix}$$

FIGURE 36.5. Global-to-local system transformation module test.